Canad. Math. Bull. Vol. **39** (2), 1996 pp. 219–226

SLICE MAPS AND MULTIPLIERS OF INVARIANT SUBSPACES

TAKAHIKO NAKAZI

ABSTRACT. Let $\overline{D^2}$ be the closed bidisc and T^2 be its distinguished boundary. For $(\alpha,\beta) \in \overline{D^2}$, let $\Phi_{\alpha\beta}$ be a slice map, that is, $(\Phi_{\alpha\beta}f)(\lambda) = f(\alpha\lambda, \beta\lambda)$ for $\lambda \in D$ and $f \in H^2(D^2)$. Then $\ker\Phi_{\alpha\beta}$ is an invariant subspace, and it is not difficult to describe $\ker\Phi_{\alpha\beta}$ and $\mathcal{M}(\ker\Phi_{\alpha\beta}) = \{\phi \in L^\infty(T^2) : \phi\ker\Phi_{\alpha\beta} \subset H^2(D^2)\}$. In this paper, we study the set $\mathcal{M}(M)$ of all multipliers for an invariant subspace M such that the common zero set of M contains that of $\ker\Phi_{\alpha\beta}$.

1. **Introduction.** Let D^2 be the open unit disc in $\mathbf{C}^2$ and T^2 be its distinguished boundary. Normalized Lebesgue measure on T^2 is denoted by dm. For $1 \leq p \leq \infty$, $H^p(D^2)$ is the Hardy space and $L^p(T^2)$ is the Lebesgue space on T^2. Let $N(D^2)$ denote the Nevanlinna class. Each f in $N(D^2)$ has radial limits f^* defined on T^2 a.e. Moreover, there is a singular measure $d\sigma_f$ on T^2 determined by f such that the least harmonic majorant $u(\log|f|)$ of $\log|f|$ is given by $u(\log|f|)(\zeta) = P_\zeta(\log|f^*|+d\sigma_f)$ where P_ζ denotes Poisson integration and $\zeta = (z,w) \in D^2$. Put $N_*(D^2) = \{f \in N(D^2)\ ;\ d\sigma_f \leq 0\}$; then $H^p(D^2) \subset N_*(D^2) \subset N(D^2)$ and $H^p(D^2) = N_*(D^2)\cap L^p(T^2) \subset N(D^2)\cap L^p(T^2)$. These facts are shown in [6, Theorem 3.3.5].

A closed subspace M of $H^2(D^2)$ is said to be *invariant* if $zM \subset M$ and $wM \subset M$. For an invariant subspace M of $H^2(D^2)$, set

$$\mathcal{M}(M) = \{\phi \in L^\infty(T^2)\ ;\ \phi M \subseteq H^2(D^2)\}.$$

If $M = qH^2(D^2)$ for some inner function q, it is trivial to see $\mathcal{M}(M) = \bar{q}H^\infty(D^2)$. In the case of one variable, an arbitrary invariant subspace M has the form $qH^2(D)$ for some inner function q by the famous Beurling theorem [1]. Hence $\mathcal{M}(M) = \bar{q}H^\infty(D)$. Hence the map : $M \longrightarrow \mathcal{M}(M)$ is one-to-one. However this result for invariant subspaces of $H^2(D^2)$ is not true. The author [4] studied the relation between M and $\mathcal{M}(M)$. To study $\mathcal{M}(M)$, R. G. Douglas and K. Yan [2] introduced the common zero set $Z(M)$ and the singular measure $Z_\partial(M)$, that is,

$$Z(M) = \{\zeta \in D^2\ ;\ f(\zeta) = 0 \text{ for } f \in M\}$$

and

$$Z_\partial(M) = \inf\{-d\sigma_f\ ;\ f \in M, f \neq 0\}.$$

This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.
Received by the editors February 27, 1995; revised November 14, 1995.
AMS subject classification: 47A15, 32A35.
Key words and phrases: Hardy space, several variables, invariant subspace, slice map.

They showed that if the real 2-dimensional Hausdorff measure of $Z(M)$ is zero and $Z_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^2)$. In this paper, we are interested in invariant subspaces M of $H^2(D^2)$ such that the real 2-dimensional Hausdorff measure of $Z(M)$ is positive and $Z_\partial(M) = 0$.

