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Constructing surfaces with first Steklov eigenvalue of arbitrarily large multiplicity

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Abstract. We construct surfaces with arbitrarily large multiplicity for their first nonzero Steklov eigenvalue. The proof is based on a technique by Burger and Colbois originally used to prove a similar result for the Laplacian spectrum. We start by constructing surfaces S_p with a specific subgroup of isometry $G_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ for each prime p. We do so by gluing surfaces with boundary following the structure of the Cayley graph of G_p. We then exploit the properties of G_p and S_p in order to show that an irreducible representation of high degree (depending on p) acts on the eigenspace of functions associated with $\sigma_1(S_p)$, leading to the desired result.

1 Introduction

Let (S, g) be a smooth, compact, connected, Riemannian surface with boundary ∂S. The Steklov eigenvalue problem consists of finding $(f, \sigma) \in C^\infty(S) \times \mathbb{R}$ with $f \neq 0$ satisfying

$$\begin{cases} \Delta f = 0 & \text{in } S, \\ \partial_\nu f = \sigma f & \text{on } \partial S, \end{cases}$$

where ∂_ν is the outward normal derivative on ∂S. The numbers σ are called the Steklov eigenvalues of (S, g). They form a sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. This problem has attracted substantial attention in recent years. See [4] for a survey of recent results and open problems. The focus of this article is on the multiplicity $m_1(S, g)$ of the first nonzero eigenvalue $\sigma_1 > 0$. Upper bounds for $m_1(S, g)$ have been obtained by many authors in recent years. Notably, the works of Fraser and Schoen [6], Jammes [9] as well as Karpukhin, Kokarev, and Polterovich [10] brought multiple results concerning upper bounds for the multiplicity of Steklov eigenvalues in terms of the topology and the number of boundary components of the surface. The following result is a particular case of [10, Theorem 1].

Theorem 1.1 *Let (S, g) be an orientable compact Riemannian surface with non-empty boundary. Let $\mathfrak{g}$ be the genus of (S, g). Then,*

$$m_1(S, g) \leq 4\mathfrak{g} + 3.$$

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All known general upper bounds on $m_1(S,g)$ involve either the genus of the surface S, or both the genus and the number of boundary components of the surface S. These bounds are linear in terms of the genus (and of the number of boundary components, when applicable). However, none of these bounds are known to be sharp for all genus and the question to know if surfaces with arbitrarily large multiplicity $m_1(S,g)$ is left open in the literature. The main result of this article gives a positive answer to that question.

Theorem 1.2 *For each prime number p, there exists a compact Riemannian surface (S_p, g_p) of genus $\mathfrak{g}_p = (p-1)^2$ with $2p$ boundary components such that $m_1(S_p, g_p) \geq p-1$.*

The proof of this result is based on a technique that was developed by Burger and Colbois [1, 3] used originally to obtain a similar result for the eigenvalues of the Laplacian on closed surfaces. The idea is to construct a surface (S_p, g) with a specific subgroup of isometry $G_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ by building the surface around the Cayley graph of G_p. Using the fact that the problem is invariant by isometry, we can define an action of G_p on the space of eigenfunctions $E_1(S_p)$ associated with σ_1. This action on $E_1(S_p)$ is also isometric for the inner product of $L_2(S_p)$. This action can then be decomposed in irreducible representations of the group G_p. By choosing a group with an irreducible representation of high degree and by showing that this representation acts on $E_1(S_p)$, we show that E_1 must be of large dimension, which implies that σ_1 has high multiplicity. The presence of boundary components for the Steklov problem forces us to alter the construction given by Burger and Colbois in a nontrivial way and allows us to get a slightly stronger result.

Remark 1.3 Note that in the provided constructions, the number of boundary components also goes to infinity. It is not known whether there exists a similar sequence of surfaces where the multiplicity and genus go to infinity and the number of boundary components remains fixed.

1.1 Multiplicity bounds for the Steklov problem with density

In his paper [9], Jammes studied a more general version of the Steklov problem:

$$\begin{cases} \operatorname{div}(\gamma \nabla f) = 0 & \text{in } S, \\ \gamma \partial_\nu f = \sigma \rho f. & \text{on } \partial S. \end{cases}$$

Here, $\gamma \in C^\infty(S)$ and $\rho \in C^\infty(\partial S)$ are strictly positive density functions. The multiplicity of the first nonzero eigenvalue is now written $m_1(S, g, \gamma, \rho)$ to stress the dependence on the density functions. Given a surface S of genus $\mathfrak{g}$ with l boundary components, let

$$M_1(\mathfrak{g}, l) := \sup_{g,\rho,\gamma} m_1(S, g, \gamma, \rho),$$

where the supremum is over all Riemannian metrics and densities. In his paper [8], Jammes proposed the following conjecture:

$$M_1(\mathfrak{g}, l) = \mathrm{Chr}_0(S) - 1,$$

where the *relative chromatic number* $\mathrm{Chr}_0(S)$ is the number of vertices of the largest complete graph that can be embedded in S with all vertices on the boundary ∂S. The conjecture is analogous to a conjecture of Colin de Verdière for closed surfaces [5]. In his paper [9], Jammes proved part of his conjecture. He showed that

$$M_1(\mathfrak{g}, l) \geq \mathrm{Chr}_0(S) - 1.$$

He also showed that $\mathrm{Chr}_0(S) = O(\sqrt{\mathfrak{g}})$ as $\mathfrak{g} \to +\infty$, providing examples where the multiplicity $m_1(S, g, \gamma, \rho)$ is arbitrarily large. However, the proof of this last inequality relies heavily on the theory of perturbation and uses the freedom afforded by the density functions in an essential way.

