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Finite data rigidity for one-dimensional expanding maps

THOMAS ALOYSIUS O'HARE

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA
(e-mail: ohare.26@osu.edu)

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Abstract. Let f, g be C^2 expanding maps on the circle which are topologically conjugate. We assume that the derivatives of f and g at corresponding periodic points coincide for some large period N. We show that f and g are 'approximately smoothly conjugate.' Namely, we construct a C^2 conjugacy h_N such that h_N is exponentially close to h in the C^0 topology, and $f_N := h_N^{-1} g h_N$ is exponentially close to f in the C^1 topology. Our main tool is a uniform effective version of Bowen's equidistribution of weighted periodic orbits to the equilibrium state.

Key words: expanding maps, smooth rigidity, thermodynamic formalism, effective equidistribution
2020 Mathematics Subject Classification: 37C15, 37C20, 37E10 (Primary); 37C30 (Secondary)

1. *Introduction*

A C^1 map $f : S^1 \to S^1$ is called expanding if $\min_{x \in S^1} |f'(x)| \geq \lambda_f > 1$. We call λ_f the minimum expansion rate. Let $\operatorname{Exp}^r(S^1)$ $(r \geq 1)$ be the subspace of all C^r uniformly expanding maps, and for $\gamma > 1$, let $\operatorname{Exp}^r_\gamma(S^1)$ be the space of all C^r expanding maps whose minimum expansion rate is greater than or equal to γ. Given any continuous map $f : S^1 \to S^1$, recall that the degree of f is defined to be value $F(x+1) - F(x)$, where $F : \mathbb{R} \to \mathbb{R}$ is any lift of f. In addition to being well-defined independent of the choice of lift and the point $x \in S^1$, it was proved by Shub in [**Shu69**] that the degree is a complete topological conjugacy invariant for expanding maps on the circle.

THEOREM 1.1. *Let $f, g : S^1 \to S^1$ be continuous expanding maps. Then there exists a homeomorphism $h : S^1 \to S^1$ such that $h \circ f = g \circ h$ if and only if* $\deg(f) = \deg(g)$.

It is easy to check that h is Hölder continuous, but there is an obstruction to h having higher regularity at the periodic orbits of f. By formally differentiating the conjugacy equation, it is clear that h will not be differentiable if $(f^n)'(x) \neq (g^n)'(h(x))$ for at least one periodic point $x \in \operatorname{Fix}(f^n)$. As the next theorem shows, satisfying this obstruction at all periodic points is sufficient to conclude differentiability of h.

THEOREM 1.2. *Suppose $f, g : S^1 \to S^1$ are $C^{1+\alpha}$ ($\alpha > 0$) expanding maps of the same degree, and fix a conjugacy h such that $h \circ f = g \circ h$. Then, the map h is $C^{1+\alpha}$ if and only if for every point $p \in \mathrm{Fix}(f^n)$, $n \in \mathbb{N}$, we have $(f^n)'(p) = (g^n)'(h(p))$.*

While not directly stated in this form, a proof of Theorem 1.2 can be found from de la Llave [**Lla92**]. A closely related result is the theorem of Shub and Sullivan [**SS85**], which proves that C^r ($r \geq 2$) expanding maps of the circle which are conjugated by an absolutely continuous homeomorphism h_1 are in fact conjugated by a C^r diffeomorphism h_2, but it may be that $h_1 \neq h_2$. Martens and de Melo prove a more general version of Theorem 1.2 [**MM99**, Corollary 2.9] applying to all C^r Markov maps in one dimension. In particular, the main theorem of [**MM99**] applies to unimodal maps (with critical points) without wild and solenoidal attractors and establishes periodic data rigidity for these systems. The picture for general unimodal maps is more complicated: Moreira and Smania proved [**MS14**] that unimodal maps with Cantor set attractor, such as Feigenbaum maps and Fibonacci maps with high order at the critical point, are always absolutely continuously conjugated, but not necessarily smoothly. Thus, Shub and Sullivan's theorem does not carry over to these maps. It would be interesting to study finite data rigidity (Theorem 1.3 below) for unimodal and Markov maps as well.

The goal of the present paper is to relax the conditions of Theorem 1.2 to hold at only finitely many periodic orbits, and then see how close f and g are to being smoothly conjugated.

THEOREM 1.3. *Let $\gamma > 1$ be fixed and let $\mathcal{W} \subset \mathrm{Exp}^2_\gamma(S^1)$ be a bounded subset. Then, there exist constants $K > 0$ and $0 < \lambda < 1$ depending only on $\gamma > 1$ and $\mathcal{W}$ such that the following holds: if $f, g \in \mathcal{W}$ are conjugated by a homeomorphism h ($h \circ f = g \circ h$) and if there exists $N \in \mathbb{N}$ such that $(f^N)'(p) = (g^N)'(h(p))$ for every $p \in \mathrm{Fix}(f^N)$, then there exists a diffeomorphism $h_N \in C^2(S^1)$ such that $d_{C^0}(h, h_N) \leq K\lambda^N$. Moreover, for every $0 < \lambda^{1/2} < \lambda_0 < 1$, there exists a constant $K' > 0$, such that if we let $f_N = h_N^{-1} \circ g \circ h_N$, then $d_{C^1}(f, f_N) \leq K'\lambda_0^N$.*

Remark 1. The conjugacy h_N we construct will not explicitly depend on the parameter N. Instead, the notation is meant to emphasize how much of the periodic data we can use for our estimates. A more descriptive notation would be $h_{f,g,N}$ to emphasize the dependence on the expanding maps f and g. However, for the sake of brevity (as well as to avoid confusing notation for f_N), we will use the simplified notation h_N.

A key step in the proof of Theorem 1.3 is to prove an effective version of Bowen's equidistribution theorem (see Theorem 2.2 below), which allows us to estimate the difference between h and h_N in terms of the periodic orbits of order N. The convergence rate λ comes from the effective equidistribution rate and depends on the degree of the expanding maps and γ. The proof of effective equidistribution is postponed until §3 and relies on the technique of Birkhoff cones for subshifts of finite type, which we recall in Appendix A. This technique is well known in the case of expanding maps; see for instance Baladi [**Bal00**].

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2. *Finite data rigidity*

The goal of this section is to generalize Theorem 1.2 to allow for the derivatives of f and g to agree only at finitely many periodic points. Of course, f and g will not be C^1 conjugate, but we can find a map C^1 close to f which is C^1 conjugate to g. Moreover, this new map converges exponentially to f as the number of periodic points that the derivatives agree on increases.

Definition 2.1. For a function $f \in C^k(S^1)$, $k \in \mathbb{N}$, let $|f|_{C^k} = \sup |D^k f|$ denote the C^k seminorm. Let $\|f\|_\infty$ denote the supremum norm of f and let $\|f\|_{C^k} = \|f\|_\infty + \sum_{i=1}^k |f|_{C^i}$ denote the C^k norm. If $f : S^1 \to S^1$ is α-Hölder continuous $(0 < \alpha \leq 1)$, define

$$|f|_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

to be the Hölder seminorm and define $\|f\|_\alpha = \|f\|_\infty + |f|_{\text{Lip}}$ to be the Hölder norm. For $\alpha = 1$, we get the Lipschitz seminorm and Lipschitz norm $|\cdot|_{\text{Lip}}$ and $\|\cdot\|_{\text{Lip}}$, respectively.

It well known that for a $C^{1+\alpha}$ expanding map f on a compact manifold M, there exists a unique invariant probability measure μ_f which is absolutely continuous with respect to Lebesgue measure, whose density $\rho_f(x)$ is C^α and strictly positive. See Baladi [**Bal00**] for full details. Moreover, μ_f is the unique equilibrium state corresponding to the geometric potential $\psi_f = -\log(|f'(x)|)$. For the uniformity claims of Theorem 1.3, it will be important to have uniform bounds on the densities ρ_f.

LEMMA 2.1. *Let $\mathcal{W}$ be a bounded set in $\text{Exp}^2_\gamma(S^1)$, $\gamma > 1$. Then, for every $0 < \alpha < 1$, there exists $C > 1$ such that for all $f \in \mathcal{W}$ and every $x \in S^1$, $C^{-1} \leq \rho_f(x) \leq C$ and $|\rho_f|_{C^\alpha} < C$.*

Proof. We prove this using a Birkhoff cone argument. See Baladi [**Bal00**, §2.2] for details and Appendix A of the present paper for a similar argument for subshifts of finite type. For $L > 0$, consider the cone

$$\Lambda_L = \left\{ \phi \in C^0(M) | \phi(x) > 0, \frac{\phi(x)}{\phi(y)} \leq e^{L d(x,y)^\alpha} \text{ for all } x, y \in S^1 \right\}.$$

Then, Baladi shows in [**Bal00**] that there exists $L > 0$ depending only on $\gamma > 1$ and $0 < \alpha < 1$ such that $\rho_f \in \Lambda_L$. Since ρ_f is the smooth density of a probability measure, there exists a point $y \in S^1$ such that $\rho_f(y) = 1$. Since $\rho \in \Lambda_L$, for any $x \in S^1$, we have

$$\rho_f(x) \leq \rho_f(y) e^{L d(x,y)^\alpha} < e^{L \,\text{diam}(S^1)^\alpha} < e^L,$$

and we likewise get the lower bound for $\rho(x)$ by swapping the roles of x and y. Finally, notice that $\rho_f \in \Lambda_L$ is equivalent to $\log(\rho_f) \in C^\alpha(S^1)$ with $|\log(\rho_f)|_\alpha \leq L$. Then, $\rho_f = \exp(\log(\rho_f))$ and

$$|\rho_f|_\alpha \leq \max_{x \in S^1} e^{\log(\rho_f(x))} |\log(\rho_f)|_\alpha \leq e^{\log(e^L)} L = L e^L. \qquad \square$$

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Remark 2. One can prove uniform bounds on the densities without a Birkhoff cone argument by instead carefully going through the arguments of Sacksteder's proof [**Sac74**] of the existence of invariant densities.