Fix $(\alpha, \beta) \in \overline{D^2}$. For f in $H^p(D^2)$,

$$(\Phi^p_{\alpha\beta} f)(\lambda) = f(\alpha\lambda, \beta\lambda) \qquad (\lambda \in D).$$

$\Phi^p_{\alpha\beta}$ is called a *slice map*. $\Phi^2_{\alpha\beta}$ maps $H^2(D^2)$ into $L^2_a(D)$, where $L^2_a(D)$ is the Bergman space (*cf.* [6, p. 53]). In this paper, we study the kernel $\ker \Phi^p_{\alpha\beta}$ and the range $\operatorname{ran} \Phi^p_{\alpha\beta}$ for $p = 2, \infty$. $\ker \Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$ and the closure of $\operatorname{ran} \Phi^2_{\alpha\beta}$ is an invariant subspace of $L^2_a(D)$. Put

$$D_{\alpha\beta} = \{(\alpha\lambda, \beta\lambda) \in D^2 \,;\, \lambda \in \mathbf{C}\} \,;$$

then $Z(\ker \Phi^2_{\alpha\beta}) = \mathcal{D}_{\alpha\beta}$ if $(\alpha, \beta) \in T^2 \cup T \times D \cup D \times T$. The 2-dimensional Hausdorff measure of $Z(\ker \Phi^2_{\alpha\beta})$ is positive and $Z_\partial(\ker \Phi^2_{\alpha\beta}) = 0$. In this paper, we show $\mathcal{M}(M) = H^\infty(D^2)$ when $Z(M) = \mathcal{D}_{\alpha\beta}$ for some $(\alpha, \beta) \in T^2$ and $Z_\partial(M) = 0$ and M satisfies some additional natural condition. The main result in this paper is Theorem 4 in Section 3. Theorem 1 of [2] has a lot of corollaries on the rigidity of invariant subspaces. Similarly Theorem 3 in this paper has such corollaries. Hence our results can be seen as the generalizations of results of R. G. Douglas and K. Yan.

For f in $N(D^2)$, $f(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f and F_j is a polynomial which is homogeneous of degree j. The smallest $j = j(f)$ such that F_j is not the zero-polynomial is called the *order of the zero* which f has at $\zeta = (0,0)$. For $p \in D^2$, the order of the zero of f at p is simply the order of the zero of $f(p + \zeta)$ at $\zeta = (0,0)$. We will write $f_p(\zeta) = f(p + \zeta)$.

2. **Slice maps.**

In this section, we study the slice map $\Phi_{\alpha\beta} = \Phi^p_{\alpha\beta}$ for $(\alpha, \beta) \in \overline{D^2}$.

PROPOSITION 1. *Let $(\alpha, \beta) \in \overline{D^2}$.*
(1) $\Phi^2_{\alpha\beta}$ is a contractive map from $H^2(D^2)$ to $L^2_a(D)$.
(2) If $(\alpha, \beta) \in D^2$, then $\operatorname{ran} \Phi^2_{\alpha\beta}$ *is a subset of analytic functions on $\bar{D}$.*
(3) If $(\alpha, \beta) \in T^2$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $L^2_a(D)$ with $\|\Phi^2_{\alpha\beta}\| = 1$.
(4) If $(\alpha, \beta) \in T \times D \cup D \times T$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $H^2(D)$ with $\|\Phi^2_{\alpha\beta}\| \le (1 - |\beta|^2)^{-1}$.

PROOF. (1) For $f \in H^2(D^2)$, let $f(z, w) = \sum_{j=0}^\infty F_j(z, w)$ be a homogeneous expansion of f. Then $F_j(z, w) = \sum_{\ell=0}^j a_\ell z^{j-\ell} w^\ell$ and $\int |F_j|^2\, dm = \sum_{\ell=0}^j |a_\ell|^2$. Moreover

$$\int |f|^2\, dm = \sum_{j=0}^\infty \int |F_j|^2\, dm = \sum_{j=0}^\infty \sum_{\ell=0}^j |a_\ell|^2 < \infty.$$

$(\Phi_{\alpha\beta} f)(\lambda) = \sum_{j=0}^\infty F_j(\alpha, \beta)\lambda^j$ and

$$|F_j(\alpha, \beta)|^2 \le \Big(\sum_{\ell=0}^j |a_\ell|^2\Big)\Big(\sum_{\ell=0}^j |\beta|^{2\ell}\Big) \le (j+1)\Big(\sum_{\ell=0}^j |a_\ell|^2\Big).$$

Hence

$$\int_0^1 \int_0^{2\pi} |\Phi_{\alpha\beta} f|^2(re^{i\theta}) r\, d\theta\, dr/\pi = \int_0^1 \sum_{j=0}^{\infty} |F_j(\alpha, \beta)|^2 r^{2j+1} 2\, dr$$
$$= \sum_{j=0}^{\infty} |F_j(\alpha, \beta)|^2 \frac{1}{j+1} \le \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} |a_\ell|^2$$
$$= \int |f|^2\, dm.$$

Thus $\Phi_{\alpha\beta} f \in L_a^2(D)$ and $\|\Phi_{\alpha\beta}\| \le 1$.