Outline of this article. In Section 2, we describe the construction of Cayley graphs Γ_p associated with the groups G_p. In Section 3, we discuss the construction of S_p by using G_p and discuss the basic properties of such surfaces. In Section 4, we catalog all irreducible representations of G_p. In Section 5, we prove the main theorem.

2 Construction of the Cayley graphs Γ_p

The Cayley graph $\Gamma(G, S)$ of a group G with set of generators S is the oriented graph with vertices $g \in G$ and edges between $g_1, g_2 \in S$ if and only if there is $\delta \in S$ such that $g_1\delta = g_2$. This relation also induces a natural orientation on each edge of the graph.

Note that the construction of the graph is dependent on the choice of set of generators S. Also, since the vertices are associated with group elements, the number of vertices is the same as the number of elements in the group. Finally, since every generator and its inverse can be applied to an element $g \in G$, the degree of each vertex in Γ is equal to $2|S|$. For more details on Cayley graphs, see [11].

Let $p \geq 3$ be a prime number and

$$G_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*,$$

where $\mathbb{Z}_p$ is the cyclic group of p elements and $\mathbb{Z}_p^*$ is the multiplicative group modulo p. Alternatively, we can represent G_p in terms of generators. Let k be an integer such that k is a primitive $p-1$-root modulo p. Then,

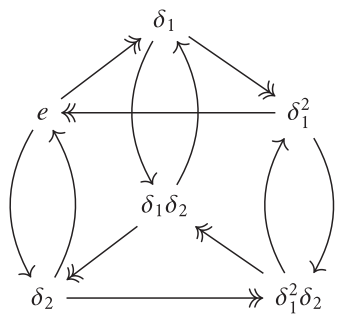
$$G_p = \langle \delta_1, \delta_2 \mid \delta_1^p = e,\ \delta_2^{p-1} = e,\ \delta_2^{-1}\delta_1\delta_2 = \delta_1^k \rangle.$$

One can verify that these two groups are isomorphic by considering the isomorphism ψ that goes from $\mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ to G_p induced by

$$\psi(\bar{1}, \bar{1}) = \delta_1,$$

$$\psi(\bar{0}, \bar{k}) = \delta_2.$$

This definition allows us to build the Cayley graph of G_p with the generator set $S = \{\delta_1, \delta_2\}$. For example, we obtain the following Cayley graph when performing this construction for $p = 3$.



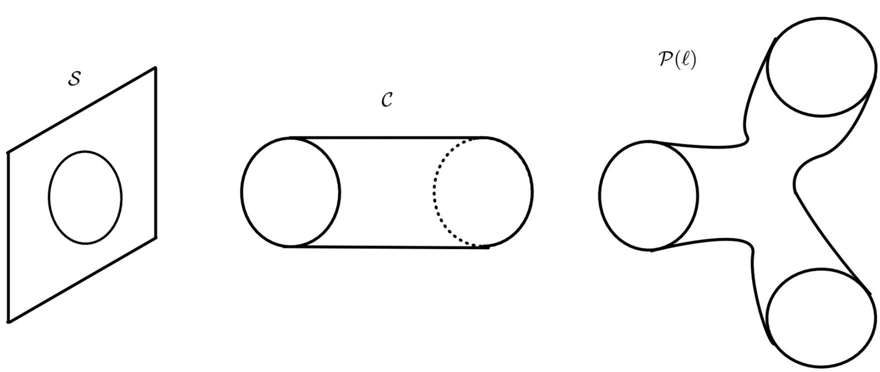
Here, the double arrows represent the action by δ_1 and the simple arrows represent the action by δ_2. Since $|G_p| = p(p-1)$, the graphs $\Gamma(G_p, \{\delta_1, \delta_2\})$ get increasingly complicated as p grows larger. As a direct consequence of this construction, the graph $\Gamma(G_p, \{\delta_1, \delta_2\})$ admits an action of G_p by isometry induced by the action of the elements of the group G on the vertices of the graph by left multiplication. For the next sections, we will opt for the more compact notation $\Gamma_p := \Gamma(G_p, \{\delta_1, \delta_2\})$.