We are now ready to introduce the main technical tools needed to prove Theorem 1.3. We will use the notation $S_{f,n}\psi(x)$ to denote the nth ergodic sum of ψ at the point x with respect to the dynamics of f:

$$S_{f,n}\psi(x) = \sum_{i=0}^{n-1} \psi(f^i(x)).$$

Definition 2.2. Let (X, f) be a dynamical system and $\psi : X \to [0, \infty)$ a non-negative potential function. Whenever $\mathrm{Fix}(f^n) \neq \emptyset$, we can define a measure $\mu^n_{f,\psi}$ by

$$\mu^n_{f,\psi} = \frac{1}{Z_n(f, \psi)} \sum_{x \in \mathrm{Fix}(f^n)} e^{S_{f,n}\psi(x)} \delta_x,$$

where

$$Z_n(f, \psi) = \sum_{x \in \mathrm{Fix}(f^n)} e^{S_{f,n}\psi(x)}$$

is a normalization constant so that $\mu^n_{f,\psi}$ is a probability measure. We call $\mu^n_{f,\psi}$ the nth weighted discrete measure associated to the dynamics of f and the potential ψ.

The measure $\mu^n_{f,\psi}$ depends on three ingredients: the length n of periodic orbits under consideration, the dynamical system $f : X \to X$, and the potential function $\phi : X \to [0, \infty)$. In our setting, we will be considering a smooth expanding map $f : S^1 \to S^1$ together with the associated geometric potential $\psi_f(x) = -\log(|f'(x)|)$. When this is the case, since the potential depends on the dynamics, we will use the more compact notation μ^n_f in place of μ^n_{f,ψ_f}.

Let us recall the following theorem of Bowen [**Bow74**].

THEOREM 2.2. (Bowen's equidistribution theorem) *Let (X, d) be a compact metric space, $f : X \to X$ an expansive homeomorphism with the specification property, and $\psi \in C^f(X)$. Then, there exists a unique equilibrium state $\mu_\psi \in \mathcal{M}(f)$ given by*

$$\mu_\psi = \lim_{n\to\infty} \mu^n_{f,\psi},$$

where the limit converges in the weak-topology.*

Remark 3. Theorem 2.2 is sufficient for proving Theorem 1.2 (though Bowen's theorem is stated for invertible systems, the same result is true for uniformly expanding maps on the circle with essentially the same proof; see also Theorem 2.3 below). Indeed, from the assumption on periodic data, it follows that $\mu^n_f = h^*\mu^n_g$ for all $n \in \mathbb{N}$, where $h^*\mu_g$ denotes the pullback measure by h. Then, taking weak*-limits and using Theorem 2.2, we conclude that $\mu_f = h^*\mu_g$. Defining the functions

$$I_f(x) = \int_0^x \rho_f(y)\, dy,\ I_g(x) = \int_0^x \rho_g(y)\, dy,$$

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and integrating $\mu_f = h^*\mu_g$ from 0 to x, we find that $I_f(x) = I_g(h(x))$ or

$$h = I_g^{-1} \circ I_f \in C^{1+\alpha}. \tag{2.1}$$

Equation (2.1) provides us with insight as to how we should define h_N in the proof of Theorem 1.3.

To estimate $d_{C^0}(h, h_N)$ in Theorem 1.3, we will need the following effective version of Theorem 2.2.

THEOREM 2.3. (Effective equidistribution) *Let $\mathcal{W}$ be as bounded set in $\mathrm{Exp}^2_\gamma(S^1)$, $\gamma > 1$. Let $f \in \mathcal{W}$, $\psi_f : S^1 \to \mathbb{C}$ be the geometric potential of f with unique equilibrium state μ_f, and let μ_f^N be the Nth weighted discrete measure associated to f and ψ_f. Then, there exist constants $C' > 0$ and $0 < \tau < 1$, depending only on $\mathcal{W}$, such that for every Lipschitz function $\phi : S^1 \to \mathbb{C}$, we have*

$$\left| \int \phi \, d\mu_f - \int \phi \, d\mu_f^N \right| \le C' \|\phi\|_{\mathrm{Lip}} \tau^N. \tag{2.2}$$

We will defer the proof of Theorem 2.3 to §3. See Kadyrov [**Kad16**, Theorem 1.5] for a more general result for the measure of maximal entropy of a subshift of finite type, and Rühr [**Rüh21**] for a version applying to equilibrium states of countable state shifts.

To prove Theorem 1.3, we will need to apply a version of Theorem 2.3 to the characteristic functions $\chi_{[0,x]}$ for every $x \in S^1$.

LEMMA 2.4. *There exist constants $K > 0$ and $0 < \lambda < 1$ such that for every $f \in \mathcal{W}$, every $x \in S^1$, and every $N \in \mathbb{N}$,*

$$\left| \int_0^x d\mu_f - \int_0^x d\mu_f^N \right| \le K\lambda^N. \tag{2.3}$$

Here and throughout the paper, we write $\int_0^x d\mu_f^N$ as shorthand notation for $\int \chi_{[0,x]} \, d\mu_f^N = \mu_f^N([0, x])$.

Proof. We would like to apply Theorem 2.3 but cannot do so directly since the characteristic function $\chi_{[0,x]}$ is not Lipschitz continuous. Instead, we will approximate $\chi_{[0,x]}$ by a Lipschitz function ϕ_x and show that the effective equidistribution in equation (2.2) of ϕ_x can be passed on to $\chi_{[0,x]}$, albeit with a slower rate of convergence. To find an appropriate choice of ϕ_x, we will construct a one-parameter family of Lipschitz functions ϕ_x^s for $s \in [0, \tau^{N/2}]$, where τ is as in Theorem 2.3, satisfying the following properties:

(1) the family ϕ_x^s varies continuously with s in the C^0-topology;
(2) $\phi_x^0 \le \chi_{[0,x]}$ and $\phi_x^{\tau^{N/2}} \ge \chi_{[0,x]}$;
(3) for every $s \in [0, \tau^{N/2}]$, $|\phi_x^s|_{\mathrm{Lip}} \le \tau^{-N/2}$;
(4) for every $s \in [0, \tau^{N/2}]$, $|\phi_x^s - \chi_{[0,x]}| = 0$ except on a set Ω_N of Lebesgue measure $m(\Omega_N) \le 2\tau^{N/2}$.

The definition of the families ϕ_x^s depends on the point $x \in S^1$, though the typical graph of the functions are all similar and illustrated in Figure 1. We split the construction of the ϕ_x^s into three general cases.

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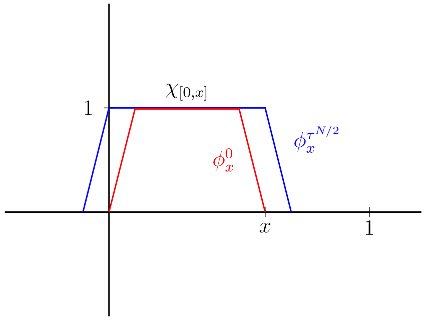

FIGURE 1. The family ϕ_x^s at $s = 0$ (below $\chi_{[0,x]}$) and $s = \tau^{N/2}$ (above $\chi_{[0,x]}$).

Case 1. Assume that $2\tau^{N/2} \le x \le 1 - 2\tau^{N/2}$: Fix $2\tau^{N/2} \le x \le 1 - 2\tau^{N/2}$ and define a family of continuous functions as follows:

$$\phi_x^s(y) = \begin{cases} \tau^{-N/2}(y+s), & -s \le y \le \tau^{N/2} - s, \\ 1, & \tau^{N/2} - s \le y \le s - (\tau^{N/2} - x), \\ -\tau^{-N/2}(y-s) + \tau^{-N/2}x, & s - (\tau^{N/2} - x) \le y \le x + s, \\ 0, & x + s \le y \le 1 - s. \end{cases}$$

It can be easily shown that the above family has the stated properties. Next, we estimate

$$\begin{aligned} &\left| \int_0^x d\mu_f - \int_0^x d\mu_f^N \right| \\ &\quad \le \left| \int_0^x d\mu_f - \int \phi_x^s \, d\mu_f \right| + \left| \int \phi_x^s \, d\mu_f - \int \phi_x^s \, d\mu_f^N \right| + \left| \int \phi_x^s \, d\mu_f^N - \int_0^x d\mu_f^N \right|. \end{aligned} \tag{2.4}$$

Let us consider each term on the right side of equation (2.4) separately. The first term can be rewritten as

$$\left| \int_0^x d\mu_f - \int \phi_x^s \, d\mu_f \right| = \left| \int (\chi_{[0,x]}(y) - \phi_x^s(y))\rho_f(y)\, dy \right|.$$

By property (4) of the family ϕ_x^s, $|\chi_{[0,x]}(y) - \phi_x^s(y)| = 0$ except on a set of Lebesgue measure less than $2\tau^{N/2}$ (depending on s), and is otherwise bounded above by 1. Furthermore, by Lemma 2.1, there exists $C > 1$ such that the density ρ_f of μ_f is bounded above by C. Hence, $|\chi_{[0,x]}(y) - \phi_x^s(y)| = 0$ except on a set of μ_f-measure less than $2C\tau^{N/2}$, and is otherwise bounded above by 1. Therefore,

$$\left| \int (\chi_{[0,x]}(y) - \phi_x^s(y))\rho(y)\, dy \right| \le 2C\tau^{N/2}.$$

For the second term on the right side of equation (2.4), we observe that by property (3), we have $\|\phi_x^s\|_{\mathrm{Lip}} = 1 + \tau^{-N/2}$. We can therefore apply Theorem 2.3 to find

$$\left| \int \phi_x^s \, d\mu_f - \int \phi_x^s \, d\mu_f^N \right| \le C(1 + \tau^{-N/2})\tau^N = C(\tau^N + \tau^{N/2}) \le C\tau^{N/2}.$$

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We claim that the final term on the right side of equation (2.4) is zero for some choice of s. Define

$$\Phi(s) = \int \phi_x^s \, d\mu_f^N - \int_0^x \, d\mu_f^N.$$

Then by property (2), since $\Phi(0)$ is the integral of a strictly negative function with respect to a positive measure, we have $\Phi(0) \leq 0$, and likewise $\Phi(\tau^{N/2}) \geq 0$. We claim that $\Phi(s)$ is a continuous function of s. Indeed, by property (1), fixing $\varepsilon > 0$, we can find a $\delta > 0$ such that if $|s_1 - s_2| < \delta$, then $\|\phi_x^{s_1} - \phi_x^{s_2}\|_\infty < \varepsilon$. Then,

$$|\Phi(s_1) - \Phi(s_2)| = \left| \int (\phi_x^{s_1} - \phi_x^{s_2}) \, d\mu_f^N \right| < \varepsilon \mu_f^N(0, 1) = \varepsilon.$$

By the intermediate value theorem, we can therefore choose some $0 < s < \tau^{N/2}$ so that $\Phi(s) = 0$. Therefore, combining these three bounds, we have the following bound on equation (2.4):

$$\left| \int_0^x \, d\mu_f - \int_0^x \, d\mu_f^N \right| \leq 2C\tau^{N/2} + C\tau^{N/2} + 0 = K\tau^{N/2},$$

which is exactly equation (2.3), with $\lambda = \tau^{1/2}$.