(2) is clear. (3): For $g \in L_a^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} \frac{b_j}{j+1} (\bar{\beta} w)^{j-\ell} (\bar{\alpha} z)^\ell$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (3). (4): We may assume $(\alpha, \beta) \in T \times D$. Then

$$|F_j(\alpha, \beta)|^2 \le (1 - |\beta|^2)^{-1} \sum_{\ell=0}^{j} |a_\ell|^2$$

and hence

$$\int_0^{2\pi} |\Phi_{\alpha\beta} f|^2(re^{i\theta})\, d\theta/2\pi \le \sum_{j=0}^{\infty} \frac{1}{1 - |\beta|^2} \sum_{\ell=0}^{j} |a_\ell|^2 \le \frac{1}{1 - |\beta|^2} \int |f|^2\, dm.$$

For $g \in H^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This implies (4).

(3) of Proposition 1 is essentially known (see [6, p. 53]). Now we study the slice map $\Phi_{\alpha\beta}^{\infty}$ on $H^\infty(D^2)$. Let L be the norm closed linear span of $\overline{H^\infty(D^2)} H^\infty(D^2)$ in $L^\infty(T^2)$. Then $L \ne L^\infty(T^2)$ (see [5]).

PROPOSITION 2. *Let* $(\alpha, \beta) \in \overline{D^2}$.

(1) $\Phi_{\alpha\beta}^{\infty}$ *is a contractive homomorphism from* $H^\infty(D^2)$ *to* $H^\infty(D)$.

(2) If $(\alpha, \beta) \in T^2 \cup T \times D \cup D \times T$, *then* $\Phi_{\alpha\beta}^{\infty}$ *is a contractive homomorphism from* $H^\infty(D^2)$ *onto* $H^\infty(D)$.

(3) If $(\alpha, \beta) \in T^2$, *there exists a contractive* $*$*-homomorphism* $\tilde{\Phi}_{\alpha\beta}^{\infty}$ *from L onto* $L^\infty(T)$ *such that* $\tilde{\Phi}_{\alpha\beta}^{\infty} \mid H^\infty(T^2) = \Phi_{\alpha\beta}^{\infty} \mid H^\infty(T^2)$.

PROOF. (1) is clear. (2): If $g \in H^\infty(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$ and $|\alpha| = 1$, then $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j \in H^\infty(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (2).

(3): For $f_j, g_j \in H^\infty(D^2)$ and $j = 1, \ldots, n$, put

$$\left\{ \tilde{\Phi}_{\alpha\beta} \left(\sum_{j=1}^{n} f_j \bar{g}_j \right) \right\}(\lambda) = \sum_{j=1}^{n} f_j(\alpha\lambda, \beta\lambda) \overline{g_j(\alpha\lambda, \beta\lambda)}$$

for $\lambda \in D$, then $\tilde{\Phi}_{\alpha\beta}(\sum_{j=1}^{n} f_j \bar{g}_j)$ can be seen as an element in $L^\infty(T)$ by its radial limits.

Hence for a.e. $\lambda \in T$

$$\begin{aligned}\left|\Phi_{\alpha\beta}\left(\sum_{j=1}^{n} f_j\bar{g}_j\right)(\lambda)\right| &\le \operatorname*{ess\,sup}_{\lambda\in T}\left|\sum_{j=1}^{n} f_j(\alpha\lambda,\beta\lambda)\overline{g_j(\alpha\lambda,\beta\lambda)}\right| \\ &\le \operatorname*{ess\,sup}_{(z,w)\in T^2}\left|\sum_{j=1}^{n} f_j(\alpha z,\beta w)\overline{g_j(\alpha z,\beta w)}\right| \\ &= \left\|\sum_{j=1}^{n} f_j\bar{g}_j\right\|_\infty\end{aligned}$$

because $(\alpha,\beta) \in T^2$. Then $\tilde{\Phi}_{\alpha\beta}$ is the extension of $\Phi_{\alpha\beta}$ from $H^\infty(D^2)$ to L, and then $\tilde{\Phi}_{\alpha\beta}$ is a contractive $*$-homomorphism from L to $L^\infty(T)$. If $U(\lambda) = \sum_{j=1}^{n} F_j(\lambda)\bar{G}_j(\lambda)$ a.e. on T where $F_j,\ G_j \in H^\infty(D)$, then $u(z,w) = \sum_{j=1}^{n} F_j(\bar{\alpha}z)\bar{G}_j(\beta w)$ belongs to L and $(\tilde{\Phi}_{\alpha\beta}u)(\lambda) = U(\lambda)$ a.e. on T. Since arbitrary function U in $L^\infty(T)$ can be approximated by such functions, $\tilde{\Phi}_{\alpha\beta}$ is onto.