3 Construction of the surface S_p

Let $\ell \in \mathbb{R}_{>0}$ and

$$\mathcal{S} := \{(x, y) \in [-2, 2] \times [-2, 2] \ : \ x^2 + y^2 > 1\}$$

be the perforated square equipped with the Euclidean metric of $\mathbb{R}^2$. Let $\mathcal{P}(\ell)$ be the pair of hyperbolic pants with one boundary component of length 2π and two boundary components of length ℓ. Note that by Theorem 3.1.7 of [2], such pants are uniquely defined (up to isometry) by the lengths of their geodesic boundary components. The building block $B(\ell)$ is then obtained by gluing one boundary component of the cylinder $\mathcal{C} := S^1 \times [0, 1]$ along the inner circular boundary component of $\mathcal{S}$ and the other boundary component of $\mathcal{C}$ along the boundary component of $\mathcal{P}(\ell)$ of length 2π. The metric on $\mathcal{C}$ is chosen so as to interpolate smoothly between the metric on $\mathcal{S}$ and the metric on $\mathcal{P}(\ell)$. Note that every hyperbolic gluing in this section onward is performed with no twist.



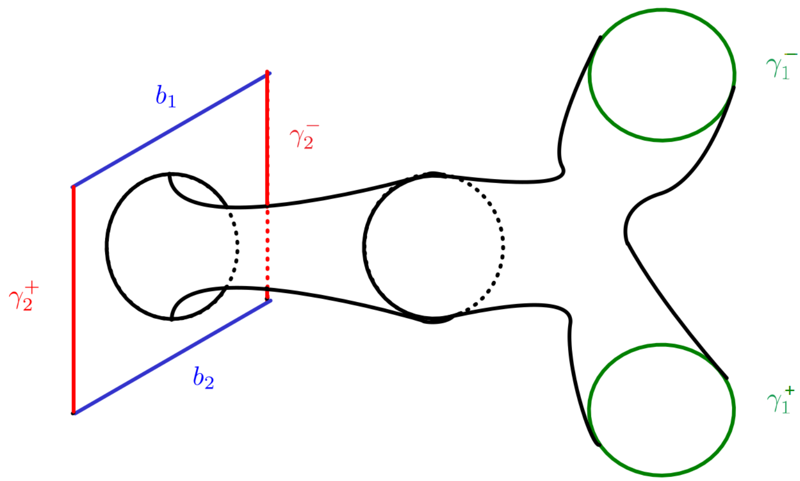


Figure 1: The building block $B(\ell)$.

Thus, the building block $B(\ell)$ posesses three boundary components: one of them corresponding to the sides of the square $\mathcal{S}$ and two of them corresponding to the boundary components of length ℓ in $\mathcal{P}(\ell)$. In what follows, we will denote by γ_1^+ and γ_1^- the two boundary components of length ℓ in $B(\ell)$. We will also denote (γ_2^+, γ_2^-) and (b_1, b_2) the two pairs of parallel sides of length 4 of the remaining boundary component in $B(\ell)$ (Figure 1).

The surface $S_p(\ell)$ is obtained by gluing copies of $B(\ell)$ along the Cayley graph Γ_p of Section 2 via the following procedure. First, we assign to each vertex v of Γ_p a copy of $B(\ell)$ labeled $B_v(\ell)$. We will denote the boundary components $\gamma_i^\pm$ of $B_v(\ell)$ by $\gamma_{i,v}^\pm$ for $i \in \{1, 2\}$. Let v, w be two vertices of Γ_p and let $B_v(\ell), B_w(\ell)$ be their respective assigned copy of $B(\ell)$. We glue $B_v(\ell)$ along $\gamma_{1,v}^+$ to $B_w(\ell)$ along $\gamma_{1,w}^-$ if there is an edge corresponding to δ_1 in Γ_p going from v to w. Similarly, we glue $B_v(\ell)$ along $\gamma_{2,v}^+$ to $B_w(\ell)$ along $\gamma_{2,w}^-$ if there is an edge corresponding to δ_2 in Γ_p going from v to w (Figure 2).

Lemma 3.1 *The surface S_p has $2p$ boundary components and genus $\mathfrak{g} = 1 + |G_p| = 1 + p(p-2) = (p-1)^2$.*

Proof For $j \in \{0, 1, \dots, p-1\}$, let

$$C_j := \{g \in G \ : \ \exists m \in \mathbb{N},\ g = \delta_1^j \delta_2^m\}$$

denote the δ_2 generated cycles of δ_1^j in Γ_p. Since

$$|G_p| \leq \sum_{j=0}^{p-1} |C_i| \leq p|C_0| = p(p-1) = |G_p|,$$

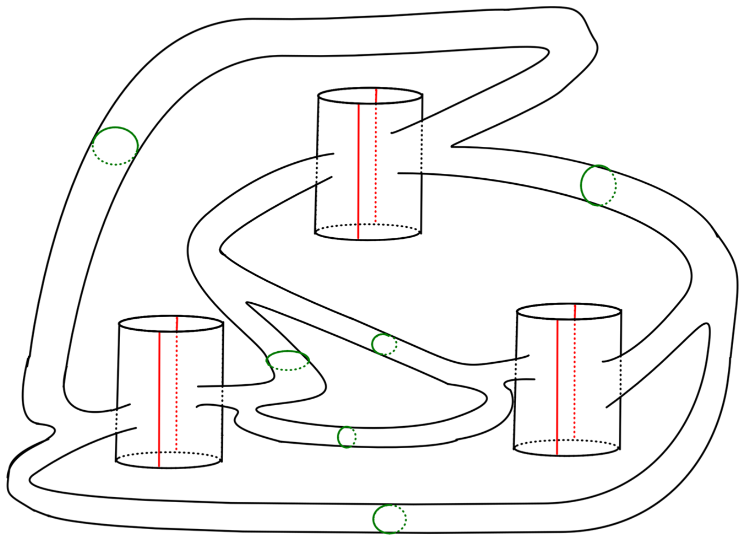


Figure 2: The surface S_3.

