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Perturbed cone theorems for proper harmonic maps

Renan Assimos, Balázs Márk Békési, and Giuseppe Gentile

Abstract. Inspired by the halfspace theorem for minimal surfaces in $\mathbb{R}^3$ of Hoffman–Meeks, the halfspace theorem of Rodriguez–Rosenberg, and the classical cone theorem of Omori in $\mathbb{R}^n$, we derive new non-existence results for proper harmonic maps into perturbed cones in $\mathbb{R}^n$, horospheres in $\mathbb{H}^n$, culminating in a generalization of Omori's theorem in arbitrary Riemannian manifolds. The technical tool proved here extends the foliated Sampson's maximum principle, initially developed in the first author's Ph.D. thesis, to a non-compact setting.

1 Introduction

The analysis of harmonic maps has a central role in Riemannian geometry. The reason for such a centrality is due to the fact that several interesting objects in Riemannian geometry are *harmonic*. For instance, minimal submanifolds are immersed submanifolds so that the immersion $u : M \to (N, h)$ is harmonic with respect to the induced metric $g = u^* h$. Geodesics are yet another prominent example of harmonic maps. Indeed, $\gamma : I \to (N, h)$, where I is an interval in $\mathbb{R}$, is a geodesic if and only if γ is a harmonic map. Similarly, one has closed geodesics as harmonic maps $\gamma : S^1 \to (N, h)$. These are just a few examples showing that, knowing the existence of harmonic maps as well as their behavior, allows for applications to several other problems in Riemannian geometry.

One of the first systematic approaches to the construction of harmonic maps between manifolds can be traced back to the groundbreaking work of Eells and Sampson in 1964 [14]. There, the authors describe a deformation method for producing harmonic maps between closed manifolds, the so-called harmonic map flow. The harmonic map flow allows for the conclusion of several existence results for harmonic maps in interesting settings, the most classical one being the existence of harmonic maps into Riemannian manifolds with non-positive sectional curvature. Once the existence of harmonic maps between Riemannian manifolds is given, a naturally arising question is the following:

Which obstructions would prevent the existence of a harmonic map?

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To the best of our knowledge, the first ever attempt to this question, although with a slightly different flavor, is the analysis performed by Omori in 1967. In [21] the author shows that the second fundamental form of an isometric immersion inside a non-degenerate cone in $\mathbb{R}^n$ is positive definite at some point. Less than 20 years later, in [4] the authors generalize Omori's work and gain a result about the non-existence of harmonic maps mapping entirely into a non-degenerate cone in $\mathbb{R}^n$ (see [4, Corollary 4]).

Related to the previously formulated question are several very interesting results and conjectures, mostly formulated in terms of minimal hypersurfaces. In 1966, Calabi [11] proposed the following conjecture.

Calabi conjecture

(i) Complete (not necessarily proper) minimal hypersurfaces in $\mathbb{R}^n$ are unbounded.
(ii) Non-flat complete minimal hypersurfaces in $\mathbb{R}^n$ have unbounded projection in any codimension 2 hyperplane.

We will not dive into the details of the Calabi conjecture. Nonetheless, it is important to point out that the conjecture, in its full generality, has been proven wrong by some counterexamples.

Historically, the first counterexample to the Calabi conjecture is due to Jorge and Xavier in [17], where they disproved claim (ii). For (i), a counterexample has been given by Nadirashvili in [20] where he constructs a minimal hypersurface entirely contained in a ball. In particular, the above counterexamples show that the Calabi conjecture cannot hold in full generality. Later on, in [12] Colding and Minicozzi showed that the Calabi conjecture is true if one requires embeddedness. Moreover, their work displays a deep connection between the Calabi conjecture and properness. We take this as an inspiration for assuming our harmonic maps to be proper as well.

As previously mentioned, in [4] the authors proved a non-existence result for harmonic maps into non-degenerate cones in $\mathbb{R}^n$. How wide can such a cone be? There are several answers to this question (see, e.g., [18]). Nonetheless, we would like to recall a very interesting result in the theory of minimal surfaces due to Hoffman and Meeks.

Theorem 1 in [15] *A connected, proper, possibly branched, non-planar minimal surface M in $\mathbb{R}^3$ is not contained in a halfspace.*

In particular, this shows that, for a special class of harmonic maps (proper minimal) into $\mathbb{R}^3$, the cone can be as wide as the whole halfspace. Interestingly enough, Hoffman and Meeks' result is extremely powerful and holds only for $\mathbb{R}^n$ with $n = 3$; indeed, for $n > 3$, the generalized catenoids in [9] are minimal hypersurfaces inside a slab.

It is important to note that Hoffman and Meeks' halfspace theorem as well as Calabi's conjecture relate to minimal submanifolds of codimension 1, that is, hypersurfaces.

Here is where our work finds its place; that is, providing answers to the following question:

> Can we characterize regions R inside the target manifold which would prevent the existence of proper harmonic maps, possibly of higher codimension, mapping entirely inside them?

1.1 Statement of the main results

Our work is based on the extension of the maximum principle in [2, 23], which we refer to as the *Foliated Maximum Principle*, which we now state and prove in Section 2.

***Theorem* 2.2** (Foliated Maximum Principle) *Let (N, h) be a complete Riemannian manifold, $\mathcal{F}$ a strictly convex foliation, and $u : (M, g) \to (N, h)$ a non-constant proper harmonic map. Let $q = u(p) \in \mathcal{F}$ be a point inside the foliation. Then, one of the following holds:*

(a) *The image of u leaves the foliation on the concave boundary $u(p^*) = q^* \in \partial\mathcal{F}_{>q}$ within finite distance $d(q, q^*) < \infty$, where the metric d is induced by the Riemannian metric $h|_{\mathcal{F}}$.*

(b) *There is a sequence $u(p_n) = q_n \in \mathcal{F}_{>q}$ with $\lim_{n\to\infty} d(q, q_n) = \infty$. In this case, $\mathcal{F}$ is necessarily unbounded.*

We refer the reader to the definitions of the objects cited in the statement in Section 2.

With this powerful tool at our disposal, we prove results, such as nice obstructions for the existence of harmonic maps into $\mathbb{R}^n$.