The following lemma will be used in the proofs in the following proposition and the main theorem. We can prove it by an approximation method as in [4] but we prove it using Proposition 2.

LEMMA. *If $\phi \in L^\infty(T^2)$, $(\alpha,\beta) \in T^2$ and $\phi(z,w)(\beta z - \alpha w) \in H^\infty(D^2)$, then $\phi \in H^\infty(D^2)$.*

PROOF. Note that $\beta z - \alpha w \in \ker\Phi_{\alpha\beta}$. If $\phi(\beta z - \alpha w) = g$ for some $g \in H^\infty(D^2)$, then g belongs to $\ker\Phi_{\alpha\beta}$. In fact, $\hat{\phi}(\beta z - \alpha w)^\wedge = \hat{g}$ on $\operatorname{Spec} L^\infty(T^2)$ which is the maximal ideal space of $L^\infty(T^2)$ and $(\beta z - \alpha w)^\wedge = 0$ on $\operatorname{hull}(\ker\tilde{\Phi}_{\alpha\beta})$. Hence $\hat{g} = 0$ on $\operatorname{hull}(\ker\tilde{\Phi}_{\alpha\beta}) \cap \operatorname{Spec} L^\infty(T^2)$. Since L is a commutative C^*-algebra, every element of $\operatorname{Spec} L$ extends to an element of $\operatorname{Spec} L^\infty(T^2)$. Therefore $\hat{g} = 0$ on $\operatorname{hull}(\ker\tilde{\Phi}_{\alpha\beta})$. Thus $g \in (\ker\tilde{\Phi}_{\alpha\beta}) \cap H^\infty(D^2) = \ker\Phi_{\alpha\beta}$. Hence if $g = \sum_{j=0}^{\infty} G_j$ and $G_j(z,w) = \sum_{\ell=0}^{j} b_\ell z^{j-\ell}w^\ell$, then

$$G_j(z,w) = z^j\sum_{\ell=0}^{j} b_\ell(\bar{z}w)^\ell = k\prod_{\ell=1}^{j}(w - k_\ell z)$$

where $k \in \mathbf{C}$ and $k_\ell \in \mathbf{C}$ for $1 \le \ell \le j$ and $G_j(\alpha\lambda,\beta\lambda) \equiv 0$ for $\lambda \in D$ because $g \in \ker\Phi^2_{\alpha\beta}$. Thus $G_j(z,w) = m(\beta z - \alpha w)\prod_{\ell=2}^{j}(w - m_\ell z)$ where $m \in \mathbf{C}$ and $m_\ell \in \mathbf{C}$ for $2 \le \ell \le j$ and hence $g/(\beta z - \alpha w)$ is analytic on D^2. Since $d\sigma_{\beta z - \alpha w} = 0$, $g/(\beta z - \alpha w) \in N_*(D^2) \cap L^\infty(T^2) = H^\infty(D^2)$ and hence ϕ belongs to $H^\infty(D^2)$.

PROPOSITION 3. *Let $(\alpha,\beta) \in \overline{D^2}$.*

(1) For any $r \in (0,1]$, $\ker\Phi^2_{\alpha\beta} = \ker\Phi^2_{r\alpha,r\beta}$.

(2) $\ker\Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$,

$$Z(\ker\Phi^2_{\alpha\beta}) = \mathcal{D}_{\alpha\beta} \text{ and } Z_\partial(\ker\Phi^2_{\alpha\beta}) = 0.$$

For any $p \in \mathcal{D}_{\alpha\beta}$, $\beta z - \alpha w \in \ker\Phi^2_{\alpha\beta}$ has a zero of order 1 at p.

(3) If $(\alpha,\beta) \in T^2$, then $(\beta z - \alpha w)H^2(D^2)$ is dense in $\ker\Phi^2_{\alpha\beta}$ but $\ker\Phi^2_{\alpha\beta} \neq (\beta z - \alpha w)H^2(D^2)$. If $(\alpha,\beta) \in T \times D \cup D \times T$, then $\ker\Phi^2_{\alpha\beta} = (\beta z - \alpha w)H^2(D^2)$.

(4) If $(\alpha,\beta) \in T^2$, then $\mathcal{M}(\ker \Phi^2_{\alpha\beta}) = H^\infty(D^2)$ and if $(\alpha,\beta) \in T \times D \cup D \times T$, then $\mathcal{M}(\ker \Phi^2_{\alpha\beta}) = (\beta z - \alpha w)^{-1} H^\infty(D^2)$.

