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Optimization of the principal eigenvalue of the Neumann Laplacian with indefinite weight and monotonicity of minimizers in cylinders

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Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open bounded connected set. We consider the indefinite weighted eigenvalue problem $-\Delta u = \lambda m u$ in Ω with $\lambda \in \mathbb{R}$, $m \in L^\infty(\Omega)$ and with homogeneous Neumann boundary conditions. We study weak* continuity, convexity and Gâteaux differentiability of the map $m \mapsto 1/\lambda_1(m)$, where $\lambda_1(m)$ is the principal eigenvalue. Then, denoting by $\mathcal{G}(m_0)$ the class of rearrangements of a fixed weight m_0, under the assumptions that m_0 is positive on a set of positive Lebesgue measure and $\int_\Omega m_0 \, dx < 0$, we prove the existence and a characterization of minimizers of $\lambda_1(m)$ and the non-existence of maximizers. Finally, we show that, if Ω is a cylinder, then every minimizer is monotone with respect to the direction of the generatrix. In the context of the population dynamics, this kind of problems arise from the question of determining the optimal spatial location of favourable and unfavourable habitats for a population to survive.

Keywords: indefinite weight; Neumann boundary conditions; optimization; principal eigenvalue monotonicity, population dynamics

2020 *Mathematics Subject Classification:* 47A75; 35J25; 35Q92

1. Introduction and main results

In this paper, we consider the weighted eigenvalue problem with homogenous Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda m u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary $\partial\Omega$, $m \in L^\infty(\Omega)$ changes sign in Ω, $\lambda \in \mathbb{R}$ and ν is the outward unit normal vector on $\partial\Omega$.

An eigenvalue λ of (1) is called *principal eigenvalue* if it admits a positive eigenfunction. Clearly, $\lambda = 0$ is a principal eigenvalue with positive constants as its eigenfunctions. Problem (1) has been studied in various papers (see, for example, [8, 9, 44]). In particular, it is known that there is a positive (respectively negative) principal eigenvalue if and only if $\int_\Omega m\,dx < 0$ (respectively $\int_\Omega m\,dx > 0$). For the sake of completeness and in order to maintain this paper self-contained, we prefer to give here (see §2) an independent proof of the result above. Moreover, we show that, under the previous hypothesis on m, there exists an increasing (respectively decreasing) sequence of positive (respectively negative) eigenvalues. The smallest positive eigenvalue is the principal eigenvalue, which will be denoted by $\lambda_1(m)$.

Problem (1) and its variants play a crucial role in studying nonlinear models from population dynamics (see [45]) and population genetics (see [26]). We illustrate in details the following model in population dynamics devised by Skellam [45]

$$\begin{cases} v_t = \Delta v + \gamma v[m(x) - v] & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(x,0) = v_0(x) \geq 0,\ v(x,0) \not\equiv 0 & \text{in } \overline{\Omega}. \end{cases} \tag{2}$$

In (2), $v(x,t)$ represents the density of a population inhabiting the region Ω at location x and time t (for that reason, only non-negative solutions of (2) are of interest), v_0 is the initial density and γ is a positive parameter. The function $m(x)$ represents the intrinsic local grow rate of the population, it is positive on favourable habitats and negative on unfavourable ones and it mathematically describes the available resources in the spatially heterogeneous environment Ω. The integral $\int_\Omega m\,dx$ can be interpreted as a measure of the total resources in Ω. The Neumann conditions in (2) are zero-flux boundary conditions: it means that no individuals cross the boundary of the habitat, i.e. the boundary acts as a barrier.

The model (2) can be also considered with Neumann boundary conditions replaced by the homogeneous Dirichlet or Robin conditions. In the first case, the environment Ω is surrounded by a completely inhospitable region, i.e. any individual reaching the boundary dies, while in the second some individuals reaching the boundary die and the others return to the interior of Ω. It is known (see [13, 15] and references therein) that (2) predicts persistence for the population if $\lambda_1(m) < \gamma$. As a consequence, determining the best spatial arrangement of favourable and unfavourable habitats for the survival, within a fixed class of environmental configurations, results in minimizing $\lambda_1(m)$ over the corresponding class of weights. Having information of this type could affect, for example, on the strategies to be adopted for the conservation of species with limited resources.

This kind of problem has been investigated by many other authors. The question of determining the optimal spatial arrangements of favourable and unfavourable habitats in Ω for the survival of the modelled population was first addressed by Cantrell and Cosner in [13, 14]. The authors considered the diffusive logistic equation (2) with homogeneous Dirichlet boundary conditions and when the weight m has fixed maximum, minimum and integral over Ω. The analogous problem with

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Neumann boundary conditions has been analysed by Lou and Yanagida in [36]. Berestycki et al. [5] investigated a model similar to (2) in the case of periodically fragmented environment ($\Omega = \mathbb{R}^N$ and $m(x)$ periodic), Roques and Hamel [42] proved the existence of a minimizer in the case of "bang-bang" configurations and also investigated the problem by using numerical computation, Jha and Porru [30], among other things, exhibited an example of symmetry breaking of the optimal arrangement of the local growth rate. Lamboley et al. [33] investigated model (2) with Robin boundary conditions. Mazari et al. [37] studied several shape optimization problems arising in population dynamics, we refer the reader to it for a review of current knowledge on the subject. We also mention Cadeddu et al. [12], which considered mixed boundary conditions, Ferreri and Verzini [25] which studied asymptotic properties for Dirichlet boundary conditions, Mazzoleni et al. [38] which considered a singular analysis for Neumann problems, Derlet et al. [22] and Cuccu et al. [19], that extended these type of results to the principal eigenvalue associated to the p-Laplacian operator for Neumann and Dirichlet boundary conditions respectively. Finally, Pellacci and Verzini [40] considered the fractional Laplacian operator and Dipierro et al. [23] a mixed local and nonlocal operator.

In order to present our work, we briefly give some notations and definitions here. We denote by $\lambda_k(m)$, $k \in \mathbb{N}$, the kth positive eigenvalue of problem (1) corresponding to the weight m (assuming $\int_\Omega m\,dx < 0$). We say that two Lebesgue measurable functions $f, g : \Omega \to \mathbb{R}$ are *equimeasurable* if the super-level sets $\{x \in \Omega : f(x) > t\}$ and $\{x \in \Omega : g(x) > t\}$ have the same measure for all $t \in \mathbb{R}$. For a fixed $f \in L^\infty(\Omega)$, we call the set $\mathcal{G}(f) = \{g : \Omega \to \mathbb{R} : g \text{ is measurable and } g \textit{ and } f \text{ are equimeasurable}\}$ the *class of rearrangements of* f (see Appendix A). Moreover, we introduce the set $L^\infty_<(\Omega) = \left\{m \in L^\infty(\Omega) : \int_\Omega m\,dx < 0\right\}$.

The present paper contains three main results. First, we study the dependence of $\lambda_k(m)$ on m, in particular we investigate continuity and, for $k = 1$, convexity and differentiability properties (see Lemmas 3, 4 and 5). Second, we examine the optimization of $\lambda_1(m)$ in the class of rearrangements $\mathcal{G}(m_0)$ of a fixed function $m_0 \in L^\infty_<(\Omega)$. Precisely, we prove the existence of minimizers, a characterization of them in terms of the eigenfunctions relative to $\lambda_1(m)$ and a non-existence result for the maximizers.

THEOREM 1. *Let $\lambda_1(m)$ be the principal eigenvalue of problem (1), $m_0 \in L^\infty_<(\Omega)$ such that the set $\{x \in \Omega : m_0(x) > 0\}$ has positive Lebesgue measure, $\mathcal{G}(m_0)$ the class of rearrangements of* m_0 *(see Definition 7) and $\overline{\mathcal{G}(m_0)}$ its weak* closure in $L^\infty(\Omega)$. Then*

(i) *the problem*

$$\min_{\substack{m \in \overline{\mathcal{G}(m_0)} \\ |\{m>0\}|>0}} \lambda_1(m) \tag{3}$$

admits solutions and any solution $\check{m}_1$ belongs to $\mathcal{G}(m_0)$;

(ii) *for every solution $\check{m}_1 \in \mathcal{G}(m_0)$ of (3), there exists an increasing function ψ such that*

$$\check{m}_1 = \psi(u_{\check{m}_1}) \quad \textit{a.e. in } \Omega, \tag{4}$$

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where $u_{\check{m}_1}$ *is the unique positive eigenfunction relative to* $\lambda_1(\check{m}_1)$ *normalized as in* (28)*;*

(iii)

$$\sup_{m\in\mathcal{G}(m_0)} \lambda_1(m) = +\infty.$$

We note that the class of weights usually considered in literature, i.e. a set of bounded functions with fixed maximum, minimum and integral over Ω, can be written in terms of $\overline{\mathcal{G}(m_0)}$ for a m_0 which takes exactly two values (functions of this kind are called of "bang-bang" type). This fact is proved in [2].

From the biological point of view, (i) of Theorem 1 says that there exists an arrangement of the resources that maximizes the chances of survival and, in this case, the population density is larger where the habitat is more favourable. On the other hand, (iii) means that there are configurations of resources as bad (i.e. inhospitable) as one prescribes.

Our third main result is the following

THEOREM 2. *Let* $\Omega = (0, h)\times\omega \subset \mathbb{R}^N$, $\mathrm{h} > 0$ *and* $\omega \subset \mathbb{R}^{N-1}$ *be a bounded polyhedral or smooth domain. Let* $m_0 \in L^\infty_<(\Omega)$ *such that the set* $\{x \in \Omega : m_0(x) > 0\}$ *has positive Lebesgue measure and* $\mathcal{G}(m_0)$ *the class of rearrangements of* m_0 *(see Definition 7). Then every minimizer of* (3) *is monotone with respect to* x_1*, where* x_1 *is the first coordinate of* $\mathbb{R}^N$*.*

Monotonicity results of this kind have been studied both theoretically and numerically by a number of authors. Theorem 2 in the one dimensional case has been proved in [14, 36] in the case m_0 is a "bang-bang" function and in [30] for general m_0. In general dimension, when the domain Ω is an orthotope and m_0 is of "bang-bang" type, Lamboley et al. in [33] show that any minimizer is monotonic with respect to every coordinate direction. Theorem 2 contains all previous results and it is coherent with numerical simulations in [31, 42] for rectangles and "bang-bang" weights.

It is worth mentioning that in the case (1) is considered with Dirichlet boundary conditions, the monotonicity of minimizers is replaced by the Steiner symmetry of them (see [2, 7, 14]). Nevertheless, in both situations these qualitative properties of the minimizers lead to an arrangement of the favourable resources fragmented as little as possible. Indeed, in the Dirichlet case they are concentrated far from the boundary, while in the Neumann case they meet the boundary.

As final remark, we observe that problem (1), with Dirichlet boundary conditions in place of Neumann and in the case of positive weight $m(x)$, also has a well-known physical interpretation: it models the vibration of a membrane Ω with clamped boundary $\partial\Omega$ and mass density $m(x)$; $\lambda_1(m)$ represents the principal natural frequency of the membrane. Therefore, physically, minimizing $\lambda_1(m)$ means to find the mass distribution of the membrane which gives the lowest principal natural frequency. Usually, the composite membrane is built using only two homogeneous materials of different densities and, then, the weights in the optimization problem take only two positive values. Among many papers that consider the optimization of the principal natural frequency, we recall [16–18].

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This paper is structured as follows. In §2, we set up the functional framework and some tools in order to investigate the spectrum of problem (1). In §3, we study the dependence of $\lambda_k(m)$ on m, in particular continuity and, for $k=1$, convexity and differentiability properties; then, we prove Theorem 1. In §4, we give the proof of Theorem 2. Finally, in Appendix A we collect some known results about rearrangements of measurable functions we need to examine the optimization problem (3).

2. Notations, preliminaries and weak formulation of (1)

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded connected open set with Lipschitz boundary $\partial\Omega$.

In this paper, we denote by $|E|$ the measure of an arbitrary Lebesgue measurable set $E \subset \mathbb{R}^N$ and by $L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$ the usual Lebesgue and Sobolev spaces. The usual norms and scalar products of these spaces are denote by

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |u| \quad \forall\, u \in L^\infty(\Omega),$$

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega uv\, dx \quad \forall\, u, v \in L^2(\Omega)$$

$$\|u\|_{L^2(\Omega)} = \langle u, u\rangle_{L^2(\Omega)}^{1/2} \quad \forall\, u \in L^2(\Omega),$$

$$\langle u, v\rangle_{H^1(\Omega)} = \int_\Omega uv\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx \quad \forall\, u, v \in H^1(\Omega),$$

$$\|u\|_{H^1(\Omega)} = \langle u, u\rangle_{H^1(\Omega)}^{1/2} \quad \forall\, u \in H^1(\Omega). \tag{5}$$

Moreover, we also use the notation $\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla v\, dx$ for all $u, v \in H^1(\Omega)$ and by weak* convergence we always mean the weak* convergence in $L^\infty(\Omega)$.

Given $m \in L^2(\Omega)$ such that $m \neq 0$, we define the spaces

$$L^2_m(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega mf\, dx = 0 \right\} \quad \text{and} \quad V_m(\Omega) = H^1(\Omega) \cap L^2_m(\Omega).$$

$L^2_m(\Omega)$ and $V_m(\Omega)$ are separable Hilbert subspaces of $L^2(\Omega)$ and $H^1(\Omega)$ respectively.

2.1. The projection P_m and norm in $V_m(\Omega)$

In this subsection, we introduce a fundamental tool in order to develop our theory: a projection from $L^2(\Omega)$ to $L^2_m(\Omega)$ (which must not be confused with the usual orthogonal projection in Hilbert spaces).