Case 2. Assume that $x > 1 - 2\tau^{N/2}$: The proof is nearly identical as in Case 1, but we use a slightly different family of Lipschitz functions:

$$\phi_x^s(y) = \begin{cases} \tau^{-N/2}(y + s), & -s \leq y \leq \tau^{N/2} - s, \\ 1, & \tau^{N/2} - s \leq y \leq s - (\tau^{N/2} - x), \\ -\tau^{-N/2}(y - s) + \tau^{-N/2}x, & s - (\tau^{N/2} - x) \leq y \leq x + s, \\ 0, & x + s \leq y \leq 1 - s, \end{cases}$$

for $0 \leq s \leq (1 + x)/2$, and

$$\phi_x^s(y) = \begin{cases} 1, & \tau^{N/2} - s \leq y \leq s - (\tau^{N/2} - x), \\ -\tau^{-N/2}(y - s) + \tau^{-N/2}x, & s - (\tau^{N/2} - x) \leq y \leq \dfrac{1 + x}{2}, \\ \tau^{-N/2}(y - 1 + s), & \dfrac{1 + x}{2} \leq y \leq 1 - s, \end{cases}$$

for $(1 + x)/2 \leq s \leq \tau^{N/2}$. The remainder of the proof is identical to Case 1.

Case 3. Assume that $x \leq 2\tau^{N/2}$: The proof is again identical to Case 1 but with the following family:

$$\phi_x^s(y) = \begin{cases} \tau^{-N/2}(y + s), & -s \leq y \leq \dfrac{x}{2}, \\ -\tau^{-N/2}(y - s) + \tau^{-N/2}x, & \dfrac{x}{2} \leq y \leq s, \\ 0, & s \leq y \leq 1 - s, \end{cases}$$

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for $0 \le s \le \tau^{N/2} - x/2$, and

$$\phi_x^s(y) = \begin{cases} \tau^{-N/2}(y+s,) & -s \le y \le \tau^{N/2} - s, \\ 1, & \tau^{N/2} - s \le y \le s + \tau^N x - \tau^{-N/2}, \\ -\tau^{-N/2}(y-s) + \tau^{-N/2}x, & s + \tau^N x - \tau^{-N/2} \le y \le s, \\ 0, & s \le y \le 1-s, \end{cases}$$

for $\tau^{N/2} - x/2 \le s \le \tau^{N/2}$. Observe that for every $0 \le s \le \tau^{N/2}$, $\|\phi_x^s\|_\infty \le 1$, so that $\|\phi_x^s\|_{\mathrm{Lip}} \le 1 + \tau^{-N/2}$, so the remainder of the argument in Case 1 carries over verbatim. □

Proof of Theorem 1.3. We begin by recalling that when we have matching of all periodic data, the conjugacy h could be expressed as the smooth composition in equation (2.1). Observe that the latter expression $I_g^{-1} \circ I_f$ is a well-defined $C^{1+\alpha}$ function without any hypotheses on the periodic data. For this reason, we define $h_N := I_g^{-1} \circ I_f$.

Observe that since $(f^N)'(p) = (g^N)'(h(p))$ for every $p \in \mathrm{Fix}(f^N)$, we have that $\mu_f^N = h^* \mu_g^N$:

$$\begin{aligned} h^*\bigg(\frac{1}{Z_N(\psi_g)} \sum_{x \in \mathrm{Fix}(g^N)} \exp(S_N \psi_g(x))\delta_x\bigg) &= \frac{1}{Z_N(\psi_g)} \sum_{x \in \mathrm{Fix}(g^N)} \exp(S_N \psi_g(h(h^{-1}x)))\delta_{h^{-1}(x)} \\ &= \frac{1}{Z_N(\psi_f)} \sum_{y \in \mathrm{Fix}(f^N)} \exp(S_N \psi_f(y))\delta_y = \mu_f^N. \end{aligned}$$

We now calculate

$$\begin{aligned} &|h_N(x) - h(x)| \\ &\quad= |I_g^{-1} \circ I_f(x) - h(x)| = |I_g^{-1} \circ I_f(x) - I_g^{-1} \circ I_g \circ h(x)| \le C|I_f(x) - I_g(h(x))| \\ &\quad= C\bigg|\int_0^x d\mu_f - \int_0^{h(x)} d\mu_g\bigg| \le C\bigg|\int_0^x d\mu_f - \int_0^{h(x)} d\mu_g^N\bigg| \\ &\qquad+ C\bigg|\int_0^{h(x)} d\mu_g^N - \int_0^{h(x)} d\mu_g\bigg| \\ &\quad= C\bigg|\int_0^x d\mu_f - \int_0^x d(h^*\mu_g^N)\bigg| + C\bigg|\int_0^{h(x)} d\mu_g^N - \int_0^{h(x)} d\mu_g\bigg| \\ &\quad= C\bigg|\int_0^x d\mu_f - \int_0^x d\mu_f^N\bigg| + C\bigg|\int_0^{h(x)} d\mu_g^N - \int_0^{h(x)} d\mu_g\bigg|, \end{aligned}$$

where $\sup|(I_g^{-1})'| = \sup|\rho_g^{-1}| \le C$. By symmetry, it suffices to show that

$$\bigg|\int_0^x d\mu_f - \int_0^x d\mu_f^N\bigg| \le K\lambda^N,$$

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uniformly in $x \in S^1$. This is precisely the content of Lemma 2.4. Therefore, we have shown that for every $x \in S^1$, $|h_N(x) - h(x)| \leq K\lambda^N$, where $\lambda = \tau^{N/2}$. This proves the first statement of Theorem 1.3.

As a first consequence, we obtain a bound on the C^0 distance between f and f_N:

$$\begin{aligned}|f(x) - f_N(x)| &= |h^{-1}(g((h(x)) - h_N^{-1}(g(h_N(x))| \\ &\leq |h^{-1}(g((h(x)) - h_N^{-1}(g(h(x))| + |h_N^{-1}(g((h(x)) - h_N^{-1}(g(h_N(x))| \\ &\leq d_{C^0}(h^{-1}, h_N^{-1}) + Lip(h_N^{-1} \circ g) d_{C^0}(h, h_N) \leq K(1 + Lip(h_N^{-1} \circ g))\lambda^N.\end{aligned}$$

Note that $Lip(h_N^{-1} \circ g) = \sup |D(h_N^{-1} \circ g)| = \sup(|D(I_f^{-1} \circ I_g \circ g)| \leq (\max \rho_g / \min \rho_f)$ $\max |g'|$, which is uniformly bounded in $\mathcal{W}$. Absorbing these uniform constants into K, we get

$$d_{C^0}(f, f_N) < K\lambda^N. \tag{2.5}$$

To finish the proof, it remains to establish the stated C^1 exponential closeness of f and f_N. We will do so by interpolating between the C^0 exponential bound in equation (2.5) and a uniform $C^{1+\alpha}$ bound we will establish below. The following interpolation lemma we will use is elementary, but we include the proof for completeness.

LEMMA 2.5. *Fix $M > 0$ and $0 < \alpha \leq 1$. Let $\phi : S^1 \to S^1$ be a $C^{1+\alpha}$ function with $|\phi'|_{C^\alpha} < M$, and let $\varepsilon, \delta > 0$ be such that*

$$\sup_{|x-y|>\delta} \frac{|\phi(x) - \phi(y)|}{|x - y|} < \varepsilon,$$

then $|\phi|_{C^1} < (M/(\alpha + 1))\delta^\alpha + \varepsilon$.

Proof of Lemma 2.5. Suppose first that $\phi(0) = 0$ and $\phi'(0) = \sup |\phi'| =: \varepsilon'$, and take $|x| > \delta$. Then, since the α-Hölder seminorm of ϕ' is bounded by M, we have

$$\begin{aligned}\max_{y \neq 0} \frac{|\phi'(0) - \phi'(y)|}{|y|^\alpha} < M \implies -M|y|^\alpha &\leq \phi'(0) - \phi'(y) \\ &\leq M|y|^\alpha \implies \phi'(y) \geq \phi'(0) - M|y|^\alpha.\end{aligned}$$

Putting this together with

$$\frac{\phi(x)}{x} = \frac{1}{x}\int_0^x \phi'(y)\, dy,$$

we find

$$\frac{\phi(x)}{x} \geq \frac{1}{x}\int_0^x (\varepsilon' - My^\alpha)\, dy = \varepsilon' - \frac{Mx^\alpha}{\alpha + 1}.$$

It follows that

$$\varepsilon' \leq \frac{M}{\alpha + 1}|x|^\alpha + \frac{|\phi(x)|}{|x|} \leq \frac{M}{\alpha + 1}|x|^\alpha + \varepsilon \to \frac{M}{\alpha + 1}\delta^\alpha + \varepsilon,$$

where in the last line, we let $|x| \to \delta$. Finally, if $|\phi|_{C^0}$ is attained at some other point $x_0 \in S^1$, then simply apply the preceding argument to $\tilde{\phi}(x) = \phi(x + x_0) - \phi(0)$. □

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We now return to finish the proof of Theorem 1.3. We will apply Lemma 2.5 to the function $F = f_N - f$. Let $\varepsilon = 2K'\lambda^{N/2}$ and $\delta = \lambda^{N/2}$. Then,

$$\sup_{|x-y|>\delta} \frac{|F(x) - F(y)|}{|x-y|} < \frac{2|F|_{C^0}}{\delta} \leq \frac{2K' d_{C^0}(h, h_N)}{\lambda^{N/2}} \leq \frac{2K'\lambda^N}{\lambda^{N/2}} = 2K'\lambda^{N/2}.$$