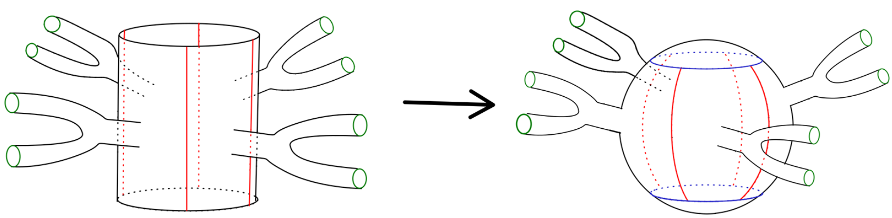


Figure 3: Transformation of Cyl_j in S_5 by gluing disks along its boundary components.

we have that the C_i are disjoint and form a partition of G. Since, by construction, we have two boundary components per δ_2 cycles C_i, the first assertion of the lemma follows.

For $j \in \{0, 1, \ldots, p-1\}$, let

$$Cyl_j := \bigcup_{g \in C_j} B_g(\ell) \subset S_p$$

be the region of S_p corresponding to the δ_2 cycles. It follows that the Cyl_j are pairwise disjoint cylinders with $2(p-1)$ additional boundary components for all $j \in \{0, 1, \ldots, p-1\}$. In order to compute the genus of S_p, we glue disks along its boundary components and compute the genus of the resulting closed surface $\hat{S}_p$. Note that such gluings transform Cyl_j into spheres Sph_j with $2(p-1)$ boundary components (Figure 3).

It follows that the Euler characteristic of Sph_j is exactly that of the sphere minus the $2(p-1)$ faces removed. Thus, for all $j \in \{0, \dots, p-1\}$,

$$\chi(Sph_j) = 2 - 2(p-1) = -2(p-2).$$

By the additivity of the Euler characteristic, we have that

$$2 - 2\mathfrak{g} = \chi(\hat{S}_p) = p \cdot \chi(Sph_j) = -2p(p-2).$$

Solving for p yields

$$\mathfrak{g} = (p-1)^2,$$

which concludes the proof. ∎

4 Irreducible representations of G_p

Note that since Γ_p is a Cayley graph of G_p, G_p acts on the vertices Γ_p by isometry via the group operation. This action can be lifted to an action by isometry of G_p on S_p. Let

$$\psi_{v \to w} \ : \ B_v(\ell) \to B_w(\ell)$$

be the canonical isometry between $B_v(\ell)$ and $B_w(\ell)$. Then, one can explicitly declare the above-mentioned action by isometry of G_p on S_p via

$$\phi_g : S_p \to S_p,$$

$$x_v \mapsto \phi_g(x_v) = \psi_{v \to g(v)}(x_v),$$

where x_v is a point of $B_v(\ell)$. This action of G_p on the surface S_p induces an action of G_p on the eigenspace of the functions associated with $\sigma_1(S_p)$ through the maps

$$\phi_g* : E_1(S_p) \to E_1(S_p),$$

$$f \mapsto f \circ \phi_g.$$

Since this action is linear, by taking a basis of $E_1(S_p)$, one can represent the action of $g \in G_p$ by a matrix $A_g \in GL_N(\mathbb{C})$, where N represents the dimension of $E_1(S_p)$. Thus, we have a representation of G_p acting on $E_1(S_p)$. By changing the basis of $E_1(S_p)$ if necessary, the action of G_p on $E_1(S_p)$ can be decomposed as a direct sum of irreducible representations. This induces a decomposition of $E_1(S_p)$ in a direct sum of subspaces associated with irreducible representations of G_p acting on $E_1(S_p)$. Thus, if we can show that an irreducible representation of high degree acts on $E_1(S_p)$, it follows that there is a subspace of $E_1(S_p)$ of high dimension, which in turn implies that $E_1(S_p)$ has high dimension and that $\sigma_1(S_p)$ has high multiplicity. All of the aforementioned results on representation theory can be seen in greater detail in [3].

Thus, we will now explicitly discuss the irreducible representations of G_p. Since it is a well-known topic, we will omit the technical details and simply state the results.

First, G_p admits $(p-1)$ irreducible representations of degree 1. Let α be a primitive $(p-1)$-root of 1 in $\mathbb{C}$. Then, for all r such that $1 \le r \le p-1$, we have the following irreducible representations:

$$\rho_r : G_p \to \mathbb{C}^*$$

induced by

$$\rho_r(\delta_1) = 1,$$
$$\rho_r(\delta_2) = \alpha^r.$$

One can verify that ρ_r is a group homomorphism for all r. These representations are all irreducible and non-equivalent. Observe that since δ_1 is in the kernel of ρ_r, the subgroup $\mathbb{Z}_p$ of G_p generated by δ_1 lies in the kernel of ρ_r for all r.