(5) If $(\alpha, 0) \in \bar{D} \times D$ and $\alpha \neq 0$, then $\ker \Phi^2_{\alpha 0} = wH^2(D^2)$ and hence $\mathcal{M}(\ker \Phi^2_{\alpha 0}) = w^{-1} H^\infty(D^2)$.

(6) Let M be an invariant subspace of $H^2(D^2)$ with $\ker \Phi^2_{\alpha\beta} \subsetneq M$, $\mathcal{M}(M) = H^\infty(D^2)$. If $(\alpha,\beta) \in T^2$, then $Z(M) = \{(\alpha a_j, \beta a_j) \in D^2 \; ; \; \sum_{j=0}^\infty (1 - |a_j|) \times [-\log(1 - |a_j|)]^{1-\varepsilon} < \infty$ for all $\varepsilon > 0\}$. If $(\alpha,\beta) \in T \times D \cup D \times T$, then $Z(M) = \{(\alpha a_j, \beta a_j) \in D^2 \; ; \; \sum_{j=1}^\infty (1 - |a_j|) < \infty\}$. If $(\alpha, 0) \in \bar{D} \times D$ and $\alpha \neq 0$, then $M = qH^2(D) \oplus wH^2(D^2)$ where q is a one variable inner function with $q = q(z)$ and hence $Z(M) = \{(s, 0) \in D^2 \; ; \; q(s) = 0$ and $s \in D\}$.

PROOF. (1) and (2) are clear. (3): Let $(\alpha,\beta) \in T^2$. If $f \in \ker \Phi_{\alpha\beta}, f = \sum_{j=0}^\infty F_j$ and $F_j(z,w) = \sum_{\ell=0}^j a_\ell z^{j-\ell} w^\ell$, then $F_j(z,w) = c(\beta z - \alpha w) \prod_{\ell=2}^j (w - c_\ell z)$ and hence f can be approximated by the functions in $(\beta z - \alpha w) H^2(D^2)$. This implies that $(\beta z - \alpha w) H^2(D^2)$ is dense in $\ker \Phi^2_{\alpha\beta}$. Suppose $\ker \Phi_{\alpha\beta} = (\beta z - \alpha w) H^2(D^2)$; then the multiplication operator by $\beta z - \alpha w$ is a left invertible operator from $H^2(D^2)$ to $\ker \Phi_{\alpha\beta}$. Hence there exists a positive constant ε such that

$$\int_{T^2} |g|^2 |\beta z - \alpha w|^2 \, dm \geq \varepsilon \int_{T^2} |g|^2 \, dm$$

for all $g \in H^\infty(D^2)$ and so

$$\int_{T^2} u |\beta z - \alpha w|^2 \, dm \geq \varepsilon \int_{T^2} u \, dm$$

for all nonnegative continuous functions u on T^2. Thus $|\beta z - \alpha w|^2 \geq \varepsilon > 0$ a.e. on T^2 and this contradiction implies that $\ker \Phi_{\alpha\beta} \neq (\beta z - \alpha w) H^2(D^2)$. Let $(\alpha,\beta) \in T \times D \cup D \times T$. Since $\beta z - \alpha w$ is invertible in L^∞, $\ker \Phi_{\alpha\beta} = (\beta z - \alpha w) H^2(D^2)$ because $(\beta z - \alpha w) H^2(D^2)$ is dense in $\ker \Phi_{\alpha\beta}$.

(4): Let $(\alpha,\beta) \in T^2$. If $\phi \in \mathcal{M}(\ker \Phi_{\alpha\beta})$, then $\phi(\beta z - \alpha w) = g$ for some $g \in H^\infty(D^2)$. By the Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(\ker \Phi_{\alpha\beta}) = H^\infty(D^2)$. Let $(\alpha,\beta) \in T \times D \cup D \times T$ and $\phi \in \mathcal{M}(\ker \Phi_{\alpha\beta})$. By (3), $\ker \Phi_{\alpha\beta} = (\beta z - \alpha w) H^2(D^2)$ and hence

$$\phi(\beta z - \alpha w) H^2(D^2) \subset H^2(D^2).$$

This implies that $\phi(\beta z - \alpha w) \in H^\infty(D^2)$ and so $\mathcal{M}(\ker \Phi_{\alpha\beta}) = (\beta z - \alpha w)^{-1} H^\infty(D^2)$. (5) is easy to see. (6): If $(\alpha,\beta) \in T^2$ and $\ker \Phi_{\alpha\beta} \subsetneq M$, then by (3) of Proposition 1 and [3, Corollary 3.6], $Z([\Phi_{\alpha\beta} M]_2) = \{a_j \in D \; ; \; \sum_{j=1}^\infty (1 - |a_j|)[-\log(1 - |a_j|)]^{1-\varepsilon} < \infty$ for all $\varepsilon > 0\}$. This and Theorem 1 in [2] imply the first part. For $(\alpha,\beta) \in T \times D \cup D \times T$, we can show similarly by (4) of Proposition 1.