***Theorem* 3.2** (Perturbed Cone Theorem) *Let C be a perturbed cone in $\mathbb{R}^n$. Then, every proper harmonic map from a complete Riemannian manifold (M, g) inside a cone region of C is constant.*

The notion of a perturbed cone in this article is vastly more general than in the known literature, where only cones with solid angle smaller than π are considered. Details are in Definition 3.2, but we can motivate the reader with an illustrative example: The graph of $f : \mathbb{R} \longrightarrow \mathbb{R}; f(x) = \log(x + 1)$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$ will be a perturbed cone in $\mathbb{R}^2$ with our generalized definition, while the part of $\mathbb{R}^2$ above this graph is going to be called a cone region, whose convex hull is the upper halfspace.

By looking at the bare minimum needed to prove Theorem 3.2, we give a definition for a perturbed cone in generic Riemannian manifolds. With such a definition at hand, we prove the following.

***Theorem* 4.2** (Perturbed Riemannian Cone Theorem) *Let $C_{\gamma,r}$ be a perturbed Riemannian cone inside the complete Riemannian manifold (N, h). Then, every proper harmonic map into $C_{r,\gamma} \backslash \partial C_{r,\gamma}$ is constant.*

The definition of such cone is given in Definition 4.2 and it is widely general. Assumptions on the curvature of the target manifold are very mild and when $N = \mathbb{R}^n$

our definition allows us to reconstruct several previously known cone theorems in Euclidean spaces.

Our results parallel several non-existence results known for harmonic or minimal immersions with their images contained in cones, wedges, or (generalized) cylinders [1, 4–8, 10, 16, 18, 21].

1.2 Structure of the article and notation

Section 2 reminds the reader of the definition of a harmonic map, and thereafter the section is devoted to the proof of Theorem 2.2. As a simple application, a Liouville-type property for $\mathbb{R}^n$ is obtained.

Section 3 is where we move to generalized definitions of *perturbed cones* in the Euclidean space and some more elaborated applications of the Foliated Maximum Principle (Theorem 2.2). In Section 3.1, we prove our second main result, Theorem 3.2. In Section 3.2, we employ Theorem 2.2 to the case of the hyperbolic n-space $\mathbb{H}^n$ resulting into a Horosphere Theorem 3.4.

Finally, in Section 4, we collect all the previously obtained results to formalize a definition of perturbed cones in a Riemannian manifold. In particular, here is where we prove our last main result, namely, Theorem 4.2.

Notation

From now on, (M, g) and (N, h) will be complete connected Riemannian manifolds of dimensions m and n, respectively. Moreover, M is always considered to be the domain and N the target of the smooth maps which we consider.

2 Strict convexity and the foliated maximum principle

2.1 Harmonic maps and foliations

A smooth map $u : M \to N$ is said to be *harmonic* if it is a critical point of the Dirichlet energy

$$E[u] = \frac{1}{2} \int_M \|du\|^2 \, d\mathrm{Vol}_g.$$

Here, $\|\cdot\|$ represents the induced Hilbert–Schmidt norm on $TM^* \otimes u^* TN$. An alternative characterization of the harmonicity of u can be obtained through the first variation of the Dirichlet energy, which leads to the elliptic partial differential equation $\Delta u = 0$. Here, $\Delta u = \mathrm{tr}_g \nabla du$ is the tension field, where ∇ denotes the induced connection on $TM^* \otimes u^* TN$ by the Levi-Civita connections of g and h. For more details, see [13].

Some of the most important examples of harmonic maps include harmonic functions, harmonic forms, geodesics, totally geodesic immersions, minimal immersions, (anti)-holomorphic maps between Kähler manifolds, and special Lagrangians. It is worth noting that any non-existence result proven for a harmonic map automatically holds for the aforementioned examples.

By a *proper harmonic* map $u : M \to N$, we mean a harmonic map u, which is also *proper* in the topological sense, that is, the preimage of compact sets is compact.[1]

Whenever M is a compact manifold, properness of a harmonic map u is automatic. Clearly, this is not true for non-compact manifolds, as an example, one can consider a geodesic from an open interval (a, b).

Another important concept that plays a central role in our work is *strict convexity* of a hypersurface $S \subseteq N$. That is, the second fundamental form $A(X, Y) := (\nabla^N_X Y)^\perp$ of S is positive definite. Taking a family of strictly convex hypersurfaces leads to the following definition.

Definition 2.1 We define a *strictly convex foliation* in N as an open, connected, and oriented subset $\mathcal{F} \subseteq N$, which is a foliation whose leaves are connected, embedded, and strictly convex hypersurfaces. Additionally, the foliation satisfies the *separating property*: every non-boundary leaf $\mathcal{L}$ separates the foliation into at least two connected components.

For each leaf $\mathcal{L}$ of a strictly convex foliation $\mathcal{F}$, the orientation determines the choice of the unit normal vector N such that the second fundamental form $A(X, Y) = h(\nabla_X Y, N)$ is positive definite. Thus, for any leaf, we can talk about its convex and concave sides.

Examples of foliations

The following are examples of strictly convex foliations:

(a) In $\mathbb{R}^n$, define the annulus $A_{r,R} = \{x \in \mathbb{R}^n \mid r^2 < \|x\|^2 < R^2\}$ for $r < R$. Take a foliation $\mathcal{F}_A$ whose leaves are spheres S_ρ of radius $r < \rho < R$, with their interior representing the convex side.

(b) Let E be a fixed half equator in S^2 with round metric, and $\Omega \subset S^2 \backslash E$ be an open set with $\mathrm{dist}(\partial\Omega, E) > 0$. Define a foliation $\mathcal{F}_{S^2\backslash E}$ on Ω using the boundaries of geodesic balls of radius $(\pi - \varepsilon)/2$, where $0 < \varepsilon < \mathrm{dist}(\partial\Omega, E)$, as follows. The centers of these geodesic balls lie on the oriented great circle passing through the midpoint of E, its antipodal point, and intersecting E orthogonally. The leaves of the foliation $\mathcal{F}$ are the subset of the boundaries of these geodesic balls, whose points have positive inner product with the velocity vector of the great circle above. Each leaf is a connected, embedded, and strictly convex hypersurface. The separating property clearly holds for the foliation. For more details, see [3].