Next, we will show that G_p admits 1 irreducible representation of degree $p-1$. Let ξ be a primitive p-root of 1 in $\mathbb{C}$ and k be a primitive $(p-1)$-root of 1 modulo p. It follows that $\xi^{k^i} \neq \xi^{k^j}$ if $j \neq i$ for all $0 \le i, j \le p-2$. We have that

$$\rho : G_p \to GL_{p-1}(\mathbb{C})$$

induced by

$$\rho(\delta_1) = \begin{pmatrix} \xi & & & 0 \\ & \xi^k & & \\ & & \cdots & \\ 0 & & & \xi^{k^{p-2}} \end{pmatrix}, \quad \rho(\delta_2) = \begin{pmatrix} 0 & 0 & \cdots & & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ & 0 & & & \\ & & & & \\ 0 & & & 1 & 0 \end{pmatrix}$$

constitutes a representation of G_p. We then prove the following lemma.

Lemma 4.1 *ρ is an irreducible representation of G_p.*

Proof By Schur's lemma (Lemma 1.7 in [7] for example), it suffices to prove that the only matrices commuting with $\rho(g)$ for all $g \in G_p$ are scalar matrices.

Let $A \in GL_{p-1}(\mathbb{C})$ such that it commutes with $\rho(g)$ for all $g \in G_p$. Then, in particular, it commutes with $\rho(\delta_1)$. Let

$$B := A\rho(\delta_1),$$
$$C := \rho(\delta_1)A.$$

It follows that $B = C$ since A and $\rho(\delta_1)$ commute. For all $i \neq j$, we have

$$b_{i,j} = a_{i,j}\xi^{k^{j-1}},$$
$$c_{i,j} = a_{i,j}\xi^{k^{i-1}}.$$

Since $\xi^{k^{j-1}} \neq \xi^{k^{i-1}}$ if $j \neq i$, it implies that A must be diagonal. Let

$$X := A\rho(\delta_2),$$
$$Y := \rho(\delta_2)A.$$

Once again, we obtain that $X = Y$ since A must commute with $\rho(\delta_2)$. For all $1 \leq i \leq p-2$, we have

$$x_{i+1,i} = a_{i,i},$$

$$y_{i+1,i} = a_{i+1,i+1}.$$

It implies that

$$a_{1,1} = a_{2,2} = \cdots = a_{p-1,p-1}.$$

It follows that A must be scalar and the proof is complete. ■

Finally, since the sum of the square of the degrees of these irreducible representations is equal to $|G_p|$, we can conclude that we have all the possible irreducible representations of G_p.

5 Proof of the main result

What is left to show is that the representation of degree $p-1$ acts on $E_1(S_p(\ell))$ for ℓ small enough. We will do so by showing that none of the irreducible representations of degree 1 act on $E_1(S_p(\ell))$ for ℓ small enough. Our strategy will consist of considering the quotient of $S_p(\ell)$ by the action of the normal subgroup $\mathbb{Z}_p$ of G_p, that we will denote $S'_p(\ell)$. Since $\mathbb{Z}_p$ lies in the kernel of every representation of degree 1, if we can show that $\sigma_1(S_p(\ell)) \neq \sigma_1(S'_p(\ell))$ for all $\ell < \varepsilon$ for a certain ε, then the result will follow.

In order to show that the first nonzero eigenvalues of $S_p(\ell)$ and $S'_p(\ell)$ do not coincide for ℓ small enough, we start by showing the following lemma.

Lemma 5.1

$$\lim_{\ell \to 0} \sigma_1(S_p(\ell)) = 0.$$

To prove this lemma, we will need two classical results.

Theorem 5.2 *(Variational characterization of the Steklov eigenvalues)*

$$\sigma_k = \min_{E \in \mathcal{H}_{k+1}} \max_{0 \neq u \in E} R_S(u),$$

where $\mathcal{H}_{k+1}$ is the set of all $k+1$-dimensional subspaces in the Sobolev space $H^1(S)$ and

$$R_S(u) = \frac{\int_S |\nabla u|^2 dA}{\int_{\partial S} |u|^2 ds}.$$

The second important result is an application of the collar theorem [2, Theorem 4.1.1]. Let

$$\mathcal{C}(S_p(\ell)) := \bigcup_{g \in G_p} \gamma_{1,g}^{\pm} \subset S_p(\ell)$$

and the neighborhood of $x \in S_p(\ell)$

$$\mathcal{T}(x) := \{q \in S_p(\ell) \mid dist(q,x) \leq w(\ell)\},$$

where

$$w(\ell) := \operatorname{arcsinh}(\coth(\tfrac{\ell}{2})).$$

Since the $\gamma^{\pm}_{1,g} \subset S_p(\ell)$ are gluing points of hyperbolic pants of constant curvature -1 the collar theorem applies locally to $\mathcal{T}(\alpha)$ and we get the following.