3. **Multipliers.** By (4) of Proposition 3, we know the set of all multipliers $\mathcal{M}(M)$ of an invariant subspace such that $\ker \Phi^2_{\alpha\beta} \subseteq M \subseteq H^2(D^2)$ when $(\alpha,\beta) \in T^2 \cup T \times D \cup D \times T$ or $(\alpha, 0) \in D \times D \setminus (0,0)$. When $(\alpha,\beta) \in D \times D$ and $|\alpha| = |\beta| > 0$, there exists $(\alpha_0, \beta_0) \in T \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. When $(\alpha,\beta) \in D \times D$ and $0 \leq |\alpha| < |\beta|$, there exists $(\alpha_0,\beta_0) \in D \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. (1) of Proposition 3 implies $\ker \Phi^2_{\alpha\beta} = \ker \Phi^2_{\alpha_0\beta_0}$. Hence for arbitrary $(\alpha, \beta) \in \bar{D} \times \bar{D} \setminus (0,0)$, we can describe $\mathcal{M}(M)$ by Proposition 3. In this section, we study $\mathcal{M}(M)$ without such a condition. In this section, for example, we study $\mathcal{M}(M)$ of an invariant subspace such that $M \subseteq \ker \Phi^2_{\alpha\beta}$. In fact, we study such a problem more generally, that is, when the 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}^c_{\alpha\beta}$ is zero. For $\Lambda \subset T^2 \cup T \times D \cup D \times T$, put

$$\mathcal{D}_\Lambda = \{\cup \mathcal{D}_{\alpha\beta}\ ;\ (\alpha,\beta) \in \Lambda\} \setminus \{(0,0)\}.$$

Note that if $Z(M) \supseteq \mathcal{D}_\Lambda$ and Λ is an infinite set such that $\mathcal{D}_{\alpha\beta} \cap \mathcal{D}_{\gamma\delta} = \{(0,0)\}$ when $(\alpha, \beta) \neq (\gamma, \delta)$, then $M = \{0\}$.

THEOREM 4. *Let Λ be a finite set of T^2. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1)–(3), then $\mathcal{M}(M) = H^\infty(D^2)$.*

(1) For any $p \in Z(M) \cap \mathcal{D}_\Lambda$, there exists a function f in M such that f has a zero of order 1 at p.

(2) The 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}^c_\Lambda$ is zero.

(3) $Z_\partial(M) = 0$.

PROOF. Suppose $\phi \in \mathcal{M}(M)$. Fix $p \in Z(M) \cap \mathcal{D}_\Lambda$. By (1), let f be a function in M such that f has a zero of order 1 at p. Let $(\alpha, \beta) \in \Lambda$ with $p \in \mathcal{D}_{\alpha\beta}$. By definition of $\mathcal{M}(M)$, $\phi f = g$ for some $g \in H^2(D^2)$. Put $k(z,w) = \beta z - \alpha w$; then $k_p(\zeta) = k(\zeta + p) = k(\zeta)$ and $k_p(\zeta)\phi_p(\zeta) f_p(\zeta) = k(\zeta) g_p(\zeta)$. Suppose $f_p(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f_p. Since $1 = s(f_p)$, $F_1(0,w) = cw$ for $c \neq 0$. By the Weierstrass preparation theorem (*cf.* [6, Theorem 1.2.1]), there exists a polydisc Δ in $\mathbf{C}^2$, centered at $(0,0)$, such that

$$f_p(z,w) = W(z,w) h(z,w)$$

for $(z,w) \in \Delta$ where h is analytic in Δ, h has no zero in Δ, $W(z,w) = w + b_0(z)$ and b_0 is analytic in Δ with $b_0(0) = 0$. Since $f_p(\alpha\lambda, \beta\lambda) \equiv 0$ on D, $\beta\lambda + b_0(\alpha\lambda) = 0$ if $(\alpha\lambda, \beta\lambda) \in \Delta$ and hence $b_0(\alpha\lambda) = -\beta\lambda$. Thus $b_0(z) = -\frac{\beta}{\alpha} z$ and $W(z,w) = -\frac{1}{\alpha}(\beta z - \alpha w)$. Therefore $k_p\phi_p = kg_p/f_p$ is analytic in Δ and so $k\phi$ is analytic in $\Delta + p$, in a sense of R. G. Douglas and K. Yan [2]. Therefore

$$\prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)\phi(z,w)$$

is analytic in a neighborhood of $Z(M) \cap \mathcal{D}_\Lambda$.