It remains to prove that $|F'|_{C^\alpha}$ is uniformly bounded for $f \in \mathcal{W}$ for every $\alpha < 1$. We have

$$|F'|_{C^\alpha} \leq |f'|_{C^\alpha} + |f'_N|_{C^\alpha},$$

and since f is uniformly bounded in the C^2 seminorm, it will be uniformly bounded in the $C^{1+\alpha}$ seminorm. So it remains to uniformly bound $|f'_N|_{C^\alpha} = |(h_N^{-1} \circ g \circ h_N)'|_{C^\alpha} = |((h_N^{-1})' \circ g \circ h_N)(g' \circ h_N)h'_N|_{C^\alpha}$. By the product rule for the $\alpha-$Hölder seminorm and symmetry between $h_N = I_g^{-1} \circ I_f$ and $h_N^{-1} = I_f^{-1} \circ I_g$, it suffices to uniformly bound $|h'_N|_{C^\alpha} = |\rho_f/(\rho_g \circ h_N)|_{C^\alpha}$. By Lemma 2.1, we can uniformly bound $|\rho_f|_{C^\alpha}$ for $f \in \mathcal{W}$ and hence, by properties of the $\alpha-$Hölder seminorm, we can uniformly bound $|\rho_f/(\rho_g \circ h_N)|_{C^\alpha}$. Therefore, we may apply Lemma 2.5 and we find that $|F|_{C^1} \leq (M/(\alpha+1))\delta^\alpha + \varepsilon = (M/(\alpha+1))(2K'\lambda^{\alpha N/2}) + \lambda^{N/2} = K''\lambda^{\alpha N/2}$. For $\alpha < 1$ such that $\lambda^{\alpha/2} = \lambda_0$, we get the desired conclusion. □

COROLLARY 2.6. *Let $f, g \in \mathrm{Exp}_\gamma^{r+1+\alpha}(S^1)$ for $r \in \mathbb{N}$, $r \geq 2$, and suppose that $d_{C^{r+1+\alpha}}(f, g) < C_0$. Under the same hypotheses of Theorem 1.3, we have that there exists a constant $K_r > 0$ independent of f such that $d_{C^r}(f, f_N) \leq K_r\lambda^{2^{-r}N}$.*

Proof. We proceed by induction. The base case $r = 2$ follows exactly as in Theorem 1.3, except the added assumption that our systems are $C^{2+\alpha}$ allow us to get uniform bounds on $|F'|_{C^1}$ (using the argument of Lemma 2.1) and apply Lemma 2.5 with $\alpha = 1$. Letting $F = f_N - f$, we assume by induction that we have proven $|F|_{C^r} \leq K_r\lambda^{2^{-r}N}$. To obtain a similar estimate for $|F|_{C^{r+1}}$, we apply Lemma 2.5 to the function $F^{(r)}$, with $\varepsilon = 2K_r\lambda^{2^{-r-1}N}$ and $\delta = \lambda^{2^{-r-1}N}$. For these choices, we find

$$\sup_{|x-y|>\delta} \frac{|F^{(r)}(x) - F^{(r)}(y)|}{|x-y|} < \frac{2|F|_{C^r}}{\lambda^{2^{-r}N}} \leq \frac{2K_r\lambda^{2^{-r}N}}{\lambda^{2^{-r-1}N}} = K_r\lambda^{2^{-r-1}N}.$$

The assumption that maps are $C^{r+2+\alpha}$ is so that we can compactly embed the set $\mathcal{W}$ in C^{r+2}, thereby getting uniform bounds on $|F^{(r+1)}|_{C^1}$. The conclusion now follows Lemma 2.5 exactly as in the proof of Theorem 1.3. □

The next corollary establishes a similar estimate on the exponential decay of $d_{C^r}(f, f_N)$ without such a loss of exponent under the stronger assumption that f and g are close in the C^k-topology for all k.

COROLLARY 2.7. *Let $f, g \in \mathrm{Exp}_\gamma^\infty(S^1)$ and suppose that $\sup_{k\geq 0} d_{C^k}(f, g) < C_0$. Then, for any $r \in \mathbb{N}$ and any $0 < \lambda^{1/2} < \lambda_0 < 1$, there exists a constant $K'_r > 0$ such that under the hypotheses of Theorem 1.3, we have $d_{C^r}(f, f_N) \leq K'_r\lambda_0^N$.*

Proof. The proof follows from interpolation theory on the spaces $C^r(S^1)$ (see Lunardi [**Lun18**, Remark 1.22]). For $k_1 < m < k_2 \in \mathbb{N}$, we have that $\|\phi\|_{C^m} \leq C_{k_1,k_2,m}\|\phi\|_{C^{k_1}}^{1-t}\|\phi\|_{C^{k_2}}^t$ for any $\phi \in C^{k_2}(S^1)$, where $t = (m-k_1)/(k_2-k_1)$.

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We will apply this with $\phi = F = f_N - f$, $k_1 = 1$, and by taking k_2 sufficiently large, we have that t can be made arbitrarily close to 0. We choose k_2 so that $\lambda^{(1-t)/2} \le \lambda_0$. We want to bound the term $\|F\|_{C^{k_2}}$ using the bound $d_{C^{k_2+2}}(f, g) < C_0$. Applying Corollary 2.1 gives us $\|F\|_{C^{k_2}} \le K_{k_2}\lambda^{tN/2^{k_2}} \le K_{k_2}$. Hence, by our interpolation inequality (collecting all constants into C),

$$\|F\|_{C^m} \le C_{k_1,k_2,m}\|F\|^{1-t}_{C^{k_1}}\|F\|^{t}_{C^{k_2}} \le C_{k_1,k_2,m}K_1^{1-t}K_{k_2}^{t}(\lambda^{(1-t)/2})^N \le C\lambda_0^N. \qquad \square$$

3. *Effective equidistribution*

We begin by reviewing the basic definitions of subshifts of finite type and transfer operators. For a more detailed treatment, see Baladi [**Bal00**], and Parry and Pollicott [**PP90**]. Let A be an irreducible and aperiodic 0, 1-matrix, which we will refer to as a transition matrix, and consider the set

$$\Sigma_A^+ := \{x \in \{1, \ldots, m\}^{\mathbb{Z}_{\ge 0}} \mid A(x_i, x_{i+1}) = 1 \text{ for all } i \ge 0\}.$$

We interpret Σ_A^+ as the set of all sequences in m-symbols that are allowed by the transition matrix A. Consider the left shift map $\sigma_A^+ : \Sigma_A^+ \to \Sigma_A^+$ defined by $(\sigma_A^+(x))_n = x_{n+1}$. We refer to the dynamical system (Σ_A^+, σ_A^+) as a one-sided subshift of finite type. We can analogously define two-sided subshifts of finite types. When our transition matrix is clear, we shall denote the left shift map simply as σ.

We topologize Σ_A^+ with the metric $d_\theta(x, y) = \theta^{\max\{n\ge 0 \mid x_i = y_i, 0\le i<n\}}$, where $0 < \theta < 1$ is a fixed constant. Notice that with respect to this metric, σ is a θ-expansion. Let $\mathcal{F}_\theta^+$ denote the Banach space of all functions $\phi : \Sigma_A^+ \to \mathbb{C}$ which are Lipschitz continuous with respect to this metric. Denote by $|\cdot|_\theta$ and $\|\cdot\|_\theta$ the Lipschitz seminorm and Lipschitz norm with respect to this metric, respectively. For $1 \le i \le m$, let $[i] = \{x \in \Sigma_A^+ | x_0 = i\}$, which we call the cylinder set at i. We will use the following important but simple estimate.

LEMMA 3.1. *If $\phi \in \mathcal{F}_\theta$ and $[i]$ is a cylinder set, then $\|\chi_{[i]}\phi\|_\theta < 2\|\phi\|_\theta$.*

Proof. Note that we always have $\|\chi_{[i]}\phi\|_\infty \le \|\phi\|_\infty \le \|\phi\|_\theta$, so it remains to prove that $|\chi_{[i]}\phi|_\theta \le \|\phi\|_\theta$. Suppose that $x \ne y$. If $x, y \in [i]$, we have

$$\frac{|(\chi_{[i]}\phi)(x) - (\chi_{[i]}\phi)(y)|}{d_\theta(x, y)} = \frac{|\phi(x) - \phi(y)|}{d_\theta(x, y)} \le |\phi|_\theta \le \|\phi\|_\theta.$$

If $x \notin [i]$ and $y \notin [i]$, then

$$\frac{|(\chi_{[i]}\phi)(x) - (\chi_{[i]}\phi)(y)|}{d_\theta(x, y)} = 0 \le \|\phi\|_\theta.$$

Finally, if $x \in [i]$ and $y \notin [i]$, then since $d_\theta(x, y) = 1$, we have

$$\frac{|(\chi_{[i]}\phi)(x) - (\chi_{[i]}\phi)(y)|}{d_\theta(x, y)} = \frac{|\phi(x)|}{d_\theta(x, y)} = |\phi(x)| \le \|\phi\|_\infty \le \|\phi\|_\theta.$$

Taking the supremum over $x \ne y$ yields $|\chi_{[i]}\phi|_\theta \le \|\phi\|_\theta$, as desired. $\square$

Our primary tool in this section will be the Ruelle transfer operator.

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Definition 3.1. Fix a weight function $\psi \in \mathcal{F}_\theta^+$ and define the Ruelle transfer operator $\mathcal{L}_\psi : \mathcal{F}_\theta^+ \to \mathcal{F}_\theta^+$ by the formula

$$\mathcal{L}_\psi(\phi)(x) = \sum_{\sigma(y)=x} e^{\psi(y)}\phi(y).$$

By the Ruelle–Perron–Frobenius theorem (see [**Bal00**, Theorem 1.5]), the operator $\mathcal{L}_\psi$ is quasi-compact, with a unique maximal positive simple eigenvalue $\lambda = e^{P(\psi)}$ corresponding to a strictly positive eigenfunction ρ, and all other points of the spectrum lie in a strictly smaller disc. Let us further assume that the transfer operator is normalized so that $\mathcal{L}_\psi(1) = e^{P(\psi)}$ (which can always be accomplished by replacing the weight ψ with $\overline{\psi} = \psi + \log(\rho) - \log(\rho) \circ \sigma$, and observing that $P(\psi) = P(\overline{\psi})$). Then the eigenmeasure μ_ψ corresponding to the eigenvalue $e^{P(\psi)}$ of the dual operator $\mathcal{L}_\psi^*$ is the unique equilibrium state of the potential ψ. Observe that

$$\mathcal{L}_\psi^n(\phi)(x) = \sum_{\sigma^n(y)=x} e^{S_n\psi(y)}\phi(y).$$

What follows is the analog of Theorem 3.2 for subshifts of finite type.