Theorem 5.3 *For connected components $\alpha, \beta \subset \mathcal{C}(S_p(\ell))$, then $\mathcal{T}(\alpha) \cap \mathcal{T}(\beta) = \varnothing$ if and only if $\alpha \cap \beta = \varnothing$. Furthermore, for all connected components $\alpha \in \mathcal{C}(S_p)$, $\mathcal{T}(\alpha)$ is isometric to the cylinder $[-w(\ell), w(\ell)] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2 \cosh^2(\rho) dt^2$ on the Fermi coordinates of the cylinder based on α parametrized with speed ℓ.*

We can now prove Lemma 5.1 using Theorem 5.2 and Lemma 5.3.

Proof Our goal will be to build two test functions $f_1, f_2 \in H^1(S_p(\ell))$ such that $R_{S_p(\ell)}(f_1)$ and $R_{S_p(\ell)}(f_2)$ go to zero as ℓ goes to zero.

Observe that $S_p - \mathcal{C}(S_p(\ell))$ has p connected components. This is a direct consequence of the construction of $S_p(\ell)$ around the Cayley graph of G_p with generators δ_1, δ_2. We will note these connected components M_i for $i \in \{1, 2, \ldots, p\}$ and $\mathcal{C}(M_i) \subset \mathcal{C}(S_p(\ell))$ the union of geodesics joining M_i to its complement in $S_p(\ell)$. Thus, $\mathcal{C}(M_i)$ is the union of $2p-2$ disjoint geodesics of lengths ℓ.

Let

$$M_1^0 := M_1 - \bigcup_{x \in \mathcal{C}(M_1)} \mathcal{T}(x) \cap M_1,$$

$$M_2^0 := M_2 - \bigcup_{x \in \mathcal{C}(M_2)} \mathcal{T}(x) \cap M_2$$

and

$$f_1(p) := \begin{cases} 1 & \text{if } q \in M_1^0 \\ 0 & \text{if } q \in M_1^c \\ \frac{dist(q,\alpha)}{w(\ell)} & \text{if } q \in \mathcal{T}(\alpha) \cap M_1 \text{ for some connected component } \alpha \in \mathcal{C}(M_1) \end{cases}$$

$$f_2(p) := \begin{cases} 1 & \text{if } q \in M_2^0 \\ 0 & \text{if } q \in M_2^c \\ \frac{dist(q,\alpha)}{w(\ell)} & \text{if } q \in \mathcal{T}(\alpha) \cap M_2 \text{ for some connected component } \alpha \in \mathcal{C}(M_2). \end{cases}$$

Both functions are continuous and weakly differentiable. Since $|\operatorname{grad}(dist(q,\alpha))| = 1$, it follows that $f_1, f_2 \in H^1(M)$. Furthermore, since both functions have disjoint supports, $span\{f_1, f_2\} \subset H^1(S_p(\ell))$ is a subspace of dimension 2. We can now calculate $R_{S_p(\ell)}(f_i)$

$$
\begin{aligned}
R_S(f_1) &= \frac{\int_{S_p(\ell)} |\nabla f_1|^2 dV}{\int_{\partial S_p(\ell)} |f_1|^2 dS} \\
&\leq \frac{\sum_\alpha \int_{\mathcal{T}(\alpha)\cap M_1} |\nabla f_1|^2 dV}{A} \\
&= \frac{(2p-2)}{A} \int_{\mathcal{T}(\hat{\alpha})\cap M_1} |\nabla f_1|^2 dV, \qquad (5.1)
\end{aligned}
$$

where the sum is taken over all the connected components $\alpha \subset \mathcal{C}(M_1)$. Here, A is a constant independent of ℓ and $\hat{\alpha}$ is a given connected component of $\mathcal{C}(M_1)$. This last integral yields

$$
\begin{aligned}
\int_{\mathcal{T}(\hat{\alpha})\cap M_1} |\nabla f_1|^2 dV &= \frac{1}{w(\ell)^2} \int_{\mathcal{T}(\hat{\alpha})\cap M_1} |\nabla dist(q,\hat{\alpha})|^2 dV \\
&= \frac{1}{w(\ell)^2} \int_{\mathcal{T}(\hat{\alpha})\cap M_1} dV = \frac{Area(\mathcal{T}(\hat{\alpha})\cap M_1)}{w(\ell)^2}. \qquad (5.2)
\end{aligned}
$$

We now need to calculate $Area(\mathcal{T}(\hat{\alpha})\cap M_1) = \frac{Area(\mathcal{T}(\hat{\alpha}))}{2}$. By Lemma 5.3, $\mathcal{T}(\hat{\alpha})$ is isometric to the cylinder $[-w(\ell), w(\ell)] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2 \cosh^2(\rho) dt^2$ on the Fermi coordinates of the cylinder based on $\hat{\alpha}$ parametrized with speed ℓ. Thus,

$$
\begin{aligned}
Area(\mathcal{T}(\hat{\alpha})) &= \ell^2 \int_0^1 \int_{-w(\ell)}^{w(\ell)} \cosh^2(\rho) d\rho dt \\
&= \frac{\ell^2}{2} \int_0^1 [\rho + \cosh(\rho)\sinh(\rho)]_{\rho=-w(\ell)}^{\rho=w(\ell)} dt = \ell^2 w(\ell). \qquad (5.3)
\end{aligned}
$$