Definition 1. *Let $m \in L^2(\Omega)$ such that $\int_\Omega m\, dx \neq 0$. We call projection $\mathrm{P_m}$ the operator*

$$P_m : L^2(\Omega) \to L^2_m(\Omega), \qquad f \mapsto f - \frac{\int_\Omega mf\, dx}{\int_\Omega m\, dx}\,.$$

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Note that $P_m(H^1(\Omega)) \subset V_m(\Omega)$. Indeed, depending on the case (which will be clear from the context), it might be more convenient to consider the projection $P_m : H^1(\Omega) \to V_m(\Omega)$. Since in our work $m(x)$ represents the local growth rate, which is a bounded function, hereafter we consider $m \in L^\infty(\Omega)$. Nevertheless, Proposition 1, Proposition 2 and Proposition 3 can also be stated for $m \in L^2(\Omega)$.

Proposition 1. *Let $m, q \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx, \int_\Omega q\,dx \neq 0$ and $\mathrm{P_m}$ the projection of Definition 1. Then*

i) $\langle mP_m(f), \varphi\rangle_{L^2(\Omega)} = \langle mf, P_m(\varphi)\rangle_{L^2(\Omega)}$ *for all* $f, \varphi \in L^2(\Omega)$;
ii) $P_m(f) = 0$ *if and only if* f *is constant;*
iii) $P_m(f) = f$ *for all* $f \in L^2_m(\Omega)$;
iv) $\nabla P_m(f) = \nabla f$ *for all* $f \in H^1(\Omega)$;
v) $\mathrm{P_m}$ *is a linear bounded operator with*

$$\|P_m\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq 1 + \frac{\|m\|_{L^\infty(\Omega)}}{\left|\int_\Omega m\,dx\right|}\,|\Omega| \tag{6}$$

and

$$\|P_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \leq 1 + \frac{\|m\|_{L^\infty(\Omega)}}{\left|\int_\Omega m\,dx\right|}\,|\Omega|; \tag{7}$$

vi) *the compositions* $P_q \circ P_m : L^2_q(\Omega) \to L^2_q(\Omega)$, $P_q \circ P_m : V_q(\Omega) \to V_q(\Omega)$ *are identities;*
vii) $P_m : L^2_q(\Omega) \to L^2_m(\Omega)$, $P_m : V_q(\Omega) \to V_m(\Omega)$ *are isomorphisms.*

Proof. (i), (ii), (iii) and (iv) are immediate consequences of the definition of the projection P_m.

(v) By the definition of P_m and straightforward calculations we find

$$\begin{aligned}
\|P_m(f)\|^2_{L^2(\Omega)} &= \int_\Omega \left[f^2 - 2\,\frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx} f + \left(\frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx}\right)^2\right]\,dx \\
&= \|f\|^2_{L^2(\Omega)} - 2\,\frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx}\int_\Omega f\,dx + \left(\frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx}\right)^2 |\Omega| \\
&\leq \|f\|^2_{L^2(\Omega)} + 2\,\frac{\|m\|_{L^2(\Omega)}\|f\|^2_{L^2(\Omega)}}{\left|\int_\Omega m\,dx\right|}|\Omega|^{1/2} + \frac{\|m\|^2_{L^2(\Omega)}\|f\|^2_{L^2(\Omega)}}{\left|\int_\Omega m\,dx\right|^2}|\Omega| \\
&= \left(\|f\|_{L^2(\Omega)} + \frac{\|m\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}}{\left|\int_\Omega m\,dx\right|}|\Omega|^{1/2}\right)^2 \\
&\leq \left(\|f\|_{L^2(\Omega)} + \frac{\|m\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}}{\left|\int_\Omega m\,dx\right|}|\Omega|\right)^2 \\
&= \left(1 + \frac{\|m\|_{L^\infty(\Omega)}}{\left|\int_\Omega m\,dx\right|}|\Omega|\right)^2 \|f\|^2_{L^2(\Omega)};
\end{aligned}$$

then (6) holds. The estimate (7) follows from (6) and (iv).

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(vi) Let $f \in L^2_q(\Omega)$; recalling that $\int_\Omega qf\,dx = 0$, we have

$$\begin{aligned}P_q(P_m(f)) &= P_q\left(f - \frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx}\right)\\ &= f - \frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx} - \frac{\int_\Omega qf\,dx - \frac{\int_\Omega mf\,dx}{\int_\Omega m\,dx}\int_\Omega q\,dx}{\int_\Omega q\,dx} = f;\end{aligned}$$

the second statement immediately follows from the first one.

(vii) It immediately follows from vi). □

For the sake of convenience, we put

$$C_1(m) = 1 + \frac{\|m\|_{L^\infty(\Omega)}}{\left|\int_\Omega m\,dx\right|}\,|\Omega|. \tag{8}$$

The previous proposition leads us to an alternative norm in the space $V_m(\Omega)$.

Proposition 2. *Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx \neq 0$. Then, for all $u \in V_m(\Omega)$ we have*

$$\|u\|_{L^2(\Omega)} \le C\left(1 + \frac{\|m\|_{L^\infty(\Omega)}}{\left|\int_\Omega m\,dx\right|}\,|\Omega|\right)\|\nabla u\|_{L^2(\Omega)}, \tag{9}$$

with C^2 equal to the constant of the Poincaré-Wirtinger's inequality (see [35, Theorem 12.23]).

Proof. Let $u \in V_m(\Omega)$. By (vi) of Proposition 1 we have $u = P_m(P_1(u))$. By (6) and the Poincaré-Wirtinger's inequality we find

$$\begin{aligned}\|u\|_{L^2(\Omega)} = \|P_m(P_1(u))\|_{L^2(\Omega)} &\le C_1(m)\|P_1(u)\|_{L^2(\Omega)}\\ &\le C_1(m)\left\|u - \frac{1}{|\Omega|}\int_\Omega u\,dx\right\|_{L^2(\Omega)}\\ &\le C_1(m)\cdot C\|\nabla u\|_{L^2(\Omega)},\end{aligned}$$

which proves the statement. □

Proposition 3. *Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx \neq 0$. Then, the bilinear form in $V_m(\Omega)$*

$$\langle u, v\rangle_{V_m(\Omega)} = \int_\Omega \nabla u\cdot\nabla v\,dx \quad \forall\, u, v \in V_m(\Omega) \tag{10}$$

is a scalar product which induces a norm equivalent to the usual norm (5). We denote by $\|u\|_{V_m(\Omega)}$ the associated norm to (10).

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Proof. Comparing $\|u\|_{V_m(\Omega)}$ with (5), we have

$$\|u\|_{V_m(\Omega)} \leq \|u\|_{H^1(\Omega)}. \tag{11}$$

By (9) and (8), we find

$$\|u\|_{H^1(\Omega)} \leq (C^2 \cdot C_1^2(m) + 1)^{1/2} \|u\|_{V_m(\Omega)}. \tag{12}$$

By (11) and (12), the thesis immediately follows. □

If not stated otherwise, we will consider V_m endowed with the norm just introduced.

2.2. The operators E_m and G_m

We study the eigenvalues of problem (1) by means of the spectrum of an operator that we will introduce in this subsection.

Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx \neq 0$. For every $f \in L^2(\Omega)$ let us consider the following continuous linear functional on $V_m(\Omega)$

$$\varphi \mapsto \langle mf, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega).$$

By the Riesz Theorem, there exists a unique $u \in V_m(\Omega)$ such that

$$\langle u, \varphi \rangle_{V_m(\Omega)} = \langle mf, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega) \tag{13}$$

holds.

Let us introduce the operator

$$E_m : L^2(\Omega) \to V_m(\Omega), \tag{14}$$

where $u = E_m(f)$ is the unique function in $V_m(\Omega)$ that satisfies (13), i.e. for all $f \in L^2(\Omega)$, $E_m(f)$ is defined by

$$\langle E_m(f), \varphi \rangle_{V_m(\Omega)} = \langle mf, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega). \tag{15}$$

E_m is clearly linear. Putting $\varphi = u$ in (13) and exploiting (9) and (8), we find

$$\|u\|_{V_m(\Omega)} \leq C \cdot C_1(m) \|m\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega)}. \tag{16}$$

Therefore E_m is a linear bounded operator such that

$$\|E_m\|_{\mathcal{L}(L^2(\Omega), V_m(\Omega))} \leq C \cdot C_1(m) \|m\|_{L^\infty(\Omega)}. \tag{17}$$

Let i_m be the inclusion of $V_m(\Omega)$ into $L^2(\Omega)$. Note that, by compactness of the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$ (see [35]), it follows that i_m is a compact operator as well. Moreover, we define a second the linear operator

$$G_m : V_m(\Omega) \to V_m(\Omega) \tag{18}$$

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by $G_m = E_m \circ i_m$, i.e. for all $f \in V_m(\Omega)$, $G_m(f)$ is defined by

$$\langle G_m(f), \varphi\rangle_{V_m(\Omega)} = \langle mf, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega). \tag{19}$$

The main properties of the operators E_m and G_m are summarized in the following Proposition.

PROPOSITION 4. *Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx \neq 0$ and P_m, E_m and $\mathrm{G_m}$ defined by Definition 1, (15) and (19) respectively. Then*

i) $E_m(f) = G_m(P_m(f))$ *for all* $f \in H^1(\Omega)$*;*
ii) $\mathrm{G_m}$ *is self-adjoint and compact;*
iii) $\mathrm{E_m}$ *restricted to* $H^1(\Omega)$ *is compact.*

Proof. (i) Let $f \in H^1(\Omega)$, then $P_m(f) \in V_m(\Omega)$. By (19), (i) and (iii) of Proposition 1 we have

$$\begin{aligned}\langle G_m(P_m(f)), \varphi\rangle_{V_m(\Omega)} &= \langle mP_m(f), \varphi\rangle_{L^2(\Omega)} = \langle mf, P_m(\varphi)\rangle_{L^2(\Omega)}\\ &= \langle mf, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega).\end{aligned}$$

Thus, by (15), $E_m(f) = G_m(P_m(f))$.

(ii) For all $f, g \in V_m(\Omega)$, by (19), we have

$$\langle G_m(f), g\rangle_{V_m(\Omega)} = \langle mf, g\rangle_{L^2(\Omega)} = \langle mg, f\rangle_{L^2(\Omega)} = \langle G_m(g), f\rangle_{V_m(\Omega)},$$

then G_m is self-adjoint. The compactness of the operator G_m is an immediate consequence of its definition $G_m = E_m \circ i_m$, the inclusion i_m being compact and the operator E_m continuous.

(iii) It follows from (i) and (ii). □

By the general theory of self-adjoint compact operators (see [21, 34]), it follows that all nonzero eigenvalues of G_m have a finite dimensional eigenspace and they can be obtained by the *Fischer's Principle*

$$\mu_k(m) = \sup_{F_k \subset V_m(\Omega)} \inf_{\substack{f \in F_k\\ f \neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} \tag{20}$$

$$= \sup_{F_k \subset V_m(\Omega)} \inf_{\substack{f \in F_k\\ f \neq 0}} \frac{\int_\Omega mf^2\,dx}{\int_\Omega |\nabla f|^2\,dx}, \quad k = 1, 2, 3, \ldots \tag{20}$$

and

$$\begin{aligned}\mu_{-k}(m) &= \inf_{F_k \subset V_m(\Omega)} \sup_{\substack{f \in F_k\\ f \neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}}\\ &= \inf_{F_k \subset V_m(\Omega)} \sup_{\substack{f \in F_k\\ f \neq 0}} \frac{\int_\Omega mf^2\,dx}{\|f\|^2_{V_m(\Omega)}}, \quad k = 1, 2, 3, \ldots,\end{aligned}$$

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where the first extrema are taken over all the subspaces F_k of $V_m(\Omega)$ of dimension k. As observed in [21], all the inf's and sup's in the above characterizations of the eigenvalues are actually assumed. Hence, they could be replaced by min's and max's and the eigenvalues are obtained exactly in correspondence of the associated eigenfunctions. The sequence $\{\mu_k(m)\}$ contains all the real positive eigenvalues (repeated with their multiplicity), is decreasing and converging to zero, whereas $\{\mu_{-k}(m)\}$ is formed by all the real negative eigenvalues (repeated with their multiplicity), is increasing and converging to zero.

We will write $\{m > 0\}$ as a short form of $\{x \in \Omega : m(x) > 0\}$ and similarly $\{m < 0\}$ for $\{x \in \Omega : m(x) < 0\}$. The following proposition is analogous to [21, Proposition 1.11].

PROPOSITION 5. *Let $m \in L^\infty(\Omega)$ and $\mathrm{G_m}$ be the operator (19). Then, the following statements hold*

i) *if $|\{m > 0\}| = 0$, then there are no positive eigenvalues;*
ii) *if $|\{m > 0\}| > 0$ and $\int_\Omega m\,dx < 0$, then there is a sequence of positive eigenvalues $\mu_k(m)$;*
iii) *if $|\{m < 0\}| = 0$, then there are no negative eigenvalues;*
iv) *if $|\{m < 0\}| > 0$ and $\int_\Omega m\,dx > 0$, then there is a sequence of negative eigenvalues $\mu_{-k}(m)$.*

Proof. (i) Let μ be an eigenvalue and u a corresponding eigenfunction. By (19) with $f = \varphi = u$ we have

$$\mu = \frac{\int_\Omega mu^2\,dx}{\|u\|^2_{V_m(\Omega)}} \le 0.$$

(ii) By measure theory covering theorems, for each positive integer k there exist k disjoint closed balls $B_1, \dots, B_k$ in Ω such that $|B_i \cap \{m > 0\}| > 0$ for $i = 1, \dots, k$. Let $f_i \in C_0^\infty(B_i)$ such that $\int_\Omega m f_i^2\,dx = 1$ for every $i = 1, \dots, k$. Note that the functions f_i are linearly independent. We put $g_i = P_m(f_i)$, where P_m is defined in Definition 1; $g_i \in V_m(\Omega)$ for all $i = 1, \dots, k$. We show that the functions g_i are linearly independent as well. Let $\alpha_1, \dots, \alpha_k$ be constants such that $\sum_{i=1}^k \alpha_i g_i = 0$; this implies $P_m\left(\sum_{i=1}^k \alpha_i f_i\right) = 0$, i.e., by (ii) of Proposition 1, $\sum_{i=1}^k \alpha_i f_i = c \in \mathbb{R}$. Evaluating $\sum_{i=1}^k \alpha_i f_i$ in $\Omega \setminus \cup_{i=1}^k B_i \neq \emptyset$ we find $c = 0$ and, therefore, $\alpha_i = 0$ for all $i = 1, \dots, k$.