THEOREM 3.2. (Effective equidistribution for equilibrium states) *Let (Σ_A^+, σ_A^+) be a subshift of finite type, where the $m \times m$ transition matrix A is irreducible and aperiodic, and let $\psi \in \mathcal{F}_\theta^+$ be a Lipschitz continuous potential. Then, there exists constants $C > 0$ and $0 < \tau < 1$ such that for any $\phi \in \mathcal{F}_\theta^+$ and all $n \in \mathbb{N}$,*

$$\left| \int \phi \, d\mu_\psi^n - \int \phi \, d\mu_\psi \right| \le C\|\phi\|_\theta \tau^n,$$

where μ_ψ is the unique equilibrium state of ψ.

Remark 4. To be consistent with our previous notation in §2, we should write $\mu_{\sigma,\psi}^n$ for the weighted discrete measure. However, since the dynamics on the shift space will always be the left shift map σ, we do not need to specify in our notation and will instead simply write μ_ψ^n.

Proof. By replacing ϕ by $\phi - \int \phi \, d\mu_\psi$, we may assume that $\int \phi \, d\mu_\psi = 0$. We need to show that

$$\left| \frac{1}{Z_n} \sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x) \right| \le C\|\phi\|_\theta \tau^n, \tag{3.1}$$

where Z_n is the normalization constant. By Katok and Hasselblatt [**KH95**, Proposition 20.3.3], there exists a constant $D > 0$ such that $(1/D)e^{nP(\psi)} \le Z_n \le De^{nP(\psi)}$ (an inspection of the proof reveals that this constant D can be made uniform). Let $[i] = \{x \in \Sigma_A^+ | x_0 = i\}$, and for a string $\underline{i} = (i_0, \ldots, i_{n-1})$, let us denote its length by $|\underline{i}| = n$ and its cylinder set by $[\underline{i}] = \{x \in \Sigma_A^+ | x_0 = i_0, \ldots, x_{n-1}\}$. For each $1 \le i \le m$, fix any point $x_i \in [i]$, and for each string $\underline{i}$, fix a point of period n, $x_{\underline{i}} \in [\underline{i}]$, if one exists, and let $x_{\underline{i}} \in [\underline{i}]$ be arbitrary otherwise. We first claim that

$$\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x) = \sum_{|\underline{i}|=n} \mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}).$$

To see this, we expand out each term in the right sum:

$$\mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) = \sum_{\sigma^n(y)=x_{\underline{i}}} e^{S_n\psi(y)}\chi_{[\underline{i}]}(y)\phi(y) = e^{S_n\psi(\underline{i}x_{\underline{i}})}\phi(\underline{i}x_{\underline{i}}) = e^{S_n\psi(x_{\underline{i}})}\phi(x_{\underline{i}}),$$

where $\underline{i}x = (i_0, \ldots, i_{n-1}, x_0, x_1, \ldots)$ denotes the only inverse branch of σ^n that contributes to the sum due to the characteristic function. Since each point of $\mathrm{Fix}(\sigma^n)$ lies in a unique cylinder set $[\underline{i}]$, and each such cylinder set contains at most one period-n point, we see that all periodic points are accounted for in the sum over $|\underline{i}| = n$. Consider the estimate

$$\bigg|\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x)\bigg| \le \bigg|\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x) - \sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg| + \bigg|\sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg|. \tag{3.2}$$

To estimate both terms on the right, we will first decompose the transfer operator as a sum of its projection onto the eigenspace of $e^{P(\psi)}$, and the orthogonal projection: $\mathcal{L}_\psi = \mathcal{P} + \mathcal{N}$, where $\mathcal{P}(\phi) = e^{P(\psi)}\int \phi\, d\mu_\psi$, and $\mathcal{N}$ has spectral radius $re^{P(\psi)}$ with $0 < r < 1$. The second term can thus be estimated as

$$\begin{aligned}
\bigg|\sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg| &\le \bigg|\sum_{k=1}^{m}\Big[\mathcal{P}^n(\chi_{[i]}\phi)(x_i) + \mathcal{N}^n(\chi_{[i]}\phi)(x_i)\Big]\bigg| \\
&= \bigg|\sum_{k=1}^{m}\Big[e^{nP(\psi)}\int_{[i]}\phi\, d\mu_\psi + \mathcal{N}^n(\chi_{[i]}\phi)(x_i)\Big]\bigg| \\
&= \bigg|e^{nP(\psi)}\int \phi\, d\mu_\psi + \sum_{k=1}^{m}\mathcal{N}^n(\chi_{[i]}\phi)(x_i)\bigg| = \bigg|\sum_{k=1}^{m}\mathcal{N}^n(\chi_{[i]}\phi)(x_i)\bigg| \\
&\le \sum_{k=1}^{m}\|\mathcal{N}^n\|\|\chi_{[i]}\phi\|_\theta \le C(r+\varepsilon)^n e^{nP(\psi)}\|\phi\|_\theta,
\end{aligned} \tag{3.3}$$

using Lemma 3.1 and the spectral radius formula (see also Lemma 3.3 for the uniform version we will need later in the case of the symbolic coding of expanding maps), where $\varepsilon > 0$ is arbitrary and $C > 0$ depends on ε (recall that m is the size of our alphabet and is independent of n).

It remains to estimate

$$\bigg|\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x) - \sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg| = \bigg|\sum_{|\underline{i}|=n}\mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg|.$$

If $\underline{i} = (i_0, \ldots, i_{n-1})$, we let $\underline{j}(\underline{i}) = (i_0, \ldots, i_{n-2})$. We now telescope our above series:

$$\begin{aligned}
&\sum_{|\underline{i}|=n}\mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \sum_{k=1}^{m}\mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i) \\
&\quad = \sum_{m=2}^{n}\bigg(\sum_{|\underline{i}|=m}\mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \sum_{|\underline{i}|=m-1}\mathcal{L}_\psi^n(\chi_{[\underline{j}]}\phi)(x_{\underline{j}})\bigg)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{m=2}^{n} \sum_{|\underline{i}|=m} (\mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{j}(\underline{i})}))\\
&= \sum_{m=2}^{n} \sum_{|\underline{i}|=m} ((\mathcal{P}^n + \mathcal{N}^n)(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - (\mathcal{P}^n + \mathcal{N}^n)(\chi_{[\underline{i}]}\phi)(x_{\underline{j}(\underline{i})}))\\
&= \sum_{m=2}^{n} \sum_{|\underline{i}|=m} (\mathcal{N}^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \mathcal{N}^n(\chi_{[\underline{i}]}\phi)(x_{\underline{j}(\underline{i})})).
\end{aligned}
$$

(The preceding expansion was not novel, and can be found for instance in [**PS01**, Lemma 3].) We now take the absolute value of both sides and observe that $d_\theta(x_{\underline{i}}, x_{\underline{j}(\underline{i})}) = \theta^{m-1}$:

$$
\begin{aligned}
&\bigg| \sum_{m=2}^{n} \sum_{|\underline{i}|=m} (\mathcal{N}^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \mathcal{N}^n(\chi_{[\underline{i}]}\phi)(x_{\underline{j}(\underline{i})}))\bigg|\\
&\quad \le \sum_{m=2}^{n} \sum_{|\underline{i}|=m} \|\mathcal{N}^n \chi_{[\underline{i}]}\phi\|_\theta \theta^{m-1} \le \sum_{m=2}^{n} \|\mathcal{N}^{n-m}\| \sum_{|\underline{i}|=m} \|\mathcal{L}_\psi^m \chi_{[\underline{i}]}\phi\|_\theta \theta^{m-1}.
\end{aligned}
$$

In this last inequality, we made use of the fact that $\mathcal{N}\mathcal{P} = 0$ to get the bound

$$
\begin{aligned}
\|\mathcal{N}^n(\chi_{[\underline{i}]}\phi)\|_\theta &= \|\mathcal{N}^{n-m}(\mathcal{N}^m(\chi_{[\underline{i}]}\phi) + \mathcal{P}^m(\chi_{[\underline{i}]}\phi))\|_\theta\\
&= \|\mathcal{N}^{n-m}(\mathcal{L}_\psi^m(\chi_{[\underline{i}]}\phi))\|_\theta \le \|\mathcal{N}^{n-m}\| \|\mathcal{L}_\psi^m(\chi_{[\underline{i}]}\phi))\|_\theta.
\end{aligned}
$$

We next estimate the term $\|\mathcal{L}_\psi^m(\chi_{[\underline{i}]}\phi))\|_\theta = \|e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}(\phi \circ \sigma_{\underline{i}}^{-1})\|_\theta$, where $\sigma_{\underline{i}}^{-1}(x) = \underline{i}x$ is an inverse branch of σ^n:

$$
\begin{aligned}
\|e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}(\phi \circ \sigma_{\underline{i}}^{-1})\|_\theta &= |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}(\phi \circ \sigma_{\underline{i}}^{-1})|_\infty + |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}(\phi \circ \sigma_{\underline{i}}^{-1})|_\theta\\
&\le |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty |\phi|_\infty + |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty |(\phi \circ \sigma_{\underline{i}}^{-1})|_\theta + |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\theta |(\phi \circ \sigma_{\underline{i}}^{-1})|_\infty\\
&\le |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty |\phi|_\infty + |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty |(\phi \circ \sigma_{\underline{i}}^{-1})|_\theta\\
&\quad + |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty |S_m\psi \circ \sigma_{\underline{i}}^{-1}|_\theta |(\phi \circ \sigma_{\underline{i}}^{-1})|_\infty\\
&= |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty (|\phi|_\infty + |\phi \circ \sigma_{\underline{i}}^{-1}|_\theta + |S_m\psi \circ \sigma_{\underline{i}}^{-1}|_\theta |(\phi \circ \sigma_{\underline{i}}^{-1})|_\infty)\\
&\le |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty (\|\phi\|_\theta + |\phi \circ \sigma_{\underline{i}}^{-1}|_\theta + |S_m\psi \circ \sigma_{\underline{i}}^{-1}|_\theta \|\phi\||_\theta).
\end{aligned}
$$

To calculate the seminorms $|\phi \circ \sigma_{\underline{i}}^{-1}|_\theta$ and $|S_m\psi \circ \sigma_{\underline{i}}^{-1}|_\theta$, we use the fact that $\sigma_{\underline{i}}^{-1}$ is a θ^m-contraction:

$$
\begin{aligned}
|S_m\psi \circ \sigma_{\underline{i}}^{-1}|_\theta &= \sup_{x\ne y} \frac{|S_m\psi \circ \sigma_{\underline{i}}^{-1}(x) - S_m\psi \circ \sigma_{\underline{i}}^{-1}(y)|}{d_\theta(x, y)}\\
&\le \sum_{i=0}^{m-1} \frac{|\psi(\sigma^i(\sigma_{\underline{i}}^{-1}(x))) - \psi(\sigma^i(\sigma_{\underline{i}}^{-1}(y)))|}{d_\theta(x, y)} \le \sum_{i=0}^{m-1} |\psi|_\theta \theta^{m-i} \le \frac{1}{1-\theta}|\psi|_\theta.
\end{aligned}
$$