Piecing equations 5.1–5.3 together, we obtain the upper bound

$$
R_S(f_1) \leq \frac{(2p-2)\ell^2}{4Aw(\ell)}.
$$

Since $w(\ell)$ goes to ∞ when ℓ goes to 0, we have that

$$
\lim_{\ell\to 0} R_S(f_1) = 0.
$$

The calculation for f_2 is similar. By the variational characterization of the Steklov eigenvalues, the result follows. ■

What remains to show is that $\sigma_1(S'_p(\ell)$ does not go to zero when ℓ goes to zero. Before proving this fact directly, we will discuss some basic properties of the compact surface

$$
S'_p(\ell) := S_p(\ell)/\mathbb{Z}_p.
$$

Since

$$
G_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*
$$

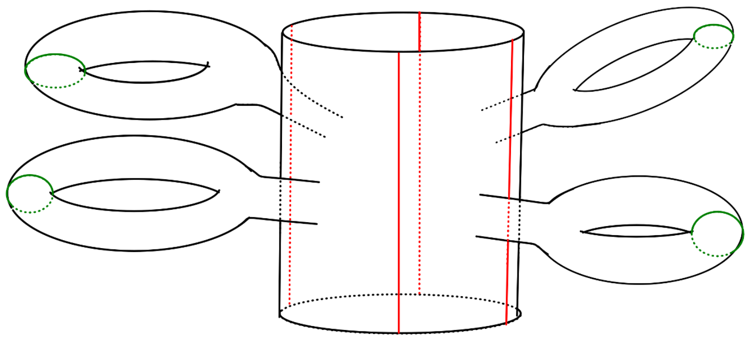


Figure 4: The surface S'_5.

it follows that

$$G_p/\mathbb{Z}_p \cong \mathbb{Z}_p^*$$

via the isomorphism induced by the quotient map. Hence, $S'_p(\ell)$ is composed of copies of $B(\ell)$ that we will note $B_{\bar{\delta}_2^i}(\ell)$ for $1 \leq i \leq p-1$ corresponding to the equivalence classes of elements δ_2^i in G_p under the quotient. Observe that the $B_{\bar{\delta}_2^i}(\ell)$ will be glued together along $\gamma^+_{2,\bar{\delta}_2}$ and $\gamma^-_{2,\bar{\delta}_2}$ in $S'_p(\ell)$ in the same way that the $B_{\delta_2^i}(\ell)$ are glued together along γ^+_{2,δ_2^i} and γ^-_{2,δ_2^i} in $S_p(\ell)$. In order to get a better idea of the properties of $S'_p(\ell)$, we will study how the $B_{\bar{\delta}_2^i}(\ell)$ are glued together along $\gamma^+_{1,\bar{\delta}_2^i}$ and $\gamma^-_{1,\bar{\delta}_2^i}$.

Let j be an integer such that $1 \leq j \leq p-1$. Then, there exist κ, an integer such that

$$j\kappa \equiv -1 \mod p.$$

Observe that since $\bar{\delta}_2^j = \overline{\delta_1^\kappa \delta_2^j}$ then $B_{\bar{\delta}_2^j}(\ell) = B_{\overline{\delta_1^\kappa \delta_2^j}}(\ell)$. Note also that in $S_p(\ell)$, $\gamma^+_{1,\delta_1^\kappa \delta_2^j}$ is glued along $\gamma^-_{1,\delta_1^\kappa \delta_2^j \delta_1}$. Since we have that

$$\delta_1^\kappa \delta_2^j \delta_1 = \delta_2^j \delta_1^{\kappa j+1} = \delta_2^j,$$

it follows that in the quotient $S'_p(\ell)$, $\gamma^+_{1,\bar{\delta}_2^j}$ is glued along $\gamma^-_{1,\bar{\delta}_2^j}$ for all j. All the gluings are done with no twist, since the original gluings had no twist.

It follows that $S'_p(\ell)$ can be constructed in the following way. Let $B'(\ell)$ be the building block obtained from $B(\ell)$ by gluing γ_1^+ along (Figure 4) γ_1^-. Then, $S'_p(\ell)$ is the surface obtained by gluing the building blocks $B'(\ell)$ along $\gamma_2^\pm$ following the Cayley graph of $\mathbb{Z}_p^*$ with one generator (Figure 5).

A direct consequence of this discussion is that $S'_p(\ell) - \cup_{g \in G_p} \gamma^\pm_{1,\bar{g}}$ possesses a single connected component. We will use this information in order to prove the following lemma and complete the proof of the main result.

Lemma 5.4 *There exists a constant* $C = C(p) > 0$ *such that for all* $\ell > 0$,

$$\sigma_1(S'_p(\ell)) \geq C.$$

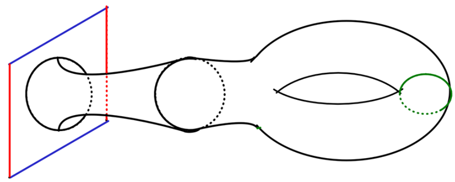


Figure 5: The building block $B'(\ell)$.