Let $F_k = \operatorname{span}\{g_1, \dots, g_k\}$. F_k is a subspace of $V_m(\Omega)$ of dimension k. For every $g \in F_k \setminus \{0\}$, $g = \sum_{i=1}^k a_i g_i$, with suitable constants $a_i \in \mathbb{R}$. Let us put $f = \sum_{i=1}^k a_i f_i$, clearly $g = P_m(f)$. Then, by (i) and (iii) of Proposition 1 and recalling that $\int_\Omega m\,dx < 0$, we have

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$$\begin{aligned}\frac{\langle G_m(g), g\rangle_{V_m(\Omega)}}{\|g\|^2_{V_m(\Omega)}} =&\frac{\langle mg, g\rangle_{L^2(\Omega)}}{\|g\|^2_{V_m(\Omega)}} = \frac{\langle mP_m(f), P_m(f)\rangle_{L^2(\Omega)}}{\|g\|^2_{V_m(\Omega)}} = \frac{\langle mf, P_m(f)\rangle_{L^2(\Omega)}}{\|g\|^2_{V_m(\Omega)}}\\ =&\frac{\langle mf, f\rangle_{L^2(\Omega)} - \left\langle mf, \int_\Omega mf\,dx/\int_\Omega m\,dx\right\rangle_{L^2(\Omega)}}{\|g\|^2_{V_m(\Omega)}}\\ =&\frac{\langle mf, f\rangle_{L^2(\Omega)} - \left(\int_\Omega mf\,dx\right)^2/\int_\Omega m\,dx}{\|g\|^2_{V_m(\Omega)}}\\ \geq&\frac{\sum_{i,j=1}^k a_i a_j \int_\Omega m f_i f_j\,dx}{\sum_{i,j=1}^k \langle g_i, g_j\rangle_{V_m(\Omega)} a_i a_j} = \frac{\sum_{i=1}^k a_i^2}{\sum_{i,j=1}^k \langle g_i, g_j\rangle_{V_m(\Omega)} a_i a_j}\\ =& \frac{\|a\|^2_{\mathbb{R}^k}}{\langle A_k a, a\rangle_{\mathbb{R}^k}} \geq \frac{1}{\|A_k\|} > 0,\end{aligned}$$

where $\|a\|_{\mathbb{R}^k}$, $\|A_k\|$ and $\langle A_k a, a\rangle_{\mathbb{R}^k}$ denote, respectively, the euclidean norm of the vector $a = (a_1, \dots, a_k)$, the norm of the non null matrix $A_k = \left(\langle g_i, g_j\rangle_{V_m(\Omega)}\right)_{i,j=1}^k$ and the inner product in $\mathbb{R}^k$. From the Fischer's Principle (20) we conclude that $\mu_k(m) \geq \dfrac{1}{\|A_k\|} > 0$ for every k.

The cases (iii) and (iv) are similarly proved. □

2.3. Weak formulation of problem (1)

The operators E_m and G_m are related to the following problem with Neumann boundary conditions

$$\begin{cases} -\Delta u = mf & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

For $m \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, a *weak solution* of problem (21) is a function $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla\varphi\,dx = \int_\Omega mf\varphi\,dx \quad \forall\,\varphi \in H^1(\Omega)$$

or, equivalently,

$$\langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)} = \langle mf, \varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi \in H^1(\Omega). \tag{22}$$

The assumptions under which problem (21) admits solutions are well known (see for example [39]). By using the tools introduced in §2.2, we find those conditions independently.

Lemma 1. *Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m\,dx \neq 0$ and $f \in L^2(\Omega)$. Then $u \in H^1(\Omega)$ satisfies*

$$\langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)} = \langle mP_m(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi \in H^1(\Omega) \tag{23}$$

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if and only if $P_m(u)$ *satisfies*

$$\langle P_m(u), \varphi \rangle_{V_m(\Omega)} = \langle mf, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in V_m(\Omega). \tag{24}$$

Proof. If $u \in H^1(\Omega)$ satisfies (23), for all $\varphi \in V_m(\Omega)$, by (iv), (i) and (iii) of Proposition 1, we have

$$\begin{aligned}\langle P_m(u), \varphi \rangle_{V_m(\Omega)} &= \langle \nabla P_m(u), \nabla \varphi \rangle_{L^2(\Omega)} = \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} \\ &= \langle m P_m(f), \varphi \rangle_{L^2(\Omega)} = \langle mf, \varphi \rangle_{L^2(\Omega)}.\end{aligned}$$

Vice versa, let u verify (24). For all $\varphi \in H^1(\Omega)$, recalling (iv) and (i) of Proposition 1 we have

$$\begin{aligned}\langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} &= \langle \nabla P_m(u), \nabla P_m(\varphi) \rangle_{L^2(\Omega)} = \langle P_m(u), P_m(\varphi) \rangle_{V_m(\Omega)} \\ &= \langle mf, P_m(\varphi) \rangle_{L^2(\Omega)} = \langle m P_m(f), \varphi \rangle_{L^2(\Omega)}.\end{aligned}$$

□

As the following proposition says, the operator E_m provides the solutions of problem (21).

PROPOSITION 6. *Let* $m \in L^\infty(\Omega)$ *such that* $\int_\Omega m\,dx \neq 0$ *and* $\mathrm{E_m}$ *be the operator (14). Then*

i) (22) *has a solution if and only if* $f \in L^2_m(\Omega)$*;*
ii) *if* $f \in L^2_m(\Omega)$*,* (22) *has a unique solution* $\overline{u} \in V_m(\Omega)$ *and any other solution is of form* $\overline{u} + c$*,* $c \in \mathbb{R}$*;*
iii) $\overline{u} = E_m(f)$*, i.e.* $\overline{u}$ *is the unique solution of*

$$\langle \overline{u}, \varphi \rangle_{V_m(\Omega)} = \langle mf, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in V_m(\Omega);$$

(iv) the estimate

$$\|\overline{u}\|_{H^1(\Omega)} \leq C \cdot C_1(m) \left(C^2 \cdot C_1^2(m) + 1 \right)^{1/2} \|m\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega)}$$

holds.

Proof. If (22) admits a solution u, choosing $\varphi \equiv 1$ we obtain $f \in L^2_m(\Omega)$.

Vice versa, let $f \in L^2_m(\Omega)$. By Lemma 1, $u \in H^1(\Omega)$ is a solution of (22) if and only if $P_m(u)$ is a solution of (24). By (13) and (15), we know that (24) admits the unique solution $\overline{u} = E_m(f) \in V_m(\Omega)$. Then the set of solutions of (22) is $\{u \in H^1(\Omega) : P_m(u) = \overline{u}\} = \{u \in H^1(\Omega) : u = \overline{u} + c,\ c \in \mathbb{R}\}$ and only $\overline{u}$ belongs to $V_m(\Omega)$. This proves (i), (ii) and (iii).

(iv) It follows immediately from (12) and (16). □

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Finally, we introduce the weak formulation of problem (1). A function $u \in H^1(\Omega)$ is said an *eigenfunction* of (1) associated to the *eigenvalue* λ if

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega m u \varphi \, dx \quad \forall \varphi \in H^1(\Omega)$$

or, equivalently,

$$\langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} = \lambda \langle m u, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega). \tag{25}$$

It is easy to check that zero is an eigenvalue and the associated eigenfunctions are all of the constant functions.

PROPOSITION 7. *Let $m \in L^\infty(\Omega)$ such that $\int_\Omega m \, dx \neq 0$ and $\mathrm{G_m}$ be the operator* (18). *Then the nonzero eigenvalues of problem* (1) *are exactly the reciprocals of the nonzero eigenvalues of the operator $\mathrm{G_m}$ and the correspondent eigenspaces coincide.*

Proof. If $\lambda \neq 0$ is an eigenvalue and u is an associated eigenfunction of problem (1), choosing $\varphi \equiv 1$ in (25), we obtain $u \in V_m(\Omega)$. By (25) and (10) we have

$$\left\langle \frac{u}{\lambda}, \varphi \right\rangle_{V_m(\Omega)} = \langle m u, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in V_m(\Omega)$$

and then, by definition (19) of G_m, $G_m(u) = \dfrac{u}{\lambda}$.

Vice versa, let $G_m(u) = \mu u$, with $\mu \neq 0$. Then we have

$$\langle \mu u, \varphi \rangle_{V_m(\Omega)} = \langle m u, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in V_m(\Omega).$$

By (iii) of Proposition 1 we obtain

$$\langle P_m(\mu u), \varphi \rangle_{V_m(\Omega)} = \langle m u, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in V_m(\Omega),$$

using Lemma 1 we find

$$\mu \, \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} = \langle m P_m(u), \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega)$$

and finally, applying (iii) of Proposition 1 again, we conclude

$$\langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} = \frac{1}{\mu} \langle m u, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega),$$

i.e. $1/\mu$ is an eigenvalue of (1). □

Consequently, in general, the eigenvalues of problem (1) form two monotone sequences

$$0 < \lambda_1(m) \leq \lambda_2(m) \leq \ldots \leq \lambda_k(m) \leq \ldots$$

and

$$\ldots \leq \lambda_{-k}(m) \leq \ldots \leq \lambda_{-2}(m) \leq \lambda_{-1}(m) < 0,$$

where every eigenvalue appears as many times as its multiplicity, the latter being finite owing to the compactness of G_m.

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The variational characterization (20) for $k = 1$, assuming that $|\{m > 0\}| > 0$ and $\int_\Omega m\,dx < 0$, becomes

$$\mu_1(m) = \max_{\substack{f\in V_m(\Omega)\\ f\neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} = \max_{\substack{f\in V_m(\Omega)\\ f\neq 0}} \frac{\int_\Omega mf^2\,dx}{\int_\Omega |\nabla f|^2\,dx}. \tag{26}$$

The maximum in (26) is obtained if and only if f is an eigenfunction relative to μ_1. Similarly, for $\lambda_1(m)$ we have

$$\lambda_1(m) = \min_{\substack{u\in V_m(\Omega)\\ \int_\Omega mu^2\,dx>0}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega mu^2\,dx} \tag{27}$$

and the minimum in (27) is obtained if and only if u is an eigenfunction relative to λ_1.

We note that the characterization

$$\lambda_1(m) = \min_{\substack{u\in H^1(\Omega)\\ \int_\Omega mu^2\,dx>0}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega mu^2\,dx}$$

also holds and it is more often used in the literature.

Proposition 8. *Let $m \in L^\infty(\Omega)$ such that $|\{m > 0\}| > 0$ and $\int_\Omega m\,dx < 0$. Then $\mu_1(m)$ is simple and any associated eigenfunction is one signed in Ω.*

Proof. Let $u \in V_m(\Omega)$ be an eigenfunction related to $\mu_1(m)$. Let us show that $|u|$ is an eigenfunction as well. Consider the projection $P_m(|u|)$ of $|u|$ on $V_m(\Omega)$, where P_m is defined in Definition 1. By (26) and (iv) of Proposition 1 we have

$$\mu_1(m) \geq \frac{\int_\Omega m(P_m(|u|))^2\,dx}{\int_\Omega |\nabla P_m(|u|)|^2\,dx} = \frac{\int_\Omega mu^2\,dx - \frac{(\int_\Omega m|u|\,dx)^2}{\int_\Omega m\,dx}}{\int_\Omega |\nabla u|^2\,dx} \geq \frac{\int_\Omega mu^2\,dx}{\int_\Omega |\nabla u|^2\,dx} = \mu_1(m).$$

Therefore, we have the equality sign in the previous chain. In particular, we find $\int_\Omega m|u|\,dx = 0$, then $|u|$ belongs to $V_m(\Omega)$ and finally, by (iii) of Proposition 1, $|u| = P_m(|u|)$ is an eigenfunction. By Proposition 7, $|u|$ satisfies the equation $-\Delta|u| = \mu_1(m)^{-1}m|u|$ and, by Harnack inequality (see [27]), we conclude that $|u| > 0$ in Ω; therefore u is one signed in Ω. Let u, v be two eigenfunctions; set $\alpha = \frac{\int_\Omega v\,dx}{\int_\Omega u\,dx}$, note that $\int_\Omega(\alpha u - v)\,dx = 0$. Note that also $\alpha u - v$ is an eigenfunction of $\mu_1(m)$. If $\alpha u - v$ was not identically zero, then, it would be one signed and hence $\int_\Omega(\alpha u - v)\,dx \neq 0$, reaching a contradiction. Therefore $v = \alpha u$ and $\mu_1(m)$ is simple. □

As a consequence of Proposition 7, we have the following

Corollary 1. *Let $m \in L^\infty(\Omega)$ such that $|\{m > 0\}| > 0$ and $\int_\Omega m\,dx < 0$. Then $\lambda_1(m)$ is simple and any associated eigenfunction is one signed in Ω.*

We call $\lambda_1(m)$ the principal eigenvalue of problem *(1)*. Throughout the paper we will denote by u_m the unique positive eigenfunction of both G_m (relative to $\mu_1(m)$)

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and problem (1) (relative to $\lambda_1(m)$), normalized by

$$\|u_m\|_{V_m(\Omega)} = 1, \tag{28}$$

which is equivalent to

$$\int_\Omega m u_m^2 \, dx = \mu_1(m) = \frac{1}{\lambda_1(m)}. \tag{29}$$

By standard regularity theory (see [27]), $u_m \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap C^{1,\beta}(\Omega)$ for all $0 < \beta < 1$.