(c) A perturbed version of example (b) can be constructed as follows (see Figure 1). Let Γ be a connected curve joining two antipodal points A and $-A$ on S^2. Consider an open subset Ω of $S^2\backslash\Gamma$ with $\mathrm{dist}(\partial\Omega, \Gamma) > \varepsilon$ for some $\varepsilon > 0$. We define a foliation $\mathcal{F}_{S^2\backslash\Gamma}$ on Ω using the boundaries of geodesic balls of radius $(\pi - \varepsilon)/2$ exactly like in the above example. The reader can notice that, as in the case, where Γ was a half equator, the separation property also holds.

[1] In some communities, properness means that the image $u(M)$ has a compact intersection with compact subsets of N. We do not use this convention.

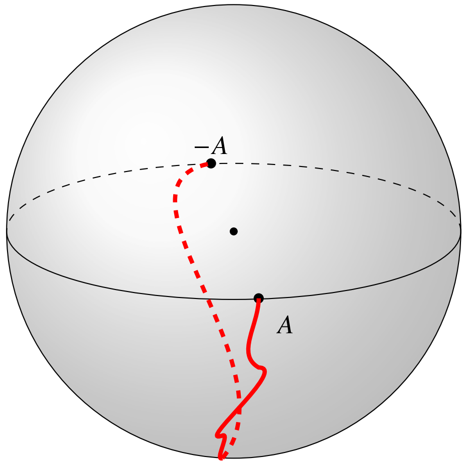

Figure 1: A perturbed half equator in S^2.

This is because the foliation $\mathcal{F}_{S^2\setminus\Gamma}$ actually foliates $S^2\setminus(B_\varepsilon(A) \cup B_\varepsilon(-A))$, and as soon as we plug in a connected curve from A to $-A$, the separation property automatically holds.

A partial order

In the presence of a strictly convex foliation $\mathcal{F}$, one can introduce a relation $p < q$ for points $p, q \in \mathcal{F}$ if p lies on the convex side of the leaf $\mathcal{L}_q$ passing through q. The relation $p \leq q$ denotes either $p < q$ or p and q lying in the same leaf. These relations do not define a (strict) partial order on $\mathcal{F}$ since $p \leq q$ and $q \leq p$ do not imply $p = q$. For $q \in \mathcal{F}$, we denote by

$$\mathcal{F}_{>q} = \bigcup_{q<r} \mathcal{L}_r$$

the concave side of $\mathcal{L}_q$ and by $\partial\mathcal{F}_{>q} := \partial\overline{(\cup_{q<r}\mathcal{L}_r)}\setminus\mathcal{L}_q$ the concave boundary of $\mathcal{F}_{>q}$.

Leaf space

The notion of leaf space for a strictly convex foliation $\mathcal{F}$ can be viewed as a directed graph $G = (V, E)$, where the vertices V correspond to

(a) separating leaves, i.e., the leaves that produce at least three connected components due to the separating property or
(b) boundary components, i.e., for each separating leaf $\mathcal{L}$ as described in (a), insert a vertex for each component of $\mathcal{F}\setminus\mathcal{L}$ which does not contain a separating leaf.

By inserting directed edges E from convex to concave leaves successively, a directed graph is obtained. If there are no separating leaves, then the leaf space consists of two vertices connected by an oriented edge.

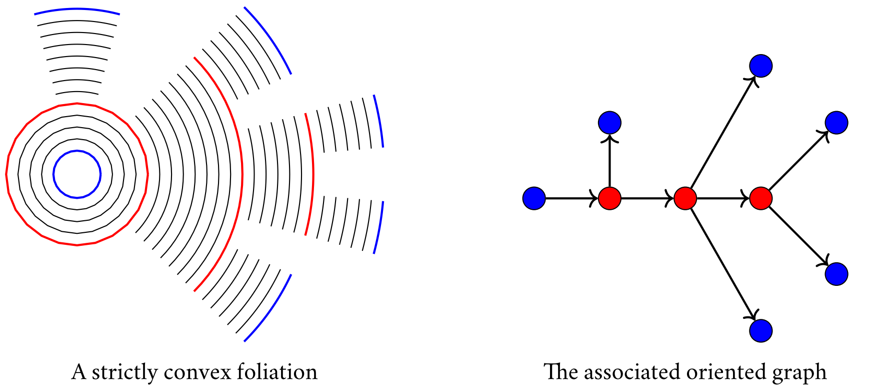

Figure 2: A foliation and its leaf space.

The orders $<$ and $\leq$ on $\mathcal{F}$ descend to G and define a strict partial order and a partial order on G, respectively. It is important to note that this graph does not contain any cycles; otherwise, the separating property would not hold for those leaves (Figure 2).

2.2 The foliated maximum principle

The maximum principle we prove is a foliated version of Sampson's maximum principle and the original result appears in [2] for compact domains. Here, we present an improved version for the non-compact case. First, let us recall the statement of Sampson's maximum principle.

Theorem 2.1 (Sampson's maximum principle) *Let $u : M \to N$ be a non-constant harmonic map, and let S be a strictly convex hypersurface of N passing through $q = u(p)$. Then, for every open neighborhood Ω of p in M, the image $u(\Omega)$ cannot lie entirely on the convex side*[2] *of S.*

Sketch of Proof Let $f : V \to \mathbb{R}$ be a convex function on the open subset $V \subseteq N$ such that $q \in f^{-1}(0) = S \cap V$. The preimage of $(-\infty, 0)$ denotes the convex side of S and the preimage of $(0, \infty)$ its concave side. The composition formula for the Laplacian (see [13, Proposition 2.20]) implies

$$\Delta_g(f \circ u) = df(\Delta u) + \mathrm{tr}_g \nabla df(du, du) = \mathrm{tr}_g \nabla df(du, du) \geq 0,$$

where the last inequality follows by the strict convexity of S. Suppose there exists a neighborhood of p that is mapped to the convex side of S, i.e., the inequality $\Delta_g(f \circ u) < 0$ is satisfied. If $du_p \neq 0$, we obtain that $\Delta_g(f \circ u) > 0$ by the composition formula, thus contradicting the classical maximum principle. For the case $du_p = 0$, we refer the reader to [23]. ∎

[2] Note that the conventions of the convex side and the concave side are reversed in Sampson's paper compared to ours.