If $p \notin Z(M)$, then there exists a function k in M such that k has no zeros in some polydisc Δ_p, centered at p. As in the proof above, $\phi(z,w)$ is analytic in Δ_p and hence ϕ is analytic in $D^2 \setminus Z(M)$. Thus $\prod(\beta z - \alpha w)\phi(z,w)$ is analytic in $D^2 \setminus Z(M) \cap \mathcal{D}^c_\Lambda$. By

(2), $Z(M) \cap \mathcal{D}_\Lambda^c$ is a removable singularity for analytic functions, and hence $\psi(z,w) = \prod(\beta z - \alpha w)\phi(z,w)$ is analytic in D^2. By the proof of [2, Theorem 1], $\psi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\psi \leq -Z_\partial(M)$ because $d\sigma_\phi = d\sigma_\psi$. By (3), $d\sigma_\psi = 0$ and hence $\psi \in H^\infty(D^2)$. By the Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(M) = H^\infty(D^2)$.

THEOREM 5. *Let Λ be a finite set of $T \times D \cup D \times T$. If M is an invariant subspace of $H^2(D^2)$ which $Z(M) \supseteq \mathcal{D}_\Lambda$ and satisfies the following (1)–(3), then*

$$\mathcal{M}(M) = \prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)^{-1} H^\infty(D^2).$$

(1) For any $p \in Z(M)$, there exists a function f in M such that f has a zero of order 1 at p.

(2) The 2-dimensional Hausdorff measure of $Z(M) \cap \mathcal{D}_\Lambda^c$ is zero.

(3) $Z_\partial(M) = 0$.

PROOF. By the proof of Theorem 4, if $\phi \in \mathcal{M}(M)$, then $\prod(\beta z - \alpha w)\phi(z,w) \in H^\infty(D^2)$ where (α,β) ranges over Λ. Hence $\phi \in \prod(\beta z - \alpha w)^{-1}H^\infty(D^2)$. Conversely if $\phi \in \prod(\beta z - \alpha w)^{-1}H^\infty(D^2)$ and $f \in M$, then $f = 0$ on $\mathcal{D}_\Lambda$; hence by the Weierstrass preparation theorem, $\prod(\beta z - \alpha w)^{-1}f(z,w)$ is analytic in D^2 and ϕ belongs to $\mathcal{M}(M)$.

4. **Two general cases and remarks.**

Let a and b be two functions in $H^\infty(D)$ with $\|a\|_\infty \leq 1$ and $\|b\|_\infty \leq 1$. For f in $H^p(D^2)$,

$$(\Phi_{ab}^p f)(\lambda) = f\big(a(\lambda), b(\lambda)\big) \qquad (\lambda \in D).$$

If $a(\lambda) = \alpha(\lambda)$ and $b(\lambda) = \beta\lambda$, then Φ_{ab}^p was called a slice map $\Phi_{\alpha\beta}^p$ in the previous sections. For an arbitrary pair a and b, we know only very trivial results. It is easy to see that Φ_{ab}^∞ maps $H^\infty(D^2)$ into $H^\infty(D)$. If $\|a\|_\infty < 1$ and $\|b\|_\infty < 1$, then Φ_{ab}^p maps $H^p(D^2)$ into $H^\infty(D)$. In general, $\ker \Phi_{ab}^2$ is still an invariant subspace of $H^2(D^2)$, and

$$Z(\ker \Phi_{ab}^2) \supseteq \mathcal{D}_{ab} = \big\{\big(a(\lambda), b(\lambda)\big) \in D^2 \; ; \; \lambda \in D\big\}.$$

The function $b(z) - a(w)$ may not belong to $\ker \Phi_{ab}^2$. If $a(\lambda) = \alpha\lambda$ and $b(\lambda) = \beta\lambda$, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$, and hence $b(z) - a(w)$ belongs to $\ker \Phi_{ab}^2$. If $a(\lambda) = \lambda$ and $b(\lambda)$ is an inner function, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$, and hence $b(z) - a(w) = b(z) - w$ belongs to $\ker \Phi_{ab}^2$. In this case, $Z_\partial(\ker \Phi_{ab}^2) = 0$. For any $p \in \mathcal{D}_{ab}$, $b(z) - w$ has a zero of order 1 at $p \in \mathcal{D}_{ab}$. If $\phi \in L^\infty(T^2)$ and $\big(b(z) - w\big)\phi(z,w) \in H^\infty(D^2)$, then $\phi \in H^\infty(D^2)$. This can be shown as in [4, Proposition 3 and Theorem 7]. This implies (4) of Proposition 3. The proof of the following theorem is almost parallel to that of Theorem 4.