As last comment, we observe that $\mu_1(m)$ is homogeneous of degree 1, i.e.

$$\mu_1(\alpha m) = \alpha \mu_1(m) \quad \forall\, \alpha > 0. \tag{30}$$

This follows immediately from (26).

3. Optimization of $\lambda_1(m)$

This section is devoted to the study of the optimization of $\lambda_1(m)$. For this purpose, we need some qualitative properties of $\mu_1(m) = 1/\lambda_1(m)$ with respect to m. We begin by proving the continuity of $\mu_1(m)$ (actually, of all of the eigenvalues of the operator G_m defined in (19)) and then showing its convexity and Gâteaux differentiability. The structure of the proofs follows the ideas contained in [3] in the case of the fractional Laplacian and Dirichlet boundary conditions. Here, we deal with Neumann boundary conditions which require, especially in proving continuity, a more sophisticated argument involving the projection P_m.

Finally, we examine the minimization and maximization of $\lambda_1(m)$.

We introduce the following convex subset of $L^\infty(\Omega)$

$$L^\infty_<(\Omega) = \left\{ m \in L^\infty(\Omega) : \int_\Omega m \, dx < 0 \right\}.$$

Observe that, by Proposition 5, $\mu_k(m)$ and u_m (the unique positive eigenfunction of $\mu_1(m)$ of problem (1) normalized as in (28)) are well defined only when $|\{m > 0\}| > 0$. We extend them to the whole set $L^\infty_<(\Omega)$ by putting

$$\widetilde{\mu}_k(m) = \begin{cases} \mu_k(m) & \text{if } |\{m > 0\}| > 0 \\ 0 & \text{if } |\{m > 0\}| = 0 \end{cases} \tag{31}$$

and

$$\widetilde{u}_m = \begin{cases} u_m & \text{if } |\{m > 0\}| > 0 \\ 0 & \text{if } |\{m > 0\}| = 0. \end{cases} \tag{32}$$

Remark 1. Note that $\widetilde{\mu}_k(m) = 0$ if and only if $|\{m > 0\}| = 0$ and, in this circumstance, the inequality

$$\sup_{F_k \subset V_m(\Omega)} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_m(f), f \rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} \leq 0 \tag{33}$$

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holds, where F_k varies among all the k-dimensional subspaces of $V_m(\Omega)$. Moreover, from (30), we have $\widetilde{\mu}_1(\alpha m) = \alpha\widetilde{\mu}_1(m)$ for every $\alpha \geq 0$.

LEMMA 2. *Let $m \in L^\infty_<(\Omega)$ and $\mathrm{E_m}$ be the linear operator (14). Then, the map $m \mapsto E_m$ is sequentially weakly* continuous from $L^\infty_<(\Omega)$ to $\mathcal{L}(H^1(\Omega), H^1(\Omega))$ endowed with the norm topology.*

Proof. Let $\{m_i\}$ be a sequence which weakly* converges to m in $L^\infty_<(\Omega)$. Being $\{m_i\}$ bounded in $L^\infty(\Omega)$, there exists a constant $M > 0$ such that

$$\|m\|_{L^\infty(\Omega)} \leq M \quad \text{and} \quad \|m_i\|_{L^\infty(\Omega)} \leq M \quad \forall\, i. \tag{34}$$

We begin by proving that $E_{m_i}(f)$ tends to $E_m(f)$ in $H^1(\Omega)$ for any fixed $f \in H^1(\Omega)$. Recalling (i) of Proposition 4, we put $u_i = E_{m_i}(f) = G_{m_i}(P_{m_i}(f))$ and $u = E_m(f) = G_m(P_m(f))$.

First, we show that $P_m(u_i)$ weakly converges to u in $V_m(\Omega)$; indeed, by (15) we have

$$\langle u_i, \varphi\rangle_{V_{m_i}(\Omega)} = \langle m_i f, \varphi\rangle_{L^2(\Omega)} \quad \forall\varphi \in V_{m_i}(\Omega).$$

By Lemma 1 and (iii) of Proposition 1, we find

$$\langle \nabla u_i, \nabla\varphi\rangle_{L^2(\Omega)} = \langle m_i P_{m_i}(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi \in H^1(\Omega). \tag{35}$$

Similarly, for u we have

$$\langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)} = \langle m P_m(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi \in H^1(\Omega). \tag{36}$$

Taking $\varphi \in V_m(\Omega)$ in (35) and (36) and by using (iii) of Proposition 1, we find

$$\langle P_m(u_i), \varphi\rangle_{V_m(\Omega)} = \langle m_i P_{m_i}(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\varphi \in V_m(\Omega) \tag{37}$$

and

$$\langle u, \varphi\rangle_{V_m(\Omega)} = \langle m P_m(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\varphi \in V_m(\Omega). \tag{38}$$

Subtracting (38) from (37), we get

$$\langle P_m(u_i) - u, \varphi\rangle_{V_m(\Omega)} = \langle m_i P_{m_i}(f) - m P_m(f), \varphi\rangle_{L^2(\Omega)} \quad \forall\varphi \in V_m(\Omega). \tag{39}$$

As a consequence of the weak* convergence of m_i to m in $L^\infty(\Omega)$, letting $i \to \infty$ we obtain $\int_\Omega m_i f\varphi\, dx \to \int_\Omega m f\varphi\, dx$, $\int_\Omega m_i f\, dx \to \int_\Omega m f\, dx$, $\int_\Omega m_i\varphi\, dx \to \int_\Omega m\varphi\, dx$ and $\int_\Omega m_i\, dx \to \int_\Omega m\, dx$, which imply that the right hand term goes to zero, thus $P_m(u_i)$ weakly converges to u in $V_m(\Omega)$. By exploiting the continuity of the inclusion $V_m(\Omega) \hookrightarrow H^1(\Omega)$ and the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that $P_m(u_i)$ weakly converges to u in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Putting $\varphi = P_m(u_i) - u$ in

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(39) we get

$$\|P_m(u_i)-u\|^2_{V_m(\Omega)} = \langle m_i P_{m_i}(f) - mP_m(f), P_m(u_i)-u\rangle_{L^2(\Omega)} \\ \leq \left(\|m_i P_{m_i}(f)\|_{L^2(\Omega))} + \|mP_m(f)\|_{L^2(\Omega)}\right)\|P_m(u_i)-u\|_{L^2(\Omega)}. \tag{40}$$

By using (34), (6) and (8), we find

$$\|mP_m(f)\|_{L^2(\Omega)} \leq M\|P_m(f)\|_{L^2(\Omega)} \leq MC_1(m)\|f\|_{L^2(\Omega)}. \tag{41}$$

Similarly, we have

$$\|m_i P_{m_i}(f)\|_{L^2(\Omega)} \leq MC_1(m_i)\|f\|_{L^2(\Omega)}. \tag{42}$$

By the weak* convergence of m_i to m we can assume

$$\left|\int_\Omega m_i\,dx\right| > \frac{\left|\int_\Omega m\,dx\right|}{2}$$

for i large enough. Therefore

$$C_1(m_i) \leq 1 + \frac{M}{\left|\int_\Omega m_i\,dx\right|}\,|\Omega| \leq 1 + \frac{2M}{\left|\int_\Omega m\,dx\right|}\,|\Omega| \tag{43}$$

and, trivially

$$C_1(m) \leq 1 + \frac{2M}{\left|\int_\Omega m\,dx\right|}\,|\Omega|. \tag{44}$$

For the sake of simplicity we put

$$D(m,M) = 1 + \frac{2M}{\left|\int_\Omega m\,dx\right|}\,|\Omega|. \tag{45}$$

Then, by (40), (41), (42), (43), (44) and (45) we find

$$\|P_m(u_i)-u\|^2_{V_m(\Omega)} \leq 2MD(m,M)\|f\|_{L^2(\Omega)}\|P_m(u_i)-u\|_{L^2(\Omega)},$$

from which the convergence of $P_m(u_i)$ to u in $V_m(\Omega)$ follows. The next step shows that, actually, u_i strongly converges to u in $L^2(\Omega)$. Indeed, by using Definition 1, (vi) of Proposition 1, (6), (43) and (45), we have

$$\begin{aligned}\|u_i-u\|_{L^2(\Omega)} &= \left\|-\frac{\int_\Omega m_i u\,dx}{\int_\Omega m_i\,dx} + P_{m_i}\left(P_m(u_i)-u\right)\right\|_{L^2(\Omega)} \\ &\leq \left\|\frac{\int_\Omega m_i u\,dx}{\int_\Omega m_i\,dx}\right\|_{L^2(\Omega)} + \|P_{m_i}\left(P_m(u_i)-u\right)\|_{L^2(\Omega)} \\ &\leq \left|\frac{\int_\Omega m_i u\,dx}{\int_\Omega m_i\,dx}\right| |\Omega|^{1/2} + D(m,M)\,\|P_m(u_i)-u\|_{L^2(\Omega)},\end{aligned}$$

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which goes to zero because $\int_\Omega m_i u\,dx/\int_\Omega m_i\,dx \to \int_\Omega mu\,dx/\int_\Omega m\,dx = 0$ (being $u \in V_m(\Omega)$) and $P_m(u_i) \to u$ in $L^2(\Omega)$. Moreover, by (iv) of Proposition 1, we have

$$\|u_i - u\|_{H^1(\Omega)} = \left(\|u_i - u\|^2_{L^2(\Omega)} + \|P_m(u_i) - u\|^2_{V_m(\Omega)}\right)^{1/2}$$

and then u_i converges to u in $H^1(\Omega)$. Summarizing, for every $f \in H^1(\Omega)$ we have

$$\|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} \to 0 \quad \text{for } i \to \infty.$$

Now, for fixed i, let $\{f_{i,j}\}$, $j = 1, 2, 3, \ldots$, be a maximizing sequence of

$$\sup_{\substack{g \in H^1(\Omega) \\ \|g\|_{H^1(\Omega)} \le 1}} \|E_{m_i}(g) - E_m(g)\|_{H^1(\Omega)} = \|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))}.$$

Then, being $\|f_{i,j}\|_{H^1(\Omega)} \le 1$, we can extract a subsequence (still denoted by $\{f_{i,j}\}$) weakly convergent to some $f_i \in H^1(\Omega)$. Since the operators E_{m_i} and E_m restricted to $H^1(\Omega)$ are compact (see iii) of Proposition 4), it follows that $E_{m_i}(f_{i,j})$ converges to $E_{m_i}(f_i)$ and $E_m(f_{i,j})$ converges to $E_m(f_i)$ strongly in $V_m(\Omega)$ and then in $H^1(\Omega)$ as j goes to ∞. Thus we find

$$\|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} = \|E_{m_i}(f_i) - E_m(f_i)\|_{H^1(\Omega)}.$$

This procedure yields a sequence $\{f_i\}$ in $H^1(\Omega)$ such that $\|f_i\|_{H^1(\Omega)} \le 1$ for all i. Then, up to a subsequence, we can assume that $\{f_i\}$ weakly converges to a function $f \in H^1(\Omega)$ and (by the compactness of the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$) strongly in $L^2(\Omega)$. By using (12), (43), (44), (45), (17) and (34) we find

$$\begin{aligned}
&\|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} = \|E_{m_i}(f_i) - E_m(f_i)\|_{H^1(\Omega)} \\
&\le \|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} + \|E_{m_i}(f_i - f)\|_{H^1(\Omega)} + \|E_m(f_i - f)\|_{H^1(\Omega)} \\
&\le \|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} + \left(C^2 \cdot C_1^2(m_i) + 1\right)^{1/2} \|E_{m_i}(f_i - f)\|_{V_{m_i}(\Omega)} \\
&+ \left(C^2 \cdot C_1^2(m) + 1\right)^{1/2} \|E_m(f_i - f)\|_{V_m(\Omega)} \\
&\le \|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} \\
&+ \left(C^2 \cdot D^2(m, M) + 1\right)^{1/2} \left(\|E_{m_i}\|_{\mathcal{L}(L^2(\Omega),V_{m_i}(\Omega))} + \|E_m\|_{\mathcal{L}(L^2(\Omega),V_m(\Omega))}\right) \\
&\|f_i - f\|_{L^2(\Omega)} \le \|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} \\
&+ \left(C^2 \cdot D^2(m, M) + 1\right)^{1/2} \left(C \cdot C_1(m_i)\|m_i\|_{L^\infty(\Omega)} + C \cdot C_1(m)\|m\|_{L^\infty(\Omega)}\right) \\
&\|f_i - f\|_{L^2(\Omega)} \le \|E_{m_i}(f) - E_m(f)\|_{H^1(\Omega)} + 2CMD(m, M) \\
&\left(C^2 \cdot D^2(m, M) + 1\right)^{1/2} \|f_i - f\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore E_{m_i} converges to E_m in the operator norm. □

Remark 2. We note that the previous lemma still holds replacing $L^\infty_<(\Omega)$ by the set of $L^\infty(\Omega)$ such that $\int_\Omega m\,dx \ne 0$.