Theorem 2.2 (Foliated Maximum Principle) *Let (N,h) be a complete Riemannian manifold, $\mathcal{F}$ a strictly convex foliation, and $u:(M,g)\to(N,h)$ a non-constant proper harmonic map. Let $q=u(p)\in\mathcal{F}$ be a point inside the foliation. Then either*

(a) *the image of u leaves the foliation on the concave boundary $u(p^*)=q^*\in\partial\mathcal{F}_{>q}$ within finite distance $d(q,q^*)<\infty$, where the metric d is induced by the Riemannian metric $h|_{\mathcal{F}}$ or*

(b) *there is a sequence $u(p_n)=q_n\in\mathcal{F}_{>q}$ with $\lim_{n\to\infty}d(q,q_n)=\infty$. In this case, $\mathcal{F}$ is necessarily unbounded.*

Proof The proof is an adaptation of Theorem 3.3 in [2]. Let $q=u(p)$ be as stated. Since q lies on the leaf $\mathcal{L}_q$ passing through q, we can apply the Sampson maximum principle, which allows us to choose $q_1=u(p_1)$ on the concave side of $\mathcal{L}_q$. We repeat this process of applying Sampson's theorem to each $q_i:=u(p_i)$ to find a point q_{i+1} in the concave side of $\mathcal{L}_{q_i}$. Thus, we obtain two sequences of points $\{p_k\}$ in M and $\{q_k\}$ in $\mathcal{F}$.

- Assume that $\{q_k\}$ accumulates to a point q^*. Due to the completeness of N, we conclude that $\{q_k\}$ converges to $q^*\in N$. Due to the properness of u, we conclude that $\{p_k\}$ is a sequence contained in the compact set $u^{-1}(\{q_k\}\cup\{q^*\})$. Due to compactness, one can consider a convergent subsequence $\{p_k'\}$ of $\{p_k\}$ converging to some p^*. Finally, continuity of u leads to $u(p^*)=q^*$. Let now $\mathcal{L}_k$ be the leaf containing q_k. The constructed sequence satisfies $\mathcal{L}_1<\mathcal{L}_2<\dots$. Set

$$\mathcal{L}^*=\inf\{\mathcal{L}\mid\mathcal{L}_k<\mathcal{L}\text{ for all }k\in\mathbb{N},\operatorname{Im}u\cap\mathcal{L}\neq\varnothing\},$$

  where the infimum is taken with respect to the partial order $\leq$. Notice that $q^*\in\mathcal{L}^*$. If $q^*\notin\partial\mathcal{F}_{>q}$, we get a contradiction by another application of Sampson's maximum principle. This implies (a).
- Assume that any sequence $\{q_k\}$ constructed via the iterative method above is not convergent. Then, the sequence $\{q_k\}$ is divergent and monotone with respect to the order $<$. Thus, the only possibility is $\lim_{n\to\infty}d(q,q_n)=\infty$. ∎

Comments on the foliated maximum principle

Firstly, (a) does not automatically mean that u has to leave the foliation. Only a part of the harmonic map might leave and could even re-enter the foliation later.

Secondly, the completeness of N and properness of u are essential. By N being non-complete, the limit q^* of the constructed sequence might not exist. An example violating properness would be a small piece of a unit-speed geodesic $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{R}^2$, which can be included in any strictly convex foliation $\mathcal{F}$. A more involved non-proper example is given by the work of Nadirashvili in [20], where he constructs a non-proper bounded complete minimal disk in $\mathbb{R}^3$.

Lastly, the Foliated Maximum Principle is valid in the context of harmonic sections s of fiber bundles $F\to M$. Here, the vertical Dirichlet energy $E[s]:=\int_M\|D^Vs\|d\mathrm{Vol}_g$ with D^Vs being the vertical projection of the differential D is used for the definition of harmonicity. In this case, the strictly convex foliation only needs to foliate the vertical directions.

Liouville's theorem

As a first illustration of the non-compact Foliated Maximum Principle, we prove a version of Liouville's theorem.

Corollary 2.3 (Liouville's theorem) *The only proper harmonic maps from a complete (not necessarily compact) Riemannian manifold (M, g) into a bounded open subset B of $\mathbb{R}^n$ for $n > 1$ are the constant maps. In particular, there are no bounded, proper minimal hypersurfaces in $\mathbb{R}^n$.*

Proof By translating B, we can assume that $r = \inf_{q \in B} \|q\| > 0$. Let $R = \sup_{q \in B} \|q\|$ after a possible translation. Then, $B \subseteq A_{r,R} = \{x \in \mathbb{R}^n \mid r^2 < \|x\|^2 < R^2\}$, and Theorem 2.2 (a) applies. ∎

3 Perturbed cone and horosphere theorems

In this section, we will prove the first two non-existence theorems for proper harmonic maps into specific regions of the Euclidean space and the hyperbolic space.

3.1 Perturbed cone theorems in $\mathbb{R}^n$

We consider the Euclidean space $\mathbb{R}^n$ with its standard flat metric defined by $ds^2 = dx_1^2 + \cdots + dx_n^2$. Recall furthermore that geodesics in $\mathbb{R}^n$ with the flat metric are given by affine lines, i.e., $\gamma(t) = p + tv$ for $p, v \in \mathbb{R}^n$.

Perturbed cones in Euclidean spaces

The basic example for the definition of a perturbed cone is going to be the classical cone C defined by the graph of $f : \mathbb{R} \to \mathbb{R}; x \mapsto |x|$ inside $\mathbb{R}^2$, henceforth called classical cones. The connected component R of $\mathbb{R}^2 \backslash C$ containing $(0, 1)$ has a special property: any $p \in R$ can be enclosed by the cone and a hyperplane, in this case a line. In particular, there are no affine lines in R. Such a property seems to be very interesting and, most importantly, crucial for the analysis of harmonic maps. Thus, we give an appropriate definition for a set possessing such an *enclosing property*.

Definition 3.1 Let $R \subseteq \mathbb{R}^n$ be a connected open set. R is said to possess the *enclosing property* if, for every $p \in R$, there is an affine hyperplane H such that the connected component B of $R \backslash H$ containing p is precompact, i.e., $\bar{B}$ is compact.

The blue horizontal lines in Figure 3 verify that the enclosing property holds for the cone C. A non-example is the upper halfspace $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. The enclosing property allows us to extend the notion of a cone to one of a *perturbed cone*.