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A similar (and simpler) calculation shows that $|\phi \circ \sigma_{\underline{i}}^{-1}|_\theta \leq \theta^m \|\phi\|_\theta$. Putting this all together, we find that

$$\begin{aligned}\|\mathcal{L}_\psi^m \chi_{[\underline{i}]}\phi\|_\theta \theta^{m-1} &\leq |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty \bigg(\|\phi\|_\theta\theta^{m-1} + \|\phi\|_\theta\theta^{2m-1} + \frac{|\psi|_\theta}{1-\theta}\|\phi\|_\theta\theta^m\bigg)\\ &\leq C'|e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty \|\phi\|_\theta\theta^m,\end{aligned}$$

where $C' = \max\{1, |\psi|_\theta/(1-\theta)\}$. Thus, using the spectral radius bound for $\mathcal{N}$, we find that

$$\begin{aligned}&\sum_{m=2}^{n} \|\mathcal{N}^{n-m}\| \sum_{|\underline{i}|=m} \|\mathcal{L}_\psi^m \chi_{[\underline{i}]}\phi\|_\theta\theta^{m-1}\\ &\quad\leq \sum_{m=2}^{n} C(r+\varepsilon)^{n-m} e^{(n-m)P(\psi)} C'\theta^m \|\phi\|_\theta \sum_{|\underline{i}|=m} |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty\\ &\quad\leq C''\|\phi\|_\theta\kappa^n \sum_{m=2}^{n} e^{(n-m)P(\psi)} \sum_{|\underline{i}|=m} |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty,\end{aligned}$$

where $\kappa = \max\{\theta, r+\varepsilon\} < 1$. We have seen previously that $|S_m\psi \circ \sigma_{\underline{i}}^{-1}(x) - S_m\psi \circ \sigma_{\underline{i}}^{-1}(y)| \leq (|\psi|_\theta/(1-\theta))\, d_\theta(x,y) \leq K < \infty$ (since the diameter of Σ_A^+ is finite). Taking the exponential of both sides, we obtain the following bounded distortion estimate:

$$e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}(x)} \leq Ce^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}(y)}$$

for any x, y in the domain of $\sigma_{\underline{i}}^{-1}$. Notice that the domain of $\sigma_{\underline{i}}^{-1}$ is completely determined by the last symbol in the string $\underline{i}$. For each $\underline{i}$, let $y_{\underline{i}}$ be such that $e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}(y_{\underline{i}})} = |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty$, and let $z_{\underline{i}}$ be any point in the domain of $\sigma_{\underline{i}}^{-1}$ that only depends on the last symbol i_m. Then,

$$\begin{aligned}\sum_{|\underline{i}|=m} |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty &\leq C\sum_{|\underline{i}|=m} e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}(z_{\underline{i}})}\\ &= C\sum_{i_m=1}^{m} \mathcal{L}_\psi^m 1(z_{\underline{i}}) \leq Cm\|\mathcal{L}_\psi^m 1\|_\theta \leq Ce^{m(P(\psi)+\varepsilon)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\bigg|\sum_{|\underline{i}|=n} \mathcal{L}_\psi^n(\chi_{[\underline{i}]}\phi)(x_{\underline{i}}) - \sum_{k=1}^{m} \mathcal{L}_\psi^n(\chi_{[i]}\phi)(x_i)\bigg|\\ &\quad\leq C''\|\phi\|_\theta\kappa^n \sum_{m=2}^{n} e^{(n-m)P(\psi)} \sum_{|\underline{i}|=m} |e^{(S_m\psi)\circ\sigma_{\underline{i}}^{-1}}|_\infty\\ &\quad\leq C'''\|\phi\|_\theta\kappa^n e^{n(P(\psi)+\varepsilon)}(n-2) \leq C'''\|\phi\|_\theta(\kappa+\varepsilon)^n e^{n(P(\psi)+\varepsilon)}. \qquad (3.4)\end{aligned}$$

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Combining the estimates in equations (3.2), (3.3), and (3.4), we have shown

$$\bigg|\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x)\bigg| \le C\|\phi\|_\theta(\kappa+\varepsilon)^n e^{n(P(\psi)+\varepsilon)}$$

with $\kappa+\varepsilon<1$. We therefore have

$$\bigg|\frac{1}{Z_n}\sum_{\sigma^n(x)=x} e^{S_n\psi(x)}\phi(x)\bigg| \le De^{-nP(\psi)}C\|\phi\|_\theta(\kappa+\varepsilon)^n e^{n(P(\psi)+\varepsilon)} \le C\|\phi\|_\theta(\kappa+\varepsilon)^n e^{n\varepsilon}.$$

Finally, letting $\tau=(\kappa+\varepsilon)e^{\varepsilon}<1$ for sufficiently small $\varepsilon>0$, this gives equation (3.1). □

Remark 5. By standard arguments, it is easy to deduce the same equidistribution result for two-sided shifts.

To pass to the proof of Theorem 2.3, we will use Markov partitions. It is well known that repellers and Axiom A diffeomorphisms admit finite Markov partitions; see Przytycki and Urbanski [**PU10**, Theorem 3.5.2] in the case of expanding maps and Bowen [**Bow70**] for the case of Axiom A diffeomorphisms. More precisely, if (J,T) is a repeller or $(\Omega(T),T)$ is an Axiom A diffeomorphism, then there exists a subshift of finite type (Σ_A^+,σ) and a semiconjugacy $\pi:\Sigma_A^+\to J$ (respectively (Σ_A,σ) for $(\Omega(T),T)$). If the transformation T is mixing, then the transition matrix A is irreducible and aperiodic. For an appropriately chosen θ, Lipschitz functions f defined on J or $\Omega(T)$ can be lifted to a Lipschitz function $f\circ\pi$. We proceed to prove Theorem 2.3 in the case of one-dimensional expanding maps, where the passage to a Markov partition is simplest. However, to obtain a uniform effective equidistribution for expanding maps, we need uniform effective equidistribution of corresponding lifted systems in Theorem 3.2. Uniformity is lost in Theorem 3.2 when we apply the spectral radius formula to get bounds on $\|\mathcal{N}^n\|_\theta$. However, for the particular systems we are considering, we can obtain uniform bounds on $\|\mathcal{N}^n\|_\theta$ by using the Birkhoff cone technique, adapted to subshifts of finite type by Naud in [**Nau04**].

LEMMA 3.3. *There exists $C_{\mathcal{W}}>0$ and $0<\tau_{\mathcal{W}}<1$ such that the following is true. Given $f\in\mathcal{W}$, let π_f be the associated semiconjugacy to the full shift on* $\deg(f)$*-symbols, and let $\psi_f=-\log(f'\circ\pi_f)$. If $\mathcal{L}_{\psi_f}h_f=e^{P(\psi_f)}h_f$, we consider the normalized potential $\overline{\psi}_f=\psi_f+\log(h_f)-\log(h_f\circ\sigma)$. Then, for all $n\in\mathbb{N}$, $\|\mathcal{N}^n_{\overline{\psi}_f}\|_\theta<C_{\mathcal{W}}\tau^n_{\mathcal{W}}e^{nP(\overline{\psi}_f)}$.*

We review the Birkhoff cone construction and prove Lemma 3.3 in Appendix A.

Proof of Theorem 2.3. Consider a degree k expanding map $f:S^1\to S^1$. Then, the Markov partition of (f,S^1) consists of k closed intervals which are each mapped under f to the entirety of S^1. Consequentially, (f,S^1) is semiconjugated to the full shift on k symbols, (σ,Σ^+). Let $\phi:S^1\to\mathbb{C}$ be a Lipschitz function. We can then lift ϕ to a Lipschitz function $\phi\circ\pi:\Sigma^+\to\mathbb{C}$. Likewise, let $\psi_f:S^1\to\mathbb{C}$ be the geometric potential with equilibrium state μ_f, and let $\psi_f\circ\pi:\Sigma^+\to\mathbb{C}$ be the lifted potential with equilibrium state $\mu_{\psi_f\circ\pi}$. We claim that the pushforward of $\mu_{\psi_f\circ\pi}$ under π is μ_f, that is, $\pi_*\mu_{\psi_f\circ\pi}=\mu_f$. Unfortunately, it is not true that $\pi_*\mu^n_{\psi_f\circ\pi}=\mu^n_f$. Since the semiconjugacy is not injective, we may have multiple distinct periodic orbits for (Σ^+, σ) get mapped under π to the same periodic orbit of (S^1, f).

We know that $|\operatorname{Fix}(\sigma^n)| = k^n$ and $|\operatorname{Fix}(f^n)| = k^n - 1$. Moreover, distinct periodic orbits of f can be lifted to distinct periodic orbits of σ with the same period, so we have that for each n, only two distinct points of Σ^+ of period n get mapped under π to the same point. Let $\nu^n_{\psi_f\circ\pi}$ be any measure on Σ^+ such that $\pi^*\nu^n_{\psi_f\circ\pi} = \mu^n_f$. Then,

$$\left|\int \phi\, d\mu^n_f - \int \phi\, d\mu_f\right| = \left|\int \phi\circ\pi\, d\nu^n_{\psi\circ\pi} - \int \phi\circ\pi\, d\mu_{\psi_f\circ\pi}\right|$$
$$\leq \left|\int \phi\circ\pi\, d\mu^n_{\psi_f\circ\pi} - \int \phi\circ\pi\, d\mu_{\psi_f\circ\pi}\right| + \left|\int \phi\circ\pi\, d\nu^n_{\psi_f\circ\pi} - \int \phi\circ\pi\, d\mu^n_{\psi_f\circ\pi}\right|. \tag{3.5}$$