Lemma 3. *Let $m \in L^\infty_<(\Omega)$, $\widetilde{\mu}_k(m)$ as defined in (31) for $k = 1, 2, 3, \ldots$ and $\widetilde{u}_m$ as in (32). Then*

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i) *the map* $m \mapsto \widetilde{\mu}_k(m)$ *is sequentially weakly* continuous in* $L^\infty_<(\Omega)$*;*

(ii) *the map* $m \mapsto \widetilde{\mu}_1(m)\widetilde{u}_m$ *is sequentially weakly* continuous from* $L^\infty_<(\Omega)$ *to* $H^1(\Omega)$ *(endowed with the norm topology). In particular, for any sequence* $\{m_i\}$ *weakly* convergent to* $m \in L^\infty_<(\Omega)$ *with* $\widetilde{\mu}_1(m) > 0$*, then* $\{\widetilde{u}_{m_i}\}$ *converges to* $\widetilde{u}_m$ *in* $H^1(\Omega)$*.*

Proof. i) Let $\{m_i\}$ be a sequence which weakly* converges to m in $L^\infty_<(\Omega)$. Being $\{m_i\}$ bounded in $L^\infty(\Omega)$, there exists a constant $M>0$ such that (34) holds. We will show that

$$|\widetilde{\mu}_k(m_i) - \widetilde{\mu}_k(m)| \leq D(m,M)(C^2 \cdot D^2(m,M)+1)^{1/2}\|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))}, \tag{46}$$

where $D(m,M)$ is the constant in (45). By Lemma 2 the claim follows. We split the argument in three cases.

Case 1. Let i be fixed and assume $\widetilde{\mu}_k(m_i),\ \widetilde{\mu}_k(m) > 0$.

Following the idea in [29, Theorem 2.3.1] and by means of the Fischer's Principle (20) we have

$$\begin{aligned}
\widetilde{\mu}_k(m_i) - \widetilde{\mu}_k(m) &= \max_{F_k \subset V_{m_i}(\Omega)} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_{m_i}(f), f\rangle_{V_{m_i}(\Omega)}}{\|f\|^2_{V_{m_i}(\Omega)}} \\
&\quad - \max_{F_k \subset V_m(\Omega)} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} \\
&\leq \min_{\substack{f\in \overline{F_k}\\ f\neq 0}} \frac{\langle G_{m_i}(f), f\rangle_{V_{m_i}(\Omega)}}{\|f\|^2_{V_{m_i}(\Omega)}} - \min_{\substack{f\in P_m(\overline{F_k})\\ f\neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} \\
&\leq \frac{\langle G_{m_i}(\overline{f}), \overline{f}\rangle_{V_{m_i}(\Omega)}}{\|\overline{f}\|^2_{V_{m_i}(\Omega)}} - \frac{\langle G_m(P_m(\overline{f})), P_m(\overline{f})\rangle_{V_m(\Omega)}}{\|P_m(\overline{f})\|^2_{V_m(\Omega)}},
\end{aligned}$$

where $\overline{F_k}$ is a k-dimensional subspace of $V_{m_i}(\Omega)$ such that

$$\max_{F_k \subset V_{m_i}(\Omega)} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_{m_i}(f), f\rangle_{V_{m_i}(\Omega)}}{\|f\|^2_{V_{m_i}(\Omega)}} = \min_{\substack{f\in \overline{F_k}\\ f\neq 0}} \frac{\langle G_{m_i}(f), f\rangle_{V_{m_i}(\Omega)}}{\|f\|^2_{V_{m_i}(\Omega)}}$$

(note that, by vii) of Proposition 1, $P_m(\overline{F_k})$ is a k-dimensional subspace of $V_m(\Omega)$) and $\overline{f}$ is a function in $\overline{F_k}$ such that

$$\min_{\substack{f\in P_m(\overline{F_k})\\ f\neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}} = \frac{\langle G_m(P_m(\overline{f})), P_m(\overline{f})\rangle_{V_m(\Omega)}}{\|P_m(\overline{f})\|^2_{V_m(\Omega)}}.$$

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By (iv) of Proposition 1 $\nabla P_m(\overline{f}) = \nabla \overline{f}$ and $\nabla G_m(P_m(\overline{f})) = \nabla P_{m_i}(G_m(P_m(\overline{f})))$ hold, thus we have

$$\begin{aligned}\widetilde{\mu}_k(m_i) - \widetilde{\mu}_k(m) &\leq \frac{\langle G_{m_i}(\overline{f}), \overline{f}\rangle_{V_{m_i}(\Omega)}}{\|\overline{f}\|^2_{V_{m_i}(\Omega)}} - \frac{\langle P_{m_i}(G_m(P_m(\overline{f}))), \overline{f}\rangle_{V_{m_i}(\Omega)}}{\|\overline{f}\|^2_{V_{m_i}(\Omega)}} \\ &= \frac{\langle (G_{m_i} - P_{m_i}\circ G_m \circ P_m)(\overline{f}), \overline{f}\rangle_{V_{m_i}(\Omega)}}{\|\overline{f}\|^2_{V_{m_i}(\Omega)}} \\ &\leq \frac{\|(G_{m_i} - P_{m_i}\circ G_m \circ P_m)(\overline{f})\|_{V_{m_i}(\Omega)}}{\|\overline{f}\|_{V_{m_i}(\Omega)}}.\end{aligned}$$

Then, taking into account the identities (recall iii) of Proposition 1)

$$G_{m_i} - P_{m_i}\circ G_m \circ P_m = P_{m_i}\circ G_{m_i}\circ P_{m_i} - P_{m_i}\circ G_m\circ P_m = P_{m_i}\circ (G_{m_i}\circ P_{m_i} - G_m \circ P_m),$$

(12), (7), (8), (43), (45) and (i) of Proposition 4 we find

$$\begin{aligned}\widetilde{\mu}_k(m_i) - \widetilde{\mu}_k(m) &\leq \frac{\|P_{m_i}\circ (G_{m_i}\circ P_{m_i} - G_m\circ P_m)(\overline{f})\|_{V_{m_i}(\Omega)}}{\|\overline{f}\|_{V_{m_i}(\Omega)}} \\ &\leq \frac{\|P_{m_i}\circ (G_{m_i}\circ P_{m_i} - G_m\circ P_m)(\overline{f})\|_{H^1(\Omega)}}{\|\overline{f}\|_{V_{m_i}(\Omega)}} \\ &\leq (C^2\cdot C_1^2(m_i)+1)^{1/2}\|P_{m_i}\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \\ &\quad\times \frac{\|(G_{m_i}\circ P_{m_i} - G_m\circ P_m)(\overline{f})\|_{H^1(\Omega)}}{\|\overline{f}\|_{H^1(\Omega)}} \\ &\leq C_1(m_i)(C^2\cdot C_1^2(m_i)+1)^{1/2}\|G_{m_i}\circ P_{m_i} \\ &\quad - G_m\circ P_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \\ &\leq D(m,M)(C^2\cdot D^2(m,M)+1)^{1/2}\|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))}.\end{aligned}$$

Interchanging the role of m_i and m and replacing (43) by (44), we also have

$$\widetilde{\mu}_k(m) - \widetilde{\mu}_k(m_i) \leq D(m,M)(C^2\cdot D^2(m,M)+1)^{1/2}\|E_{m_i} - E_m\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))}$$

and finally (46).

Case 2. Let i be fixed and assume $\widetilde{\mu}_k(m_i) > 0$, $\widetilde{\mu}_k(m) = 0$ (and similarly in the case $\widetilde{\mu}_k(m) > 0$, $\widetilde{\mu}_k(m_i) = 0$).

Note that in this case (33) holds for the weight function m. Then the previous argument still applies provided that we replace the first step of the inequality chain by

$$\begin{aligned}|\widetilde{\mu}_k(m_i) - \widetilde{\mu}_k(m)| = \widetilde{\mu}_k(m_i) &\leq \max_{F_k\subset V_{m_i}(\Omega)} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_{m_i}(f), f\rangle_{V_{m_i}(\Omega)}}{\|f\|^2_{V_{m_i}(\Omega)}} \\ &\quad - \sup_{F_k\subset V_m(\Omega)} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_m(f), f\rangle_{V_m(\Omega)}}{\|f\|^2_{V_m(\Omega)}}.\end{aligned}$$

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Case 3. $\widetilde{\mu}_k(m_i) = \widetilde{\mu}_k(m) = 0$.

In this case, (46) is obvious.

Therefore statement (i) is proved.

(ii) Let $\{m_i\}$ be such that m_i is weakly* convergent to $m \in L^\infty_<(\Omega)$. By using (12), (43) and (45), for i sufficiently large, we have

$$\|\widetilde{u}_{m_i}\|_{H^1(\Omega)} \leq \left(C^2 \cdot D^2(m, M) + 1\right)^{1/2},$$

up to a subsequence we can assume that $\widetilde{u}_{m_i}$ is weakly convergent to $z \in H^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwisely a.e. in Ω.

First suppose $\widetilde{\mu}_1(m) = 0$. Then, by (i), $\widetilde{\mu}_1(m_i)\widetilde{u}_{m_i}$ weakly converges in $H^1(\Omega)$ to $\widetilde{\mu}_1(m)z = 0 = \widetilde{\mu}_1(m)\widetilde{u}_m$. Moreover, $\|\widetilde{\mu}_1(m_i)\widetilde{u}_{m_i}\|_{H^1(\Omega)} = \widetilde{\mu}_1(m_i)\|\widetilde{u}_{m_i}\|_{H^1(\Omega)}$ tends to $0 = \|\widetilde{\mu}_1(m)\widetilde{u}_m\|_{H^1(\Omega)}$. Therefore $\widetilde{\mu}_1(m_i)\widetilde{u}_{m_i}$ strongly converges to $\widetilde{\mu}_1(m)\widetilde{u}_m$ in $H^1(\Omega)$.

Next, consider the case $\widetilde{\mu}_1(m) > 0$. By (i) we have $\widetilde{\mu}_1(m_i) > 0$ for all i large enough. This implies $\widetilde{\mu}_1(m_i) = \frac{1}{\lambda_1(m_i)}$ and $\widetilde{u}_{m_i} = u_{m_i}$. Positiveness and pointwise convergence of u_{m_i} to z imply $z \geq 0$ a.e. in Ω. Moreover, by (29) we have

$$\int_\Omega m_i u_{m_i}^2 \, dx = \frac{1}{\lambda_1(m_i)}$$

and by (i), passing to the limit, we find

$$\int_\Omega m z^2 \, dx = \frac{1}{\lambda_1(m)},$$

which implies $z \neq 0$. By using (25) for u_{m_i} we have

$$\langle \nabla u_{m_i}, \nabla \varphi \rangle_{L^2(\Omega)} = \lambda_1(m_i) \langle m_i u_{m_i}, \varphi \rangle_{L^2(\Omega)} = \lambda_1(m_i) \int_\Omega m_i u_{m_i} \varphi \, dx \quad \forall \, \varphi \in H^1(\Omega),$$

and, letting i to ∞, we deduce $z = u_m$. By (i) $\mu_1(m_i)u_{m_i}$ weakly converges in $H^1(\Omega)$ to $\mu_1(m)u_m$ and $\|\mu_1(m_i)u_{m_i}\|_{H^1(\Omega)} = \mu_1(m_i)$ tends to $\mu_1(m) = \|\mu_1(m)u_m\|_{H^1(\Omega)}$. Hence $\mu_1(m_i)u_{m_i}$ strongly converges to $\mu_1(m)u_m$ in $H^1(\Omega)$. The last claim is immediate provided one observes that $\widetilde{\mu}_1(m) > 0$ implies $\widetilde{\mu}_1(m_i) > 0$ for all i large enough. □

LEMMA 4. *Let $m, q \in L^\infty_<(\Omega)$, $\widetilde{\mu}_1(m)$ be defined as in (31) for* $\mathrm{k} = 1$. *Then*

(i) *the map $m \mapsto \widetilde{\mu}_1(m)$ is convex on $L^\infty_<(\Omega)$;*

(ii) *if* m *and* q *are linearly independent and $\widetilde{\mu}_1(m), \widetilde{\mu}_1(q) > 0$, then*

$$\widetilde{\mu}_1(tm + (1-t)q) < t\widetilde{\mu}_1(m) + (1-t)\widetilde{\mu}_1(q)$$

for all $0 < t < 1$.

Proof. (i) The Fischer's Principle (20) and (33) both for $k=1$ yield

$$\sup_{\substack{f\in V_m(\Omega)\\ f\neq 0}} \frac{\int_\Omega mf^2\,dx}{\int_\Omega |\nabla f|^2\,dx} \leq \widetilde{\mu}_1(m) \tag{47}$$

for every $m\in L^\infty_<(\Omega)$. Moreover, if $\widetilde{\mu}_1(m)>0$, then the equality sign holds and the supremum is attained when f is an eigenfunction of $\widetilde{\mu}_1(m)=\mu_1(m)$. Let $m,q\in L^\infty_<(\Omega)$, $0\leq t\leq 1$. We show that

$$\widetilde{\mu}_1(tm+(1-t)q)\leq t\widetilde{\mu}_1(m)+(1-t)\widetilde{\mu}_1(q). \tag{48}$$

If $\widetilde{\mu}_1(tm+(1-t)q)=0$, (48) is obvious. Suppose $\widetilde{\mu}_1(tm+(1-t)q)>0$. Then, for all $f\in V_{tm+(1-t)q}(\Omega)$, $f\neq 0$, we have

$$\begin{aligned}\frac{\int_\Omega (tm+(1-t)q)f^2\,dx}{\int_\Omega |\nabla f|^2\,dx} &= t\,\frac{\int_\Omega mf^2\,dx}{\int_\Omega |\nabla f|^2\,dx}+(1-t)\,\frac{\int_\Omega qf^2\,dx}{\int_\Omega |\nabla f|^2\,dx}\\ &\leq t\,\frac{\int_\Omega mf^2\,dx-\frac{(\int_\Omega mf\,dx)^2}{\int_\Omega m\,dx}}{\int_\Omega |\nabla f|^2\,dx}\\ &\quad +(1-t)\,\frac{\int_\Omega qf^2\,dx-\frac{(\int_\Omega qf\,dx)^2}{\int_\Omega q\,dx}}{\int_\Omega |\nabla f|^2\,dx}\\ &= t\,\frac{\int_\Omega m(P_m(f))^2\,dx}{\int_\Omega |\nabla P_m(f)|^2\,dx}+(1-t)\,\frac{\int_\Omega q(P_q(f))^2\,dx}{\int_\Omega |\nabla P_q(f)|^2\,dx}\\ &\leq t\widetilde{\mu}_1(m)+(1-t)\widetilde{\mu}_1(q),\end{aligned} \tag{49}$$

where we used (iv) of Proposition 1 and (47) for m and q. Taking the supremum in the left-hand term of (49) and using (47) again with equality sign, we find (48).