Definition 3.2 Let C be a closed, path-connected subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash C$ consists of at least two connected components. Let R be one of the connected components of $\mathbb{R}^n \backslash C$. If R satisfies the enclosing property, then C is called a *perturbed cone* in Euclidean space and R a *cone region*.

Lemma 3.1 *Let C be a perturbed cone and R a cone region. Then, R cannot contain affine lines.*

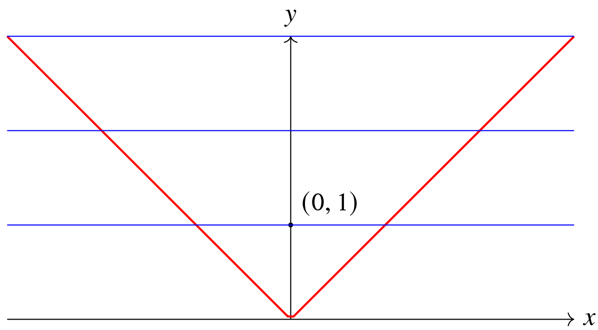

Figure 3: The graph of $f(x) = |x|$ and enclosing hyperplanes.

Proof Assume there is an affine line L in R. Let $p \in L$ and choose a separating hyperplane H and the region B via the enclosing property of R. If L is parallel to H then it would be fully contained in B and hence contradicting precompactness of B. If L intersects H then an unbounded part of L would be in B again contradicting precompactness of B. Since L can either be parallel to H or intersect it, we exhausted all the possibilities. ■

The enclosing property can be weakened in such a way that the previous lemma does not hold. In this case, the new notion of a perturbed cone needs the assumption of *not containing affine lines*.

Examples of perturbed cones

The following are examples of perturbed cones in the Euclidean space.

(a) Let $C \subseteq \mathbb{R}^2 \cong \mathbb{C}$ be defined by the union of the negative x-axis and the line $e^{i\theta}\mathbb{R}_{\geq 0}$ for some angle $\theta \in (0, \pi)$. Note that for $\theta = 0$, this is not a perturbed cone, since there are geodesics in the upper half-plane.

(b) A family of perturbed cones in $\mathbb{R}^2$ can be defined by gluing the negative x-axis and the graph of any continuous, injective, unbounded function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$. For example, (See Figure 4) $f(x) = \log(x + 1)$. This can be seen as a perturbation of the cone defined in (a). Note that the example using the logarithm is not contained in any classical cone with an angle smaller than π, since the convex hull of the graph is the upper halfspace.

A three-dimensional version of this construction is the following: Equip $\mathbb{R}^2$ with polar coordinates (r, θ) and fix an angle $0 \leq \theta_0 < \pi$. Now, the union $\mathrm{gr}\,(f) \cup S_{\theta_0}$ of the graph of the function

$$f(r, \theta) = \begin{cases} \log(r+1) & \theta \in [0, 2\pi - \theta_0] \\ 0 & \theta \in (2\pi - \theta_0, 2\pi) \end{cases}$$

and the sides $S_{\theta_0} = \{(r, \theta, z) \in \mathbb{R}^3 \mid 0 \leq z \leq \log(r+1), \theta \in \{0, 2\pi - \theta_0\}\}$ defines a perturbed cone. For $\theta_0 = 0$, this recovers the rotational surface of $\log(r + 1)$ and here the sides S_{θ_0} are not needed.

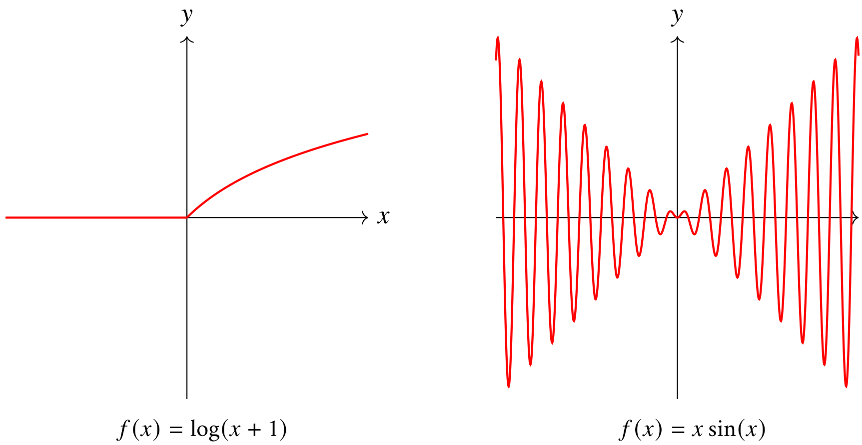

Figure 4: Examples of perturbed cones.

(c) The graph of $f(x) = x\sin(x)$ in $\mathbb{R}^2$ is an example such that both components of $\mathbb{R}^2\backslash\mathrm{gr}(f)$ are cone regions (Figure 4).

(d) Let $p, v \in \mathbb{R}^n$ with $v \neq 0$ and an angle $0 < \theta < \pi/2$. Then, the boundary of the cone

$$C(p, v, \theta) = \{p + x \mid x \in \mathbb{R}^n, |\langle x, v\rangle| \leq \cos(\theta)\|x\|\|v\|\}$$

obviously defines a perturbed cone with its inside being the cone region.

(e) Let K be a compact set with $K^\circ = K\backslash\partial K \neq \varnothing$. Then, K is a perturbed cone with cone region K°.

Next, we prove the main theorem of this section.

Theorem 3.2 (Perturbed Cone Theorem) *Let C be a perturbed cone in $\mathbb{R}^n$. Then, every proper harmonic map from a complete Riemannian manifold (M, g) inside a cone region of C is constant.*

Proof Let $u : M \to \mathbb{R}^n$ be a non-constant proper harmonic map and a point $p \in M$ chosen such that $q = u(p)$ lies in a cone region of C. Take now the associated enclosing compact set B and the hyperplane H such that $q \in B^\circ$.

Let v be the unit normal with respect to H pointing inward to B° and the half-sphere

$$S = S_r^{n-1} \cap \{x \in \mathbb{R}^n \mid 0 \leq \langle x, v\rangle\},$$

where S_r^{n-1} is the sphere of radius $r = \mathrm{diam}(B)$. Then, for $\gamma : (-r-\varepsilon, r+\varepsilon) \to \mathbb{R}^n$ defined by $\gamma(t) = q + t\,v$ for $\varepsilon > 0$, we obtain an associated strictly convex foliation

$$\mathcal{F} = \bigcup_{t\in(-r-\varepsilon, r+\varepsilon)} \mathcal{L}_{\gamma(t)}$$

with leaves $\mathcal{L}_{\gamma(t)} = \gamma(t) + S$.