THEOREM 6. *Let $a(\lambda) = \lambda$ and $b(\lambda)$ be an inner function. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1)–(3), then $\mathcal{M}(M) = H^\infty(D^2)$.*

(1) For any $p \in Z(M) \cap \mathcal{D}_{ab}$, there exists a function f in M such that f has a zero of order 1 at $p \in Z(M) \cap \mathcal{D}_{ab}$.

(2) *The* 2-*dimensional Hausdorff measure of* $\mathcal{Z}(M) \cap \mathcal{D}_{ab}^c$ *is zero.*
(3) $\mathcal{Z}_\partial(M) = 0$.

If $a(\lambda) = \lambda$ and $b(\lambda) = cq(\lambda)$ where c is a constant with $|c| < 1$ and q is an inner function, we can show a version of Theorem 5 as Theorem 6 which is that of Theorem 4.

Let D^n be the open unit polydisc in $\mathbf{C}^n$ and T^n be its distinguished boundary. Fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \overline{D^n}$. For f in $H^p(D^n)$

$$(\Phi_\alpha^p f)(\lambda) = f(\alpha_1\lambda, \dots, \alpha_n\lambda) \qquad (\lambda \in D).$$

(1), (2) and (3) of Proposition 1 can be proved for arbitrary n. If $\alpha_j \in T$ for some j with $1 \leq j \leq n$ and $\alpha_i \in D$ for all i with $1 \leq i \leq n$ and $i \neq j$, we can show that Φ_α^2 is an onto map from $H^2(D^n)$ to $H^2(D)$ with $\|\Phi_\alpha^2\| \leq \prod_{i \neq j}(1 - |\alpha_j|^2)^{-1}$. This is a generalization of (4) of Proposition 1. Similarly we can generalize Proposition 2. If $\phi \in L^\infty(T^n)$ and $(\alpha_i z_j - \alpha_j z_i)\phi(z_1, \dots, z_n) \in H^\infty(D^n)$ where $1 \leq i \neq j \leq n$ and $\alpha = (\alpha_i, \dots, \alpha_n) \in T^n$, then $\phi \in H^\infty(D^n)$. This also can be shown as in [4, Proposition 3 and Theorem 7]. $\ker \Phi_\alpha^2$ is an invariant subspace and a generalization of (1) and (2) of Proposition 3 is true. Suppose $n > 2$. If M is an invariant subspace of $H^2(D^n)$, $\mathcal{Z}(M) = \mathcal{D}_\alpha = \{(\alpha_1\lambda, \dots, \alpha_n\lambda)\ ;\ \lambda \in D\}$ for $\alpha \in T^n$ and $\mathcal{Z}_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$. For it is a result of R. G. Douglas and K. Yan [2, Theorem 1] because the real $2n-2$ dimensional Hausdorff measure of $\mathcal{Z}(M)$ is zero.

REMARK. (i): As in Theorem 1 of [2], Theorem 4 can be stated as the following: If M is an invariant subspace of $H^2(D^2)$ which satisfies (1) and (2), then $\phi \in \mathcal{M}(M)$ if and only if $\phi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\phi \leq \mathcal{Z}_\partial(M)$. (ii): By Lemma 7 in [2] and Theorem 4, if M and N are quasi-similar invariant subspaces of $H^2(D^2)$ and M satisfies (1)–(3) in Theorem 4, then $M \subseteq N$. This is a generalization of Theorem 2 in [2]. Similarly we can generalize Corollaries 9 and 12. (iii): Let M, N be invariant subspaces of $H^2(D^2)$ satisfying (a) the 2-dimensional Hausdorff measures of $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda^c$ and $\mathcal{Z}(N) \cap \mathcal{D}_\Lambda^c$ are zero. (b) $\mathcal{Z}_\partial(M) = \mathcal{Z}_\partial(N)$. (c) M and N satisfy the condition (1) in Theorem 4 about $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda$ and $\mathcal{Z}(N) \cap \mathcal{D}_\Lambda$. If M and N are quasi-similar, then $M = N$.

REFERENCES

1. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81**(1949), 239–255.
2. R. G. Douglas and K. Yan, *On the rigidity of Hardy submodules*, Integral Eq Op. Th. **13**(1990), 350–363.
3. C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41**(1974).
4. T. Nakazi, *Multipliers of invariant subspaces in the bidisc*, Proc. Edinburgh Math. Soc. **37**(1994), 193–199.
5. M. Range, *A small boundary for H^∞ on the polydisc*, Proc. Amer. Math. Soc. **32**(1972), 253–255.
6. W. Rudin, *Function Theory in Polydisks*, Benjamin, New York (1969).

Department of Mathematics
Hokkaido University
Sapporo 060
Japan