(ii) Arguing by contradiction, we suppose that equality holds in (48). We will conclude that m and q are linearly dependent. Equality sign in (48) implies $\widetilde{\mu}_1(tm+(1-t)q)>0$, then (by (47)) equalities also occur in (49) with $f=u=u_{tm+(1-t)q}$. We get $\int_\Omega mu\,dx=\int_\Omega qu\,dx=0$, thus $u\in V_m(\Omega)\cap V_q(\Omega)$, and then, by (iii) of Proposition 1,

$$\frac{\int_\Omega mu^2\,dx}{\int_\Omega |\nabla u|^2\,dx}=\widetilde{\mu}_1(m) \quad \text{and} \quad \frac{\int_\Omega qu^2\,dx}{\int_\Omega |\nabla u|^2\,dx}=\widetilde{\mu}_1(q).$$

The simplicity of the principal eigenvalue, the positiveness of u and the normalization (28) imply that $u=u_m=u_q$. By using (25) with $\lambda=\frac{1}{\widetilde{\mu}_1(m)}$ and $\lambda=\frac{1}{\widetilde{\mu}_1(q)}$ we have

$$\langle \nabla u,\nabla\varphi\rangle_{L^2(\Omega)}=\frac{1}{\widetilde{\mu}_1(m)}\langle mu,\varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi\in H^1(\Omega),$$

and

$$\langle \nabla u,\nabla\varphi\rangle_{L^2(\Omega)}=\frac{1}{\widetilde{\mu}_1(q)}\langle qu,\varphi\rangle_{L^2(\Omega)} \quad \forall\,\varphi\in H^1(\Omega)$$

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respectively. Taking the difference of these identities we find

$$\left\langle \left(\frac{m}{\widetilde{\mu}_1(m)} - \frac{q}{\widetilde{\mu}_1(q)} \right) u, \varphi \right\rangle_{L^2(\Omega)} = 0 \quad \forall\, \varphi \in H^1(\Omega),$$

which gives $m\widetilde{\mu}_1(q) - q\widetilde{\mu}_1(m) = 0$ a.e. in Ω, i.e. m and q are linearly dependent. □

COROLLARY 2. *Let* $m_0 \in L^\infty_<(\Omega)$, $\widetilde{\mu}_1(m)$ *be defined as in* (31) *for* $\mathrm{k}=1$ *and* $\overline{\mathcal{G}(m_0)}$ *the weak* closure in* $L^\infty(\Omega)$ *of the class of rearrangements* $\mathcal{G}(m_0)$ *introduced in Definition 7. Then the map* $m \mapsto \widetilde{\mu}_1(m)$ *is convex but not strictly convex on* $\overline{\mathcal{G}(m_0)}$.

Proof. By (ii) of Proposition 15 and (i) of Corollary 3, we have that $\overline{\mathcal{G}(m_0)}$ is convex and $\overline{\mathcal{G}(m_0)} \subset L^\infty_<(\Omega)$. Then, by Lemma 4, the map $m \mapsto \widetilde{\mu}_1(m)$ is convex on $\overline{\mathcal{G}(m_0)}$.

Applying Proposition 13, we find that the constant function $c = \frac{1}{|\Omega|}\int_\Omega m_0\,dx$ is in $\overline{\mathcal{G}(m_0)}$. By convexity of $\overline{\mathcal{G}(m_0)}$, $tm_0 + (1-t)c \in \overline{\mathcal{G}(m_0)}$ for every $t \in [0,1]$. From the inequality

$$tm_0 + (1-t)c \le t\|m_0\|_{L^\infty(\Omega)} + (1-t)c \quad \text{a.e. in } \Omega,$$

we obtain

$$tm_0 + (1-t)c \le 0 \quad \text{a.e. in } \Omega \quad \forall\, t \le \frac{c}{c - \|m_0\|_{L^\infty(\Omega)}}.$$

Note that $c/\big(c - \|m_0\|_{L^\infty(\Omega)}\big) \in (0,1)$. Therefore, by (31), we conclude that $\widetilde{\mu}_1(m) = 0$ for any m in the line segment, contained in $\overline{\mathcal{G}(m_0)}$, that joins c and

$$\frac{c}{c - \|m_0\|_{L^\infty(\Omega)}} m_0 + \left(1 - \frac{c}{c - \|m_0\|_{L^\infty(\Omega)}}\right) c = \frac{\|m_0\|_{L^\infty(\Omega)} - m_0}{\|m_0\|_{L^\infty(\Omega)} - c} c.$$

This shows that the map $m \mapsto \widetilde{\mu}_1(m)$ is not strictly convex. □

For the definitions and some basic results on the Gâteaux differentiability we refer the reader to [24].

LEMMA 5. *Let* $m \in L^\infty_<(\Omega)$, $\widetilde{\mu}_1(m)$ *be defined as in* (31) *for* $\mathrm{k}=1$ *and* $\mathrm{u_m}$ *denote the relative unique positive eigenfunction of problem* (1) *normalized as in* (28)*. Then, the map* $m \mapsto \widetilde{\mu}_1(m)$ *is Gâteaux differentiable at any* m *such that* $\widetilde{\mu}_1(m) > 0$, *with Gâteaux differential equal to* u_m^2*. In other words, for every direction* $v \in L^\infty(\Omega)$ *we have*

$$\widetilde{\mu}_1'(m; v) = \int_\Omega u_m^2 v\,dx. \tag{50}$$

Proof. Let us compute

$$\lim_{t\to 0} \frac{\widetilde{\mu}_1(m+tv) - \widetilde{\mu}_1(m)}{t}.$$

Note that $m+tv \in L^\infty_<(\Omega)$ for $|t|$ sufficiently small and by (i) of Lemma 3, $\widetilde{\mu}_1(m+tv)$ converges to $\widetilde{\mu}_1(m)$ as t goes to zero for any $m \in L^\infty_<(\Omega)$ and $v \in L^\infty(\Omega)$. Therefore,

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$\widetilde{\mu}_1(m+tv) > 0$ for $|t|$ small enough. The eigenfunctions u_m and u_{m+tv} satisfy (see (25))

$$\widetilde{\mu}_1(m)\langle \nabla u_m, \nabla \varphi\rangle_{L^2(\Omega)} = \langle m u_m, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H^1(\Omega)$$

and

$$\widetilde{\mu}_1(m+tv)\langle \nabla u_{m+tv}, \nabla \varphi\rangle_{L^2(\Omega)} = \langle (m+tv) u_{m+tv}, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H^1(\Omega).$$

By choosing $\varphi = u_{m+tv}$ in the former equation, $\varphi = u_m$ in the latter and comparing we get

$$\widetilde{\mu}_1(m+tv)\langle m u_m, u_{m+tv}\rangle_{L^2(\Omega)} = \widetilde{\mu}_1(m)\langle (m+tv) u_{m+tv}, u_m\rangle_{L^2(\Omega)}.$$

Rearranging we find

$$\frac{\widetilde{\mu}_1(m+tv) - \widetilde{\mu}_1(m)}{t} \int_\Omega m\, u_m u_{m+tv}\, dx = \widetilde{\mu}_1(m) \int_\Omega u_m u_{m+tv} v\, dx. \tag{51}$$

If t goes to zero, then by (ii) of Lemma 3 it follows that u_{m+tv} converges to u_m in $H^1(\Omega)$ and therefore in $L^2(\Omega)$. Passing to the limit in (51) and using (29) we conclude

$$\lim_{t\to 0} \frac{\widetilde{\mu}_1(m+tv) - \widetilde{\mu}_1(m)}{t} = \int_\Omega u_m^2 v\, dx,$$

i.e. (50) holds. □

THEOREM 3. *Let $m_0 \in L^\infty_<(\Omega)$, $\overline{\mathcal{G}(m_0)}$ be the weak* closure in $L^\infty(\Omega)$ of the class of rearrangements $\mathcal{G}(m_0)$ introduced in Definition 7 and $\widetilde{\mu}_1(m)$ defined as in (31) for* k = 1. *Then*

i) *there exists a solution of the problem*

$$\max_{m\in\overline{\mathcal{G}(m_0)}} \widetilde{\mu}_1(m); \tag{52}$$

(ii) *if $|\{m_0 > 0\}| > 0$, any solution $\check{m}_1$ of (52) belongs to $\mathcal{G}(m_0)$, more explicitly, we have $\widetilde{\mu}_1(m) < \widetilde{\mu}_1(\check{m}_1)$ for all $m \in \overline{\mathcal{G}(m_0)} \smallsetminus \mathcal{G}(m_0)$ (note that, in this case, by Proposition 5 $\widetilde{\mu}_1(\check{m}_1) = \mu_1(\check{m}_1) > 0$);*

(iii) *if $|\{m_0 > 0\}| > 0$, for every solution $\check{m}_1 \in \mathcal{G}(m_0)$ of (52) there exists an increasing function ψ such that*

$$\check{m}_1 = \psi(u_{\check{m}_1}) \quad a.e.\ in\ \Omega,$$

where $u_{\check{m}_1}$ is the positive eigenfunction relative to $\mu_1(\check{m}_1)$ normalized as in (28).

Proof. i) By (i) of Corollary 3 we have $\overline{\mathcal{G}(m_0)} \subset L^\infty_<(\Omega)$. By (iii) of Proposition 14 and (i) of Lemma 3, $\overline{\mathcal{G}(m_0)}$ is sequentially weakly* compact and the map $m \mapsto \widetilde{\mu}_1(m)$ is sequentially weakly* continuous respectively. Therefore, there exists $\check{m}_1 \in \overline{\mathcal{G}(m_0)}$ such that

$$\widetilde{\mu}_1(\check{m}_1) = \max_{m\in\overline{\mathcal{G}(m_0)}} \widetilde{\mu}_1(m).$$

(ii) Note that, by Proposition 5, the condition $|\{m_0 > 0\}| > 0$ guarantees $\widetilde{\mu}_1(m) > 0$ on $\mathcal{G}(m_0)$ and then $\widetilde{\mu}_1(\check{m}_1) > 0$. Let $\check{m}_1$ be an arbitrary solution of (52), let us

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show that $\check{m}_1$ actually belongs to $\mathcal{G}(m_0)$. Proceeding by contradiction, suppose that $\check{m}_1 \notin \mathcal{G}(m_0)$. Then, by (iii) of Proposition 15 and by Definition 8, $\check{m}_1$ is not an extreme point of $\overline{\mathcal{G}(m_0)}$ and thus there exist $m, q \in \overline{\mathcal{G}(m_0)}$ such that $m \neq q$ and $\check{m}_1 = \frac{m+q}{2}$. By (i) of Lemma 4 and, being $\check{m}_1$ a maximizer, we have

$$\widetilde{\mu}_1(\check{m}_1) \leq \frac{\widetilde{\mu}_1(m) + \widetilde{\mu}_1(q)}{2} \leq \widetilde{\mu}_1(\check{m}_1)$$

and then, equality sign holds. This implies $\widetilde{\mu}_1(m) = \widetilde{\mu}_1(q) = \widetilde{\mu}_1(\check{m}_1) > 0$, that is m and q are maximizers as well. Now, applying (ii) of Lemma 4 to m and q with $t = \frac{1}{2}$, we conclude that m and q are linearly dependent and then, by (ii) of Corollary 3, we reach the contradiction $m = q$. Thus, we conclude that $\check{m}_1 \in \mathcal{G}(m_0)$ and (ii) is proved. https://epubs.siam.org/doi/book/10.1137/1.9781611971088

(iii) Let $\check{m}_1 \in \mathcal{G}(m_0)$ be a solution of (52). We prove the claim by using Proposition 17; more precisely, we show that

$$\int_\Omega \check{m}_1 u_{\check{m}_1}^2 \, dx > \int_\Omega m \, u_{\check{m}_1}^2 \, dx \tag{53}$$

for every $m \in \overline{\mathcal{G}(m_0)} \setminus \{\check{m}_1\}$. By exploiting the convexity of $\widetilde{\mu}_1(m)$ (see Lemma 4) and its Gâteaux differentiability in $\check{m}_1$ (see Lemma 5) we have (for details see [24])

$$\widetilde{\mu}_1(m) \geq \widetilde{\mu}_1(\check{m}_1) + \int_\Omega (m - \check{m}_1) u_{\check{m}_1}^2 \, dx \tag{54}$$

for all $m \in \overline{\mathcal{G}(m_0)}$.

First, let us suppose $\widetilde{\mu}_1(m) < \widetilde{\mu}_1(\check{m}_1)$. Comparing with (54) we find

$$\int_\Omega (m - \check{m}_1) u_{\check{m}_1}^2 \, dx < 0,$$

that is (53).