By definition, $q \in \mathcal{L}_{\gamma(-r)}$ and $\partial B \cap H$ lies on the convex side of $\mathcal{L}_{\gamma(-r)}$. Hence, by Theorem 2.2 (a), we obtain a contradiction of $\operatorname{Im} u$ being contained in a cone region of C (Figure 5). ∎

Local cones

A corollary derived from the proof of our main theorem, which can be interpreted as a local manifestation of a cone theorem, goes as follows (Figure 6).

Definition 3.3 A *local cone* C in $\mathbb{R}^n$ is a closed subset such that there is a hyperplane H and a precompact connected component B of $\mathbb{R}^n \backslash (C \cup H)$. B is called the *local cone region*.

Note that a local cone can always be extended to a cone in several non-unique ways.

Corollary 3.3 (Local Cone Theorem) *Let C be a local cone with cone region B. Then, a non-constant proper harmonic map $u : M \to \mathbb{R}^n$ with $u(p) \in B$ also possesses a point $p' \in M$ such that $u(p') \in C$. In other words, a harmonic map entering B has to leave through C.*

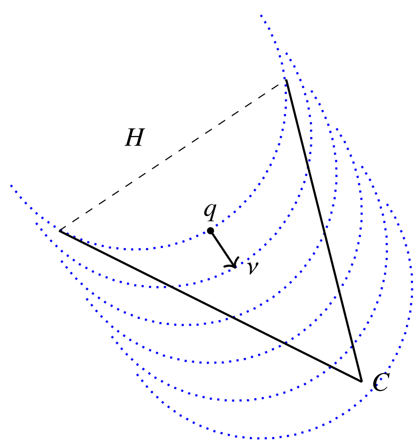

Figure 5: The constructed foliation.

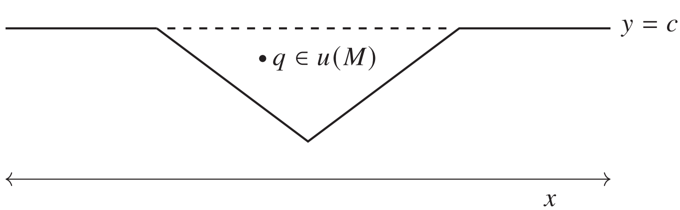

Figure 6: A local cone.

Proof Since B is an open set and $p' \in B$, we can just repeat the construction in the proof of Theorem 3.2. ∎

3.2 The horosphere theorem

To further demonstrate the flexibility of our methods on obtaining Perturbed Cone Theorems, we proceed to explore hyperbolic upper halfspaces. We exhibit a 1-parameter family of hypersurfaces that foliates the space above a horosphere. More precisely and with the nomenclature of our article, we consider the upper halfspace model of the hyperbolic space $\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$ equipped with the metric $ds^2 = \frac{1}{x_{n+1}^2}(dx_1^2 + \cdots + dx_{n+1}^2)$ of constant sectional curvature -1. We will show that a cone in the region $\{x_{n+1} > c > 0\}$ in the hyperbolic space $\mathbb{H}^{n+1}$ can be as wide as a halfspace; that is, a cone with angle π.

Geometry of graphical hypersurfaces in $\mathbb{H}^{n+1}$

We shall briefly discuss the geometry of hypersurfaces of $\mathbb{H}^{n+1}$. Let $F : \mathbb{R}^n \to \mathbb{H}^{n+1}$ be a graphical hypersurface $F(x) = (x, f(x))$ for some smooth $f : \mathbb{R}^n \to \mathbb{R}$. The tangent vectors are spanned by $F_i = \partial_i + f_i \partial_{n+1}$. Denote by $\nabla f = (f_1, \ldots, f_n)$ the Euclidean gradient and by $\|\nabla f\|$ its Euclidean norm. Then, we can choose the upward pointing unit normal

$$N = \frac{f}{\sqrt{1 + \|\nabla f\|^2}} \begin{bmatrix} -\nabla f \\ 1 \end{bmatrix}.$$

The covariant derivatives with respect to the ambient hyperbolic metric are

$$\nabla_{F_i} F_j = f^{-1}((\delta_{ij} + f f_j - f_i f_j)\partial_{n+1} - f_i \partial_j - f_j \partial_i),$$

which then implies that the second fundamental form is

$$A = \frac{1}{f^2\sqrt{1 + \|\nabla f\|^2}}(\mathrm{Id} + \nabla f \otimes \nabla f + f\,\mathrm{Hess}(f)), \tag{3.1}$$

where $\mathrm{Hess}(f)$ is the Euclidean Hessian and $[\nabla f \otimes \nabla f]_{ij} = f_i f_j$. Thus, for the convexity of the hypersurfaces, we only need to investigate the positive definiteness of A.

Spheres beyond infinity

Our aim is to construct a foliation of an open set in $\mathbb{H}^{n+1}$. Such a foliation and the Foliated Maximum Principle will lead to the *Horosphere Theorem*. Such a foliation will consist of graphical hypersurfaces in $\mathbb{H}^{n+1}$ whose convexity can be investigated by making use of equation (3.1). Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \sqrt{4q^2 - \|x\|^2} - q$ for some $q > 0$. Consider the embedding $F : U \to \mathbb{H}^{n+1}$, where $U = \{x \in \mathbb{R}^n \mid f(x) > 0\}$ with $F(x) = (x, f(x))$. The defined submanifold is the piece of the Euclidean sphere of radius $2q$ and center $(0, \ldots, 0, -q)$ as a subset of (Figure 7) $\mathbb{H}^{n+1}$. The gradient is

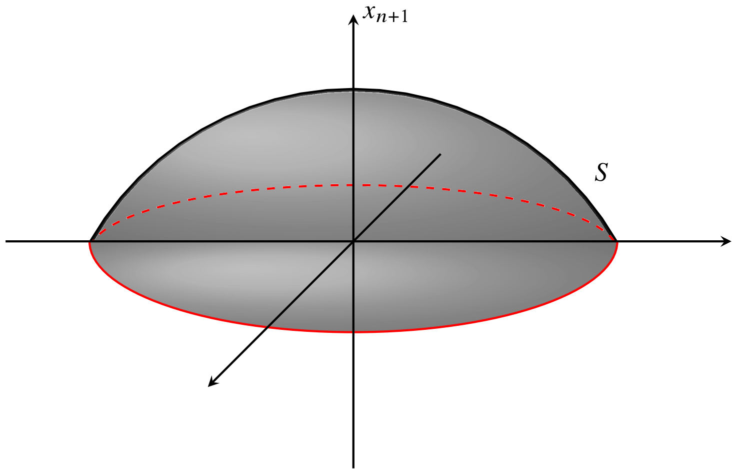

Figure 7: S defined by $f(x) = \sqrt{4q^2 - \|x\|^2} - q$.