The first term in equation (3.5) can be bounded by $C\|\phi\circ\pi\|_\theta\tau^n$ using Theorem 3.2. To estimate the last term of equation (3.5), we write

$$\nu^n_{\psi_f\circ\pi} = \frac{1}{Z_n(\psi_f)} \sum_{x\in A_n} e^{S_n(\psi_f)(\pi(x))}\delta_x,$$

where A_n is any subset of $\operatorname{Fix}(\sigma^n)$ that is mapped bijectively to $\operatorname{Fix}(f^n)$, and $Z_n(\psi)$ is a normalization constant. Then, if we let $y \in \operatorname{Fix}(\sigma^n)/A_n$, we have

$$\begin{aligned}
&\left|\int \phi\circ\pi\, d\nu^n_{\psi_f\circ\pi} - \int \phi\circ\pi\, d\mu^n_{\psi_f\circ\pi}\right| \\
&\quad= \left|\frac{1}{Z_n(\psi_f)} \sum_{x\in A_n} e^{S_n(\psi_f)(\pi(x))}\phi(\pi(x)) - \frac{1}{Z_n(\psi_f\circ\pi)} \sum_{x\in \operatorname{Fix}(\sigma^n)} e^{S_n(\psi_f)(\pi(x))}\phi(\pi(x))\right| \\
&\quad\leq \left|\left(\frac{1}{Z_n(\psi_f)} - \frac{1}{Z_n(\psi_f\circ\pi)}\right) \sum_{x\in \operatorname{Fix}(\sigma^n)} e^{S_n(\psi_f)(\pi(x))}\phi(\pi(x))\right| \\
&\qquad+ \frac{1}{Z_n(\psi_f)}\left| \sum_{x\in A_n} e^{S_n(\psi_f)(\pi(x))}\phi(\pi(x)) - \sum_{x\in \operatorname{Fix}(\sigma^n)} e^{S_n(\psi_f)(\pi(x))}\phi(\pi(x))\right| \\
&\quad\leq \left(\frac{Z_n(\psi_f\circ\pi)}{Z_n(\psi_f)} - 1\right)\|\phi\circ\pi\|_\infty + \frac{1}{Z_n(\psi_f)} e^{S_n(\psi_f)(\pi(y))}|\phi(\pi(y))| \\
&\quad\leq \left(\frac{Z_n(\psi_f) + e^{S_n(\psi_f)(\pi(y))}}{Z_n(\psi_f)} - 1\right)\|\phi\circ\pi\|_\infty + \frac{1}{Z_n(\psi_f)} e^{S_n(\psi_f)(\pi(y))}|\phi(\pi(y))| \\
&\quad\leq \frac{2}{Z_n(\psi_f)} e^{S_n(\psi_f)(\pi(y))}\|\phi\circ\pi\|_\theta \leq D\|\phi\circ\pi\|_\theta e^{S_n(\psi_f)(\pi(y))} e^{-nP(\psi_f)}.
\end{aligned}$$

In the case of expanding maps, we have that $P(\psi_f) = 0$ and $S_n(\psi_f)(\pi(y)) \leq -n\log\lambda_f$, where $\lambda_f > 1$ is the expansion constant for f, that is, $|f'(x)| \geq \lambda_f$ for all $x \in S^1$. This proves that

$$\left|\int \phi\, d\mu^n_f - \int \phi\, d\mu_f\right| \leq C'\|\phi\circ\pi\|_\theta\tau^n$$

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for some $0 < \tau < 1$ (not necessarily the same τ as in Theorem 3.2). Clearly, $\|\phi \circ \pi\|_\infty \leq \|\phi\|_\infty$. Moreover,

$$|\phi \circ \pi|_\theta = \sup_{x \neq y} \frac{|\phi(\pi(x) - \phi(\pi(y))|}{d_\theta(x, y)} \leq |\phi|_{\mathrm{Lip}} \frac{d(\pi(x), \pi(y))}{d_\theta(x, y)} \leq |\pi|_\theta |\phi|_{\mathrm{Lip}},$$

so that $\|\phi \circ \pi\|_\theta \leq \max\{1, |\pi|_\theta\}\|\phi\|_{\mathrm{Lip}}$. Thus, absorbing all constants into C, we have

$$\left| \int \phi \, d\mu_f^n - \int \phi \, d\mu_f \right| \leq C\|\phi\|_{\mathrm{Lip}} \tau^N,$$

as desired. □

Remark 6. One can prove effective equidistribution for general expanding repellers and Axiom A diffeomorphism using Markov partitions in the same way, but more care is needed to handle the difference between the measure $\nu^n_{\psi_f \circ \pi}$ and $\mu^n_{\psi_f \circ \pi}$.

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A. *Appendix. Birkhoff cones for subshifts of finite type*

Rather than deducing our desired bounds on $\|\mathcal{N}^n_{\psi_f}\|_\theta$ as a consequence of quasi-compactness of the transfer operator (the standard approach of the Ruelle–Perron–Frobenius theorem), we use the technique of Birkhoff cones. The idea is to show that the transfer operator contracts a certain cone of Lipschitz functions with respect to a 'pseudo-metric' and to then establish the leading eigenfunction as a fixed point with respect to this pseudo-metric. The benefit to this approach is that we can establish explicit bounds on $\|\mathcal{N}^n_{\psi_f}\|_\theta$ which will be uniform in our set $\mathcal{W}$. Then, one can actually deduce quasi-compactness as a consequence of this bound. This approach is standard for uniformly expanding maps; see Baladi [**Bal00**, §2.2]. For subshifts of finite type, we follow Naud [**Nau04**] closely, applied to the specific case of the full shift that we need, and for which certain technical difficulties vanish.

Definition A.1. A subset $\Lambda \subset \mathcal{B}/\{0\}$ of a Banach space $\mathcal{B}$ is called a cone if $\lambda\phi \in \Lambda$ for all $\phi \in \Lambda$ and all $\lambda > 0$. The cone is called closed if $\Lambda \cup \{0\}$ is closed, and Λ is called convex if $\psi_1 + \psi_2 \in \Lambda$ for every $\psi_1, \psi_2 \in \Lambda$. A cone Λ induces a partial order $\leq_\Lambda$ on $\mathcal{B}$ by defining $\psi \leq_\Lambda \phi \iff \phi - \psi \in \Lambda \cup \{0\}$.

Definition A.2. For ψ and ϕ in a cone Λ, define

$$\alpha(\phi, \psi) = \sup\{\lambda > 0 \mid \lambda\phi \leq_\Lambda \psi\}, \beta(\phi, \psi) = \inf\{\lambda > 0 \mid \psi \leq_\Lambda \lambda\phi\}.$$

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Then, we define the Hilbert pseudo-metric Θ_Λ on λ by

$$\Theta_\Lambda(\phi, \psi) = \log \frac{\beta(\phi, \psi)}{\alpha(\phi, \psi)}.$$

THEOREM A.1. (Birkhoff's inequality) *Let Λ be a convex cone in a Banach space $\mathcal{B}$. If $T : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $T(\Lambda) \subset \Lambda$, then for each $\phi, \psi \in \Lambda$, we have*

$$\Theta_\Lambda(T\phi, T\psi) \leq \tanh\left(\frac{\operatorname{diam}_{\Theta_\Lambda}(T\Lambda)}{4}\right)\Theta_\Lambda(\phi, \psi).$$

LEMMA A.2. *Let Λ be a closed convex cone in a Banach space $\mathcal{B}$ endowed with two (not necessarily equivalent norms $\|\cdot\|_i$, $i = 1, 2$, and assume that for all $\phi, \psi \in \mathcal{B}$,*

$$-\phi \leq_\Lambda \psi \leq_\Lambda \phi \implies \|\psi\|_i \leq \|\phi\|_i, \quad i = 1, 2.$$

Then, for any $\phi, \psi \in \Lambda$ with $\|\phi\|_1 = \|\psi\|_1$, we have

$$\|\phi - \psi\|_2 \leq (e^{\Theta_\Lambda(\phi,\psi)} - 1)\|\phi\|_2.$$

If Σ^+ is the one-sided full shift on k-symbols, and if $\mathcal{F}_\theta^+$ is the Banach space of Lipschitz continuous functions on Σ^+ with respect to the d_θ-metric, then given any $L > 0$, we have a cone in $\mathcal{F}_\theta^+$ given by

$$\mathcal{C}_L = \{\phi \in \mathcal{F}_\theta^+ | \phi \geq 0, \phi \not\equiv 0, d_\theta(x, y) \leq \theta \implies \phi(x) \leq e^{Ld_\theta(x,y)}\phi(y)\}.$$

To apply Birkhoff's inequality, we will need the following lemmas.

LEMMA A.3. *Fix $0 < \xi < 1$. Then, for every $\phi, \psi \in \mathcal{C}_{\xi L}$ with $\phi, \psi > 0$, we have*

$$\Theta_L(\phi, \psi) \leq 2\log\left(\frac{1+\xi}{1-\xi}\right) + \log \sup_{x,y\in\Sigma^+}\left(\frac{\phi(x)\psi(y)}{\phi(y)\psi(x)}\right).$$

See Naud [**Nau04**, Proposition 5.3] for the proof, which is unchanged in our setting.

LEMMA A.4. *Fix $\theta < \xi < 1$. Then, for every $L \geq \theta|\psi|_\theta/(\xi - \theta)$, we have $\mathcal{L}_\psi(\mathcal{C}_L) \subset \mathcal{C}_{\xi L}$ and we have*

$$\operatorname{diam}_{\Theta_L}(\mathcal{L}_\psi(\mathcal{C}_L)) \leq 2\log\left(\frac{1+\xi}{1-\xi}\right) + 2\xi L.$$

Proof. Let $\phi \in \mathcal{C}_L$ and let $x, y \in \Sigma^+$ be such that $d_\theta(x, y) \leq \theta$. We obtain

$$\begin{aligned}\mathcal{L}_\psi\phi(x) &= \sum_{i=1}^{k} e^{\psi(ix)}\phi(ix) \leq e^{\theta(|\psi|_\theta + L)d_\theta(x,y)}\sum_{i=1}^{k} e^{\psi(iy)}\phi(iy)\\ &= e^{\theta(|\psi|_\theta+L)d_\theta(x,y)}\mathcal{L}_\psi\phi(y).\end{aligned}$$

The condition $\theta|\psi|_\theta + L\theta \leq \xi L$ holds if and only if $L \geq \theta|\psi|_\theta/(\xi - \theta)$. Notice that $\phi \in \mathcal{C}_L$ implies that there exists at least one cylinder set $C_i = \{x \in \Sigma^+ | x_0 = i\}$ such that $\phi|_{C_i} > 0$. Thus, $\mathcal{L}_\psi\phi(x) \geq e^{\psi(ix)}\phi(ix) > 0$. We may therefore apply Lemma A.3

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to functions $\mathcal{L}_\psi\phi_1$ and $\mathcal{L}_\psi\phi_2$:

$$\Theta_L(\mathcal{L}_\psi\phi_1, \mathcal{L}_\psi\phi_2 \leq 2\log\left(\frac{1+\xi}{1-\xi}\right) + \log\sup_{x,y\in\Sigma^+}\left(\frac{\mathcal{L}_\psi\phi_1(x)\mathcal{L}_\psi\phi_2(y)}{\mathcal{L}_\psi\phi_1(y)\mathcal{L}_\psi\phi_2(x)}\right)$$
$$\leq 2\log\left(\frac{1+\xi}{1-\xi}\right) + \log(e^{2\xi L}) = 2\log\left(\frac{1+\xi}{1-\xi}\right) + 2\xi L. \qquad \square$$

For $\phi \in \mathcal{F}^+$, define the seminorm

$$V(\phi) := \sup_{d_\theta(x,y)\leq\theta, x\neq y}\frac{|\phi(x)-\phi(y)|}{d_\theta(x,y)},$$

and set $\|\phi\|_L := \max(\|\phi\|_\infty, (1/2L)V(\phi))$. The next lemma gives the essential properties of the norm $\|\cdot\|_L$.