Next, let us consider the case $\widetilde{\mu}_1(m) = \widetilde{\mu}_1(\check{m}_1)$, $m \in \overline{\mathcal{G}(m_0)} \setminus \{\check{m}_1\}$. By (ii) there are not maximizers of $\widetilde{\mu}_1$ in $\overline{\mathcal{G}(m_0)} \setminus \mathcal{G}(m_0)$, therefore $m \in \mathcal{G}(m_0)$. Being $\check{m}_1 \neq m$, by (ii) of Corollary 3, they are linearly independent. Then, (ii) of Lemma 4 implies

$$\widetilde{\mu}_1 \left(\frac{\check{m}_1 + m}{2} \right) < \frac{\widetilde{\mu}_1(\check{m}_1) + \widetilde{\mu}_1(m)}{2} = \widetilde{\mu}_1(\check{m}_1).$$

Then, arguing as in the previous case with $\frac{\check{m}_1 + m}{2}$ in place of m we find (53). This completes the proof. □

We are now able to prove Theorem 1.

Proof of Theorem 1. Being $|\{m_0 > 0\}| > 0$, we have

$$\lambda_1(m) = \frac{1}{\mu_1(m)} = \frac{1}{\widetilde{\mu}_1(m)}$$

for all $m \in \mathcal{G}(m_0)$. Therefore, (i) and (ii) immediately follow by Theorem 3.

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(iii) Given that $\int_\Omega m_0\,dx < 0$, then, by Proposition 13 and Proposition 15, the negative constant function $c = \frac{1}{|\Omega|}\int_\Omega m_0\,dx$ belongs to $\overline{\mathcal{G}(m_0)}$. Therefore, by definition of $\widetilde{\mu}_1(m)$, $\min_{m\in\overline{\mathcal{G}(m_0)}}\widetilde{\mu}_1(m) = 0$ which, in turns, being $\mathcal{G}(m_0)$ dense in $\overline{\mathcal{G}(m_0)}$ and $\widetilde{\mu}_1(m)$ sequentially weak* continuous, implies $\inf_{m\in\mathcal{G}(m_0)}\mu_1(m) = 0$ and, finally, $\sup_{m\in\mathcal{G}(m_0)}\lambda_1(m) = +\infty$. □

4. Monotonicity of the minimizers in cylinders

In this section, we consider the optimization problem (3) in cylindrical domains. Here, by (generalized) cylinder we mean a domain of the type $\Omega = (0,h)\times\omega \subset \mathbb{R}^N$, where $h > 0$ and $\omega \subset \mathbb{R}^{N-1}$ is a bounded polyhedral or smooth domain. In the sequel, for $x \in \mathbb{R}^N$ we will write $x = (x_1, x')$, with $x_1 \in \mathbb{R}$ and $x' = (x_2,\ldots,x_N) \in \mathbb{R}^{N-1}$. Exploiting the notion of monotone (decreasing and increasing) rearrangement, we will able to prove that in a cylinder any minimizer of problem (3) is monotone with respect to x_1. For a comprehensive survey of the monotone rearrangement we use here, we refer the reader to the work of Kawhol (see [32]) and Berestycki and Lachand-Robert (see [6]). In our paper, for the sake of simplicity, we choose to define this rearrangement only when the domain is a cylinder and, in order to deduce easily some of its properties, as a particular case of the Steiner symmetrization. For a brief summary of the Steiner symmetrization see [2].

Definition 2. *Let $\Omega = (0,h)\times\omega$ where $\mathrm{h} > 0$ and $\omega \subset \mathbb{R}^{N-1}$ is a bounded polyhedral or smooth domain, and $u : \Omega \to \mathbb{R}$ a measurable function bounded from below. Let* U *be the "*x_1*-even" extension of* u *onto $(-h,h)\times\omega$ obtained by reflection with respect to the hyperplane $\{x \in \mathbb{R}^N : x_1 = 0\}$ (i.e. $U(x_1,x') := U(-x_1,x')$, $x_1 \in (-h,0)$, $x' \in \omega$) and $U^\sharp$ its Steiner symmetrization relative to the same hyperplane. We define the* monotone decreasing rearrangement $u^\star : \Omega \to \mathbb{R}$ *of* u *to be the restriction of $U^\sharp$ on Ω.*

In a similar way it can be defined the *monotone increasing rearrangement* $u_\star$. Note that if $m \in \mathcal{G}(m_0)$, then $m^\star, m_\star \in \mathcal{G}(m_0)$.
From the theory of the Steiner symmetrization (see, for example [2]) and by Definition 2, we obtain the following first two properties of the monotone decreasing rearrangement.

a) Let $\Omega = (0,h)\times\omega$, ω as above, $u : \Omega \to \mathbb{R}$ be a measurable function bounded from below and $\Phi : \mathbb{R} \to \mathbb{R}$ an increasing function. Then

$$(\Phi(u))^\star = \Phi(u^\star) \quad \text{a.e. in } \Omega. \tag{55}$$

b) Let $\Omega = (0,h)\times\omega$, ω as above, $u, v : \Omega \to \mathbb{R}$ two measurable functions bounded from below such that $uv \in L^1(\Omega)$, then the *Hardy-Littlewood inequality* holds

$$\int_\Omega uv\,dx \le \int_\Omega u^\star v^\star\,dx. \tag{56}$$

Moreover, from [6, Theorem 2.8 and Lemma 2.10] and [32, Corollary 2.14] we have

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c) Let $\Omega = (0, h) \times \omega$, ω as above and $u \in H^1(\Omega)$ a nonnegative function. Then $u^\star \in H^1(\Omega)$ and the *Pòlya-Szegö inequality* holds

$$\int_\Omega |\nabla u^\star|^2\, dx \le \int_\Omega |\nabla u|^2\, dx. \tag{57}$$

More generally, we have

$$\int_\Omega |\nabla_{x'} u^\star|^2\, dx \le \int_\Omega |\nabla_{x'} u|^2\, dx \quad \text{and} \quad \int_\Omega |(u^\star)_{x_1}|^2\, dx \le \int_\Omega |u_{x_1}|^2\, dx. \tag{58}$$

An important ingredient of the next proof is the characterization of the equality case of (57), which is addressed in Theorem 3.1 of [6].

We prove Theorem 2.

Proof of Theorem 2. In what follows we use the ideas of the proof of Theorem 2 in [2]. Let $\check{m}$ be a minimizer of problem (3). By (ii) of Theorem 1, there exists an increasing function ψ such that $\check{m} = \psi(u_{\check{m}})$ a.e. in Ω, where $u_{\check{m}} \in V_{\check{m}}(\Omega)$ denotes the unique positive eigenfunction normalized by $\|u_{\check{m}}\|_{V_{\check{m}}(\Omega)} = 1$. Therefore, the monotonicity of $\check{m}$ is an immediate consequence of the monotonicity of $u_{\check{m}}$. In other words, it suffices to show that either $u_{\check{m}} = u^\star_{\check{m}}$ or $u_{\check{m}} = (u_{\check{m}})_\star$. By using (26) we find

$$\check{\mu}_1 = \mu_1(\check{m}) = \frac{\int_\Omega \check{m} u_{\check{m}}^2\, dx}{\int_\Omega |\nabla u_{\check{m}}|^2\, dx}.$$

The inequality (56), property (55) and Definition 1 yield

$$\int_\Omega \check{m} u_{\check{m}}^2\, dx \le \int_\Omega \check{m}^\star (u^\star_{\check{m}})^2\, dx = \int_\Omega \check{m}^\star \big(P_{\check{m}^\star}(u^\star_{\check{m}})\big)^2\, dx + \frac{1}{\int_\Omega \check{m}^\star\, dx} \left(\int_\Omega \check{m}^\star u^\star_{\check{m}}\, dx\right)^2$$

and (57) and (iv) of Proposition 1 give

$$\int_\Omega |\nabla u_{\check{m}}|^2\, dx \ge \int_\Omega |\nabla u^\star_{\check{m}}|^2\, dx = \int_\Omega |\nabla P_{\check{m}^\star}(u^\star_{\check{m}})|^2\, dx.$$

Note that, $\check{m}^\star \in \mathcal{G}(m_0)$ (in particular $\int_\Omega \check{m}^\star\, dx < 0$) and $P_{\check{m}^\star}(u^\star_{\check{m}}) \in V_{\check{m}^\star}$. Exploiting (26) and the maximality of $\check{\mu}_1$, we can write

$$\begin{aligned}
\check{\mu}_1 &= \frac{\int_\Omega \check{m} u_{\check{m}}^2\, dx}{\int_\Omega |\nabla u_{\check{m}}|^2\, dx} \le \frac{\int_\Omega \check{m}^\star (u^\star_{\check{m}})^2\, dx}{\int_\Omega |\nabla u^\star_{\check{m}}|^2\, dx} \\
&\le \frac{\int_\Omega \check{m}^\star \big(P_{\check{m}^\star}(u^\star_{\check{m}})\big)^2\, dx + \frac{1}{\int_\Omega \check{m}^\star\, dx} \left(\int_\Omega \check{m}^\star u^\star_{\check{m}}\, dx\right)^2}{\int_\Omega |\nabla P_{\check{m}^\star}(u^\star_{\check{m}})|^2\, dx} \\
&\le \frac{\int_\Omega \check{m}^\star \big(P_{\check{m}^\star}(u^\star_{\check{m}})\big)^2\, dx}{\int_\Omega |\nabla P_{\check{m}^\star}(u^\star_{\check{m}})|^2\, dx} \le \frac{\int_\Omega \check{m}^\star (u_{\check{m}^\star})^2\, dx}{\int_\Omega |\nabla u_{\check{m}^\star}|^2\, dx} = \mu_1(\check{m}^\star) \le \check{\mu}_1.
\end{aligned} \tag{59}$$

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Therefore, all the previous inequalities become equalities and yield $\int_\Omega \check{m}^\star u^\star_{\check{m}}\, dx = 0$, which implies $u^\star_{\check{m}} \in V_{\check{m}^\star}$, and

$$\int_\Omega \check{m} u^2_{\check{m}}\, dx = \int_\Omega \check{m}^\star (u^\star_{\check{m}})^2\, dx, \qquad \int_\Omega |\nabla u_{\check{m}}|^2\, dx = \int_\Omega |\nabla u^\star_{\check{m}}|^2\, dx. \tag{60}$$

Furthermore, by (26), $u^\star_{\check{m}}$ is an eigenfunction associated to $\mu_1(\check{m}^\star)$. By the simplicity of $\mu_1(\check{m}^\star)$, $u^\star_{\check{m}}$ being positive in Ω and, by (60), $\|u^\star_{\check{m}}\|_{V_{\check{m}^\star}(\Omega)} = \|u_{\check{m}}\|_{V_{\check{m}}(\Omega)} = 1$, we conclude that $u^\star_{\check{m}} = u_{\check{m}^\star}$. For simplicity of notation, we put $v = u^\star_{\check{m}} = u_{\check{m}^\star}$. By (59), $\check{m}^\star$ is a minimizer of (3) and v is the normalized positive eigenfunction of problem (1) associated to $1/\mu_1(\check{m}^\star) = \lambda_1(\check{m}^\star) = \check{\lambda}_1$. Moreover, by (ii) of Theorem 1, there exists an increasing function Ψ such that $\check{m}^\star = \Psi(v)$ a.e. in Ω. Thus v satisfies the problem

$$\begin{cases} -\Delta v = \check{\lambda}_1 \Psi(v) v & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{61}$$

Let $C^\infty_{0,+}(\Omega) = \{\varphi \in C^\infty_0(\Omega) : \varphi \text{ is nonnegative}\}$.

From (61) in weak form we have

$$\int_\Omega \nabla v \cdot \nabla \varphi_{x_1}\, dx = \check{\lambda}_1 \int_\Omega \Psi(v) v\, \varphi_{x_1}\, dx \quad \forall \varphi \in C^\infty_{0,+}(\Omega).$$

Being $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (see [27]), we can rewrite the previous equation as

$$-\int_\Omega \nabla v_{x_1} \cdot \nabla \varphi\, dx = \check{\lambda}_1 \int_\Omega \Psi(v) v\, \varphi_{x_1}\, dx.$$

Adding $\check{\lambda}_1 \int_\Omega \Psi(v) v_{x_1} \varphi\, dx$ to both sides and since $v \in C^{1,\beta}(\Omega)$ for all $\beta \in (0,1)$ (see [27]), it becomes

$$-\int_\Omega \nabla v_{x_1} \cdot \nabla \varphi\, dx + \check{\lambda}_1 \int_\Omega \Psi(v) v_{x_1}\, \varphi\, dx = \check{\lambda}_1 \int_\Omega \Psi(v) (v\, \varphi)_{x_1}\, dx. \tag{62}$$

Let us show that $\int_\Omega \Psi(v)(v\,\varphi)_{x_1}\, dx \geq 0$. By Fubini's Theorem we get

$$\int_\Omega \Psi(v)(v\,\varphi)_{x_1}\, dx = \int_\omega dx' \int_0^h \Psi(v)(v\,\varphi)_{x_1}\, dx_1. \tag{63}$$

For any fixed $x' \in \omega$, let $\alpha = \alpha(x_1) = v(x_1, x')\, \varphi(x_1, x')$. Since φ has compact support, we can consider α trivially defined on the whole $[0, h]$. Since $\alpha(x_1)$ is continuous and $\Psi(v)$ is decreasing with respect to x_1, the Riemann-Stieltjes integral $\int_0^h \Psi(v)\, d\alpha(x_1)$ is well defined (see Theorem 7.27 and the subsequent note in [4]).