$\nabla f = \frac{-x}{f(x)+q}$ and the Hessian is

$$\mathrm{Hess}(f) = -\frac{\mathrm{Id}}{f(x)+q} - \frac{x \otimes x}{(f(x)+q)^3},$$

where $[x \otimes x]_{ij} = x_i x_j$. A short computation shows that the second fundamental form is

$$A = \frac{1}{2f(x)}\left(\mathrm{Id} + \frac{x \otimes x}{(f(x)+q)^2}\right).$$

The positive definiteness of A follows by noticing that a matrix of the form $\mathrm{Id} + \alpha^2 x \otimes x$ has the eigenvalue $1 + \alpha^2\|x\|^2$ with multiplicity one and the eigenvalue 1 with multiplicity n.

Theorem 3.4 (Horosphere Theorem) *Let $H = \{x \in \mathbb{H}^{n+1} \mid x_{n+1} = c\}$ for some $c > 0$ be a horosphere in the hyperbolic space $\mathbb{H}^{n+1}$. Then, every proper harmonic map into the set $\{x \in \mathbb{H}^{n+1} \mid x_{n+1} > c\}$, known as the horoball, is constant.*

Proof Let u be a proper harmonic map and $p \in M$ such that $q' = u(p)$ lies above the horosphere H. By a hyperbolic isometry, we can assume that $q' = (0, \dots, 0, q)$ for some $q > c$. Define the foliation $\mathcal{F}$ by the maps $f_t(x) = \sqrt{4t^2 - \|x\|^2} - t$ for $t \in [\varepsilon, q]$ for some $0 < \varepsilon < c$. By the previous discussion, $\mathcal{F}$ is a strictly convex foliation, hence, Theorem 2.2 applies, implying that u needs to intersect the horosphere of height $\varepsilon < c$. Thus, u cannot lie above H. ■

The proof of Theorem 3.4 actually shows that if $(0, \dots, 0, q) = q' = u(p) > 0$, then for every $c < q$, there is a point $p' \in M$ such that $u(p') \in H_c \cap B_{2q}(-q')$.

Let $c > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}^n$, the inequality $-\varepsilon \leq f(x)$ holds for $0 < \varepsilon < c$. Then, the graph $\mathrm{gr}(f + c)$

in $\mathbb{R}^{n-1} \times (0, \infty) = \mathbb{H}^{n+1}$ defines a perturbed version of the horosphere. The same conclusion holds for this perturbed version as well.

Comments on the horosphere theorem

We would like to highlight two important theorems that influenced the statement of Theorem 3.4. The first, by Rodriguez and Rosenberg [22], proves a halfspace theorem for mean curvature one surfaces in the hyperbolic space $\mathbb{H}^3$, whose analogy with our result is obvious.

The second, by Mazet [19], covers a general halfspace theorem for constant mean curvature (CMC) surfaces in 3-manifolds. This article explores conditions allowing two surfaces with equal mean curvature to coexist in the same 3-space. Specifically: If Σ_H is a parabolic CMC surface with mean curvature H and any equidistant surface on the non-mean convex side has mean curvature less than H, then any CMC H surface on the non-mean convex side of Σ_H is an equidistant surface to Σ_H. This general result, however, requires the target space to be three-dimensional and the domain to be parabolic, which limits its application in higher dimensions; for example, the Euclidean space $\mathbb{R}^k$ with a flat metric is parabolic if and only if $k = 1$ or $k = 2$. In contrast, our weaker theorem applies to proper harmonic maps without any restrictions on the dimensions of either the domain or the hyperbolic space target.

4 A Riemannian cone theorem

In this section, we move to the setting of Riemannian manifolds and prove a non-existence result for perturbed Riemannian cones. Employing again a foliation argument, we will prove that the only proper harmonic maps into R are the constant ones.

Riemannian halfspaces

As a first step, we need a generalized definition of a halfspace in a Riemannian manifold.

Definition 4.1 Let (N, h) be a non-compact complete Riemannian manifold. A *Riemannian halfspace* at $p \in N$ in the direction $v \in T_pN$ is

$$H_\gamma := \bigcup_{0<t} \overline{B_t(\gamma(t))},$$

where $\gamma : [0, \infty) \to N$ is a unit-speed minimizing geodesic ray $\gamma(t) = \exp_p(tv)$ starting at p in the direction of v.

Actually, the completeness is not strictly necessary. Take, for example, $\mathbb{R}^2 \backslash (0, -1)$ with $p = (0, 0)$ and $v = (0, 1)$. Then, we still obtain the usual halfspace, although $\mathbb{R}^2 \backslash (0, -1)$ is not complete with the Euclidean metric. Another example of a halfspace is a horosphere inside hyperbolic space. For this purpose, take $p = (0, \ldots, 0, c) \in \mathbb{H}^n$ and $v = (0, \ldots, 0, 1)$.

Perturbed Riemannian cones

Perturbed Riemannian cones are going to be an abstraction of the idea of a cone being a subset inside a halfspace defined by a point, a direction, and a solid angle.

Definition 4.2 Let (N, h) be non-compact, $\gamma : [0, \infty) \to N$ a unit-speed *minimizing geodesic* ray and $r : (0, \infty) \to (0, \infty)$ a continuous function satisfying $r(t) < \min\{t, r_c(\gamma(t))\}$, where $r_c(p)$ is the convexity radius of N at p. The subset

$$C_{\gamma,r} := \bigcup_{0<t} \overline{B_{r(t)}(\gamma(t))}$$

is a *perturbed Riemannian cone* if the following separation property holds: For every $0 < t_1 < t_2$, the condition $B_{r(t_1)}(\gamma(t_1)) \notin B_{r(t_2)}(\gamma(t_2))$ holds and $C_{\gamma,r} \backslash \overline{B_{r(t)}(\gamma(t))}$ has two components for $t \in (0, \infty)$. The function r is referred to as the *cone radius function*.