In order to prove this, we will need to introduce the mixed Steklov–Neumann problem. Let A be a domain on the surface S such that $\partial S \subset A$. We note $\partial_I A = \partial A - \partial S$ the interior boundary of A. The Steklov–Neumann problem on A is the eigenvalue problem that consists of finding $\sigma^N \in \mathbb{R}$ and $f : \bar{A} \to \mathbb{R}$ nonzero such that:

$$\begin{cases} \Delta f = 0 & \text{in } A, \\ \partial_\nu f = \sigma^N f & \text{on } \partial S, \\ \partial_\nu f = 0 & \text{on } \partial_I A. \end{cases}$$

Here, ∂_ν is the outward normal derivative on ∂A. As in the Steklov problem, the eigenvalues of this mixed problem form a discrete sequence $0 = \sigma_0^N(A) < \sigma_1^N(A) \leq \cdots \nearrow \infty$. We also have the following characterization.

Theorem 5.5 (Variational characterization of the Steklov–Neumann eigenvalues)

$$\sigma_k^N(A) = \min_{E \in \mathcal{H}_{k+1}} \max_{0 \neq u \in E} R_A^N(u),$$

where $\mathcal{H}_{k+1}$ is the set of all $k+1$-dimensional subspaces in the Sobolev space $H^1(A)$ and

$$R_A^N(u) = \frac{\int_A |\nabla u|^2 dV_A}{\int_{\partial S} |u|^2 dV_{\partial S}}.$$

With this result in hand, we can prove Lemma 5.4.

Proof By a direct comparison of R_A^N and R_S, we obtain the inequality

$$\sigma_1^N(A) \leq \sigma_1(S)$$

for all suitable $A \subset S$. Thus, it suffices to find a candidate $A \subset S'_p$ such that A is invariant under change of ℓ. We will start by identifying such a domain in the building block $B'(\ell)$ and then lift it to $S'_p(\ell)$.

From the discussion above, $B'(\ell)$ is obtained by gluing the Euclidian punctured square $\mathcal{S}$ along a cylinder $\mathcal{C}$ and a pair of hyperbolic paints $\mathcal{P}(\ell)$ with one boundary component on length 1 and two of length ℓ. Thus, $\mathcal{S}$ when seen as a subset of $B'(\ell)$ is invariant under changes of ℓ. Let

$$\mathcal{S}_{\tilde{g}} \subset B_{\tilde{g}}(\ell) \subset S'_p(\ell)$$

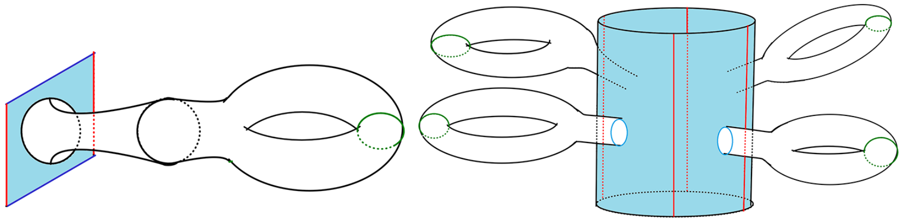


Figure 6: $\mathcal{S}$ in $B'(\ell)$ and A in S'_5, respectively.

be the copy of $\mathcal{S}$ in $B_{\bar{g}}(\ell)$ for $\bar{g} \in \mathbb{Z}_p^*$. Let

$$A := \bigcup_{\bar{g} \in \mathbb{Z}_p^*} \mathcal{S}_{\bar{g}}$$

and

$$\mathcal{C}(S'_p(\ell)) := \bigcup_{\bar{g} \in \mathbb{Z}_p^*} \gamma_{1,\bar{g}}^{\pm} \subset S'_p(\ell).$$

Note that A is still connected since $S'_p - \mathcal{C}(S'_p)$ is connected. We also have $\partial S'_p \subset A$. Furthermore, we can observe that A is invariant under the change of (Figure 6) ℓ. Thus, Theorem 5.5 yields

$$0 < \sigma_1^N(A) \leq \sigma_1(S'_p)$$

completing the proof. ■

All that is left to do to obtain Theorem 1.2 is to use Lemmas 5.1 and 5.4 in conjunction with a representation theory argument. By the two aforementioned lemmas, we know that for ℓ small enough,

$$\sigma_1(S_p) \neq \sigma_1(S'_p).$$

Fix this ℓ as such. Since $\mathbb{Z}_p$ is in the kernel of every representation of degree 1 of G_p, if these representations were to act on $E_1(S_p)$, we would have that

$$\sigma_1(S_p) = \sigma_1(S_p/\mathbb{Z}_p) = \sigma_1(S'_p).$$

However, it is not the case. It follows that the representation of degree 1 of G_p cannot be acting on $E_1(S_p)$. Thus, the only representation remaining, which is the representation of degree $p-1$, acts on $E_1(S_p)$. It follows that the dimension of $E_1(S_p)$ is at least $p-1$. This completes the proof of Theorem 1.2.

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