LEMMA A.5. *The norm $\|\cdot\|_L$ is equivalent to $\|\cdot\|_\theta$, and for all $\phi, \psi \in \mathcal{F}^+$, we have that $-\phi_2 \leq_{\mathcal{C}_L} \phi_1 \leq_{\mathcal{C}_L} \phi_2$ implies that $\|\phi_1\|_L \leq \|\phi_2\|_L$.*

Proof. Given $\phi \in \mathcal{C}_L$, $\varepsilon > 0$, and $x, y \in \Sigma^+$ such that $d_\theta(x, y) \leq \theta$, we have

$$|\phi(x) - \phi(y)| = |e^{(\phi(x)+\varepsilon)} - e^{(\phi(y)+\varepsilon)}| \leq (\|\phi\|_\infty + \varepsilon)\left|\frac{\phi(x)+\varepsilon}{\phi(y)+\varepsilon}\right| \leq (\|\phi\|_\infty + \varepsilon)Ld_\theta(x, y).$$

Letting $\varepsilon \to 0$ and taking the supremum, we get the estimate $V(\phi) \leq \|\phi\|_\infty$. This gives us that $\|\phi\|_L \leq C\|\phi\|\theta$ for some $C > 0$. Likewise, it is easy to see that $|\phi|_\theta \leq 2\|\phi\|_L$, so we have that the norms are equivalent.

Now suppose that $-\phi_2 \leq_L \phi_1 \leq_L \phi_2$. Then, $\phi_2 - \phi_1 \geq 0$ and $\phi_2 + \phi_1 \geq 0$. In other words, for every $x \in \Sigma^+$, $-\phi_2(x) \leq \phi_1(x) \leq \phi_2(x)$, which implies that $\|\phi_1\|_\infty \leq \|\phi_2\|_\infty$. To prove that $\|\phi_1\|_L \leq \|\phi_2\|_L$, it suffices to prove that $V(\phi_1) \leq 2L\|\phi_2\|_\infty$. We have

$$V(\phi_1) = V\left(\frac{\phi_1-\phi_2}{2} - \frac{\phi_1+\phi_2}{2}\right) \leq \frac{1}{2}(V(\phi_2-\phi_1) + V(\phi_2+\phi_1))$$
$$\leq \frac{L}{2}(\|\phi_2-\phi_1\|_\infty + \|\phi_2+\phi_1\|_\infty) \leq 2L\|\phi_2\|_\infty. \qquad \square$$

Observe that for $\phi \in \mathcal{F}^+$, $\phi \geq 0$, and $\alpha = |\phi|_\theta/L > 0$. we have for all $x, y \in \Sigma^+$,

$$\frac{\phi(x)+\alpha}{\phi(y)+\alpha} = e^{\log(\phi(x)+\alpha)-\log(\phi(y)+\alpha)} \leq e^{(|\phi|_\theta/\alpha)d_\theta(x,y)} = e^{Ld_\theta(x,y)}.$$

Hence, $\phi + \alpha \in \mathcal{C}_L$.

Proof of Lemma 3.1. Note that for every $f \in \mathcal{W}$, if we take $\theta > 1/\gamma$, we have $|\pi_f|_\theta \leq 1$. Thus, for uniformity in Lemma 3.3, we need to take

$$L \geq \frac{\theta M}{\xi - \theta},$$

where $M := \max\{|\log(f')|_{C^1} \mid f \in \mathcal{W}\}$. Let h_f be such that $\mathcal{L}_{\psi_f}h_f = e^{P(\psi_f)}h_f$, and let ν_f be the corresponding measure such that $\mathcal{L}^*_{\psi_f}\nu_f = e^{P(\psi_f)}\nu_f$. Note that for all $f \in \mathcal{W}$, $P(\psi_f) = 0$. It can be shown that $h_f \in \mathcal{C}_L$ for L taken as above. Hence, for any $x, y \in \Sigma^+$,

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we have

$$h_f(x) = \mathcal{L}_{\psi_f} h_f(x) = \sum_{i=1}^{k} e^{\psi_f(ix)} h_f(ix)$$
$$\leq e^{\theta(L+|\psi_f|_\theta) d_\theta(x,y)} \sum_{i=1}^{k} e^{\psi_f(iy)} h_f(iy) \leq e^{\theta(L+M) d_\theta(x,y)} h_f(y),$$

and hence $|\log(h_f)|_\theta \leq \theta(L+M)$, and so $|\overline{\psi}_f|_\theta \leq M + \theta(L+M) + L + M = (2+\theta)M + (1+\theta)L$. Thus, for the normalized operators $\mathcal{L}_{\overline{\psi}_f}$, we take

$$L_0 \geq \frac{\theta((2+\theta)M + (1+\theta)L)}{\xi - \theta}.$$

Moreover, one can show that $-\phi \leq_{\mathcal{C}_L} \psi \leq_{\mathcal{C}_L} \phi$ implies that

$$\int \psi \, d\mu_f \leq \int \phi \, d\mu_f,$$

where μ_f is the equilibrium state corresponding to $\overline{\psi}_f$.

Observe that for every $n \in \mathbb{N}$,

$$\int \mathcal{L}_{\overline{\psi}_f} \phi \, d\mu_f = \int \phi \, d\mu_f.$$

Therefore, we may apply Lemma A.2 with $\|\cdot\|_1 = \|\cdot\|_{L^1}$, $\|\cdot\|_2 = \|\cdot\|_L$, $\phi = \mathcal{L}^n_{\overline{\psi}_f}\phi$, and $\psi = \int \phi \, d\mu_f = \mathcal{L}^n_{\overline{\psi}_f}(\int \phi \, d\mu_f)$:

$$\left\| \mathcal{L}^n_{\overline{\psi}_f}\phi - \int \phi \, d\mu_f \right\|_L \leq \left(e^{\Theta_L(\mathcal{L}^n_{\overline{\psi}_f}\phi, \mathcal{L}^n_{\overline{\psi}_f}(\int \phi \, d\mu_f))} - 1 \right) \left\| \int \phi \, d\mu_f \right\|_L$$
$$\leq \left(e^{\Theta_L(\mathcal{L}^n_{\overline{\psi}_f}\phi, \mathcal{L}^n_{\overline{\psi}_f}(\int \phi \, d\mu_f))} - 1 \right) \|\phi\|_L.$$

Let $\Delta = \mathrm{diam}_{\Theta_L}(\mathcal{L}_{\overline{\psi}_f}(\mathcal{C}_L))$, and observe that Birkhoff's inequality implies that for $\phi \in \mathcal{C}_L$,

$$\Theta_L\left(\mathcal{L}^n_{\overline{\psi}_f}\phi, \mathcal{L}^n_{\overline{\psi}_f}\left(\int \phi \, d\mu_f \right) \right) \leq \left(\tanh\left(\frac{\Delta}{4} \right) \right)^{n-1} \Delta \leq \Delta \tau_{\mathcal{W}}^{n-1}$$

for uniform $\tau_{\mathcal{W}}$. Therefore, for $\phi \in \mathcal{C}_L$, we have

$$\left\| \mathcal{L}^n_{\overline{\psi}_f}\phi - \int \phi \, d\mu_f \right\|_L \leq \left(\sum_{j=1}^{\infty} \frac{(\Delta \tau_{\mathcal{W}}^{n-1})^j}{j!} \right) \|\phi\|_L = C_{\mathcal{W}} \tau_{\mathcal{W}}^n \|\phi\|_L,$$

where $C_{\mathcal{W}}$ is uniform. Since the norms are equivalent, we may replace $\|\cdot\|_L$ by $\|\cdot\|_\theta$. It remains to extend this bound to all $\phi \in \mathcal{F}_\theta^+$. If $\phi \geq 0$, then $\phi + |\phi|_\theta / L \in \mathcal{C}_L$, so

$$\left\| \mathcal{L}^n_{\overline{\psi}_f}\phi - \int \phi \, d\mu_f \right\|_\theta = \left\| \mathcal{L}^n_{\overline{\psi}_f}\left(\phi + \frac{|\phi|_\theta}{L} \right) - \int \left(\phi + \frac{|\phi|_\theta}{L} \right) d\mu_f \right\|_\theta$$
$$\leq C_{\mathcal{W}} \tau_{\mathcal{W}}^n \left(\|\phi\|_\theta + \frac{|\phi|_\theta}{L} \right) \leq C_{\mathcal{W}} \tau_{\mathcal{W}}^n \|\phi\|_\theta$$

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for a different (but still uniform) $C_{\mathcal{W}}$. For general real-valued $\phi \in \mathcal{F}_\theta^+$, we decompose ϕ as $\phi = \phi^+ - \phi^-$, where $\phi^+, \phi^- \geq 0$. Then,

$$\left\| \mathcal{L}_{\psi_f}^n \phi - \int \phi \, d\mu_f \right\|_\theta \leq \left\| \mathcal{L}_{\psi_f}^n \phi^+ - \int \phi^+ \, d\mu_f \right\|_\theta + \left\| \mathcal{L}_{\psi_f}^n \phi^- \int \phi^- \, d\mu_f \right\|_\theta$$
$$\leq C_{\mathcal{W}} \tau_{\mathcal{W}}^n (\|\phi^+\|_\theta + \|\phi^-\|_\theta) \leq 2C_{\mathcal{W}} \tau_{\mathcal{W}}^n \|\phi\|_\theta.$$

The case of complex-valued ϕ is handled similarly. □

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