The property $r(t) < t$ assures that the perturbed Riemannian cone $C_{\gamma,r}$ is a subset of the Riemannian halfspace H_γ and the property $r(t) < r_c(\gamma(t))$ is necessary for $S_{r(t)}(\gamma(t)) := \partial B_{r(t)}(\gamma(t))$ being strictly convex. Thus, a perturbed Riemannian cone intrinsically possesses a strictly convex foliation by the boundaries of geodesic balls. Note that cones in Euclidean space are a canonical example of a perturbed Riemannian cone, where in addition, the radius function r has constant derivative smaller than 1.

The following lemma characterizes when geodesic balls along a minimizing geodesic rays are contained in each other, i.e., when the separating property of a cone fails. Since the proof relies on the mean value theorem, we will assume differentiability for the cone radius function.

Lemma 4.1 *Let $\gamma : [0, \infty) \to N$ be a unit-speed minimizing geodesic ray and $r : [0, \infty) \to [0, \infty)$ a continuously differentiable function satisfying $r(0) = 0$ and $r(t) < t$. Fix an open subset $(a, b) \subseteq (0, \infty)$. The following are equivalent:*

(a) *$r' \geq 1$ on (a, b) and*
(b) *$B_{t_1} \subseteq B_{t_2}$ for all $a < t_1 < t_2 < b$, where $B_t = \overline{B_{r(t)}(\gamma(t))}$ is the closed geodesic ball in N.*

Proof Assuming (a) the mean value theorem implies $t_2 - t_1 \leq r(t_2) - r(t_1)$. Let $p \in B_{t_1}$, that is, $d(p, \gamma(t_1)) \leq r(t_1)$. Now,

$$\begin{aligned} d(p, \gamma(t_2)) &\leq d(p, \gamma(t_1)) + d(\gamma(t_1), \gamma(t_2)) \\ &\leq r(t_1) + t_2 - t_1 \\ &\leq r(t_1) + r(t_2) - r(t_1) \\ &= r(t_2). \end{aligned}$$

Note that the second inequality follows by γ being a unit-speed minimizing geodesic. Hence, $p \in B(t_2)$ and (b) follows.

Assuming (b), we obtain $d(\gamma(t_2), \gamma(t_1 - r(t_1))) \leq r(t_2)$ and by the properties of γ, we get $d(\gamma(t_2), \gamma(t_1 - r(t_1))) = t_2 - t_1 + r(t_1)$. Rearranging implies that,

$$1 \leq \frac{r(t_2) - r(t_1)}{t_2 - t_1}$$

and assertion (a) follows. ∎

Theorem 4.2 (Perturbed Riemannian Cone Theorem) *Let $C_{\gamma,r}$ be a perturbed Riemannian cone inside the complete Riemannian manifold (N, h). Then, every proper harmonic map into $C_{r,\gamma} \backslash \partial C_{r,\gamma}$ is constant.*

Proof The convexity radius assumption assures that the boundaries of the geodesic balls are strictly convex. Hence, by the above remarks, we know that there is a strictly convex foliation present and Theorem 2.2 applies. ∎

Applications of the perturbed Riemannian cone theorem

There are many possible applications of Theorem 4.2 to obtain new cone-type theorems for Riemannian geometry. We point out a couple of interesting ones to show the generality of the theorem in the Riemannian setting.

(a) Let $r(t) = \cos(\theta)t$ for a fixed angle $\theta \in (0, \pi/2)$ and $\gamma(t) = (0, t)$ with $t \in [0, \infty)$ inside $\mathbb{R}^2$. Then, $C_{r,\gamma}$ is a perturbed Riemannian cone and $\partial C_{r,\gamma}$ is a perturbed Euclidean cone. Thus, Theorem 4.2 recovers Theorem 3.2 for certain cones.

(b) In the case of hyperbolic spaces, Theorem 4.2 is strictly weaker than Theorem 3.4 since, in that case, we can prove the theorem for the halfspace, i.e., for the radius function $r(t) = t$.

(c) Let $M = \mathbb{H}^k \times \mathbb{R}^l$ equipped with the product metric. Since the convexity radius of M is infinity, one can take any geodesic ray γ and a radius function $r(t) = \cos(\theta)t$ with $\theta \in (0, \pi/2)$.

(d) Consider $M = (S^1)^n \times \mathbb{R}$ as $[-\pi, \pi]^n \times \mathbb{R}$ with opposite sides identified and equipped with the flat metric. The diffeomorphism $\psi : \mathbb{R}^{n+1} \to (-\pi, \pi)^n \times \mathbb{R}$ given by

$$(x_1, \ldots, x_n, s) \mapsto (2\arctan(x_1), \ldots, 2\arctan(x_n), s)$$

allows the definition of perturbed Riemannian cones in M via taking the image of a cone inside $\mathbb{R}^{n+1}$. For instance, let $C_{\gamma,r} \subseteq \mathbb{R}^{n+1}$ be defined by $\gamma(t) = (0, \ldots, 0, t)$ with $t \in [0, \infty)$ and $r(t) = \cos(\theta)t$ for $\theta \in (0, \pi/2)$. Then, under ψ, the cone gets mapped to $C_{\gamma,R}$ inside M, where γ is just like before and the transformed radius function is $R(t) = 2\arctan(r(t)) = \arctan(\cos(\theta)t)$.

(e) Let $M = S^n \times \mathbb{R}$ be equipped with the product metric. Denote by S the south pole of S^n and by $\gamma(t) = (S, t)$ with $t \in [0, \infty)$ the minimizing geodesic. Since the convexity radius of S^n is $\pi/2$, we need to stay below that value. Again, we can take the radius function $r(t) = \arctan(t)$ and define a perturbed Riemannian cone inside M.

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Institute of Differential Geometry, Leibniz University Hannover, Germany
e-mail: renan.assimos@math.uni-hannover.de balazs.bekesi@unibocconi.it giuseppe.gentile@uni-oldenburg.de