Moreover, by using [4, Theorem 7.8] we have

$$\int_0^h \Psi(v)(v\,\varphi)_{x_1}\,dx_1 = \int_0^h \Psi(v)\,d\alpha(x_1). \tag{64}$$

By [4, Theorems 7.31 and 7.8] there exists a point x_0 in $[0,h]$ such that

$$\begin{aligned} -\int_0^h \Psi(v)\,d\alpha(x_1) &= -\Psi(v(0,x'))\int_0^{x_0} d\alpha(x_1) - \Psi(v(h,x'))\int_{x_0}^h d\alpha(x_1) \\ &= -\Psi(v(0,x'))\int_0^{x_0}(v\,\varphi)_{x_1}\,dx_1 - \Psi(v(h,x'))\int_{x_0}^h (v\,\varphi)_{x_1}\,dx_1. \end{aligned}$$

Computing the integrals and recalling that $\varphi \in C^\infty_{0,+}(\Omega)$, v is positive and $\Psi(v)$ is decreasing with respect to x_1, we conclude that

$$-\int_0^h \Psi(v)\,d\alpha(x_1) = v(x_0,x')\varphi(x_0,x')\left[\Psi(v(h,x')) - \Psi(v(0,x'))\right] \le 0.$$

Therefore, by the previous inequality and (64) it follows $\int_0^h \Psi(v)(v\,\varphi)_{x_1}\,dx_1 \ge 0$ for any $x' \in \omega$ and, in turn, from (63) we obtain $\int_\Omega \Psi(v)(v\,\varphi)_{x_1}\,dx \ge 0$. Hence, by (62), v_{x_1} satisfies the differential inequality

$$\Delta v_{x_1} + \check{\lambda}_1\Psi(v)v_{x_1} \ge 0 \quad \text{in } \Omega$$

in weak form. Then, applying [57, Theorem 2.5.3] and since $v_{x_1} \le 0$ in Ω, we conclude that either $v_{x_1} \equiv 0$ or $v_{x_1} < 0$.

In the first case, v, and then $u_{\check{m}}$, is constant with respect to x_1.

Let $v_{x_1} < 0$. By the second equality of (60) and (58) we obtain

$$\int_\Omega |(u_{\check{m}})_{x_1}|^2\,dx = \int_\Omega |(u^\star_{\check{m}})_{x_1}|^2\,dx.$$

By Fubini's Theorem it becomes

$$\int_\omega dx' \int_0^h |(u_{\check{m}})_{x_1}|^2\,dx_1 = \int_\omega dx' \int_0^h |(u^\star_{\check{m}})_{x_1}|^2\,dx_1. \tag{65}$$

Being $u_{\check{m}}$ and $u^\star_{\check{m}} = u_{\check{m}^\star}$ of class $C^{1,\beta}(\Omega)$, the functions $x' \mapsto \int_0^h |(u_{\check{m}})_{x_1}|^2\,dx_1$ and $x' \mapsto \int_0^h |(u^\star_{\check{m}})_{x_1}|^2\,dx_1$ are continuous on ω. Therefore, using identity (65) and (57) in the one dimensional case we find

$$\int_0^h |(u_{\check{m}})_{x_1}|^2\,dx_1 = \int_0^h |(u^\star_{\check{m}})_{x_1}|^2\,dx_1 \quad \forall x' \in \omega.$$

From Theorem 3.1 in [6], again in the one dimensional case, we conclude that for all $x' \in \omega$ either $u_{\check{m}} = u^\star_{\check{m}}$ or $u_{\check{m}} = (u_{\check{m}})_\star$. This implies that for all $x' \in \omega$ either $(u_{\check{m}})_{x_1} = (u^\star_{\check{m}})_{x_1} = v_{x_1} < 0$ or $(u_{\check{m}})_{x_1} = ((u_{\check{m}})_\star)_{x_1} = -(u^\star_{\check{m}})_{x_1} = -v_{x_1} > 0$. Being $(u_{\check{m}})_{x_1}$ continuous in Ω and Ω an open connected set, it follows that $(u_{\check{m}})_{x_1}$ does not change sign in Ω. Equivalently either $u_{\check{m}} = u_{\check{m}}^{\star}$ or $u_{\check{m}} = (u_{\check{m}})_{\star}$ in the whole Ω. Finally, by $\check{m} = \psi(u_{\check{m}})$, we conclude that either $\check{m} = \check{m}^{\star}$ or $\check{m} = \check{m}_{\star}$. This proves the theorem. □

REMARK 3. Similar results on the monotonicity of the minimizers can be found in [30, Theorem 2.4] in the one dimensional case and for an arbitrary m_0 and in [33, Proposition 5] in general dimension, for an orthotope and m_0 of "bang-bang" type. Both the previous results can be obtained from Theorem 2. We also mention that other qualitative features of the minimizers are known. We refer the reader again to [30], where a symmetry breaking result in dimension two when Ω is an annulus is given, and to [33] for further qualitative properties of the minimizers in the case of the orthotope and the ball. In particular, in this last case it is proved that a minimizer cannot be a ball concentric to Ω. Finally, it is worth noting that the monotonicity property stated in Theorem 2 has also been numerically observed by some authors (see [31, 33, 42]).

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Appendix A. Rearrangements of measurable functions

In this appendix we introduce the concept of rearrangement of a measurable function and summarize some related results we use in the previous sections. The idea of rearranging a function dates back to the book [28] of Hardy, Littlewood and Pólya, since than many authors have investigated both extensions and applications of this notion. Here we relies on the results in [1, 10, 11, 20, 32, 43].

Let Ω be an open bounded set of $\mathbb{R}^N$.

Definition 3. *For every measurable function* $f:\Omega\to\mathbb{R}$ *the function* $d_f:\mathbb{R}\to[0,|\Omega|]$ *defined by*

$$d_f(t)=|\{x\in\Omega : f(x)>t\}|$$

is called distribution function of f.

The symbol μ_f is also used. It is easy to prove the following properties of d_f.

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PROPOSITION 9. *For each* f *the distribution function* d_f *is decreasing, right continuous and the following identities hold true*

$$\lim_{t\to-\infty} d_f(t) = |\Omega|, \qquad \lim_{t\to\infty} d_f(t) = 0.$$

DEFINITION 4. *Two measurable functions* $f, g : \Omega \to \mathbb{R}$ *are called* equimeasurable *functions or* rearrengements *of one another if one of the following equivalent conditions is satisfied*

(i) $|\{x \in \Omega : f(x) > t\}| = |\{x \in \Omega : g(x) > t\}| \quad \forall\, t \in \mathbb{R}$;
(ii) $d_f = d_g$.

Equimeasurability of f and g is denoted by $f \sim g$. Equimeasurable functions share global extrema and integrals as it is stated precisely by the following proposition.

PROPOSITION 10. *Suppose* $f \sim g$ *and let* $F : \mathbb{R} \to \mathbb{R}$ *be a Borel measurable function, then*

i) $|f| \sim |g|$;
ii) $\operatorname{ess\,sup} f = \operatorname{ess\,sup} g$ *and* $\operatorname{ess\,inf} f = \operatorname{ess\,inf} g$;
iii) $F \circ f \sim F \circ g$;
iv) $F \circ f \in L^1(\Omega)$ *implies* $F \circ g \in L^1(\Omega)$ *and* $\int_\Omega F \circ f\, dx = \int_\Omega F \circ g\, dx$.

For a proof see, for example, [20, Proposition 3.3] or [11, Lemma 2.1].
In particular, for each $1 \le p \le \infty$, if $f \in L^p(\Omega)$ and $f \sim g$ then $g \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} = \|g\|_{L^p(\Omega)}.$$

DEFINITION 5. *For every measurable function* $f : \Omega \to \mathbb{R}$ *the function* $f^* : (0, |\Omega|) \to \mathbb{R}$ *defined by*

$$f^*(s) = \sup\{t \in \mathbb{R} : d_f(t) > s\}$$

is called decreasing rearrangement of f.

An equivalent definition (used by some authors) is $f^*(s) = \inf\{t \in \mathbb{R} : d_f(t) \le s\}$.

PROPOSITION 11. *For each* f *its decreasing rearrangement* f^* *is decreasing, right continuous and we have*

$$\lim_{s\to 0} f^*(s) = \operatorname{ess\,sup} f \quad \textit{and} \quad \lim_{s\to|\Omega|} f^*(s) = \operatorname{ess\,inf} f.$$

Moreover, if $F : \mathbb{R} \to \mathbb{R}$ *is a Borel measurable function then* $F \circ f \in L^1(\Omega)$ *implies* $F \circ f^* \in L^1(0, |\Omega|)$ *and*

$$\int_\Omega F \circ f\, dx = \int_0^{|\Omega|} F \circ f^*\, ds.$$

Finally, $d_{f^*} = d_f$ *and, for each measurable function* g *we have* $f \sim g$ *if and only if* $f^* = g^*$.

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Some of the previous claims are simple consequences of the definition of f^*, for more details see [20, Chapter 2].

As before, it follows that, for each $1 \le p \le \infty$, if $f \in L^p(\Omega)$ then $f^* \in L^p(0, |\Omega|)$ and $\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p(0,|\Omega|)}$.

Definition 6. *Given two functions $f, g \in L^1(\Omega)$, we write $g \prec f$ if*

$$\int_0^t g^*\,ds \le \int_0^t f^*\,ds \quad \forall\, 0 \le t \le |\Omega| \qquad \text{and} \qquad \int_0^{|\Omega|} g^*\,ds = \int_0^{|\Omega|} f^*\,ds.$$

Note that $g \sim f$ if and only if $g \prec f$ and $f \prec g$. Among many properties of the relation $\prec$ we mention the following (a proof is in [20, Lemma 8.2]).

Proposition 12. *For any pair of functions $f, g \in L^1(\Omega)$ and real numbers α and β, if $\alpha \le f \le \beta$ a.e. in Ω and $g \prec f$ then $\alpha \le g \le \beta$ a.e. in Ω.*

Proposition 13. *For $f \in L^1(\Omega)$ let $g = \frac{1}{|\Omega|}\int_\Omega f\,dx$. Then we have $g \prec f$.*

Definition 7. *Let $f : \Omega \to \mathbb{R}$ a measurable function. We call the set*

$$\mathcal{G}(f) = \{g : \Omega \to \mathbb{R} : g \text{ is measurable and } g \sim f\}$$

the class of rearrangements of f *or the* set of rearrangements of f.

Note that, for $1 \le p \le \infty$, if f is in $L^p(\Omega)$ then $\mathcal{G}(f)$ is contained in $L^p(\Omega)$.

In this paper we are interested in studying the optimization of a functional which is defined on a class of rearrangements $\mathcal{G}(m_0)$, where m_0 belongs to $L^\infty(\Omega)$. For this reason, although almost all of what follows holds in a much more general context, hereafter we restrict our attention to classes of rearrangements of functions in $L^\infty(\Omega)$. We need compactness properties of the set $\mathcal{G}(m_0)$, with a little effort it can be shown that this set is closed but in general it is not compact in the norm topology of $L^\infty(\Omega)$. Therefore we focus our attention on the weak* compactness. By $\overline{\mathcal{G}(m_0)}$ we denote the closure of $\mathcal{G}(m_0)$ in the weak* topology of $L^\infty(\Omega)$.

Proposition 14. *Let m_0 be a function of $L^\infty(\Omega)$. Then $\overline{\mathcal{G}(m_0)}$ is*

i) *weakly* compact;*
ii) *metrizable in the weak* topology;*
iii) *sequentially weakly* compact.*

For the proof see [3, Proposition 3.6].

Moreover, the sets $\mathcal{G}(m_0)$ and $\overline{\mathcal{G}(m_0)}$ have further properties.

Definition 8. *Let* C *be a convex set of a real vector space. An element* v *in* C *is said to be an* extreme point of C *if for every* u *and* w *in* C *the identity $v = \frac{u+w}{2}$ implies* u = w.

A vertex of a convex polygon is an example of extreme point.

Proposition 15. *Let m_0 be a function of $L^\infty(\Omega)$, then*

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i) $\overline{\mathcal{G}(m_0)} = \{f \in L^\infty(\Omega) : f \prec m_0\}$,
ii) $\overline{\mathcal{G}(m_0)}$ *is convex,*
iii) $\mathcal{G}(m_0)$ *is the set of the extreme points of* $\overline{\mathcal{G}(m_0)}$.

Proof. The claims follow from [20, Theorems 22.13, 22.2, 17.4, 20.3]. □

An evident consequence of the previous theorem is that $\overline{\mathcal{G}(m_0)}$ is the weakly* closed convex hull of $\mathcal{G}(m_0)$.

Corollary 3. *Let* $m_0 \in L^\infty(\Omega)$ *and* $m, q \in \overline{\mathcal{G}(m_0)}$. *Then*

i) $\int_\Omega m\,dx = \int_\Omega m_0\,dx$;
ii) assuming $\int_\Omega m_0\,dx \neq 0$, $\mathrm{m} = \mathrm{q}$ *if and only if* m *and* q *are linearly dependent.*

Proof. (i) It follows immediately by (i) of Proposition 15, Definition 6 and Proposition 11 with F equal to the identity function.

(ii) If m and q are linearly dependent, then, without loss of generality we can assume that $m = \alpha q$, for some $\alpha \in \mathbb{R}$. Integrating over Ω and using (i) we find $m = q$. □

The following is [20, Theorem 11.1] rephrased for our case.

Proposition 16. *Let* $u \in L^1(\Omega)$ *and* $m_0 \in L^\infty(\Omega)$. *Then*

$$\int_0^{|\Omega|} m_0^*(|\Omega| - s)u^*(s)\,ds \leq \int_\Omega m\,u\,dx \leq \int_0^{|\Omega|} m_0^*(s)u^*(s)\,ds \qquad \forall m \in \mathcal{G}(m_0), \tag{66}$$

and moreover both sides of (66) are achieved.

The previous proposition implies that the linear optimization problems

$$\sup_{m \in \mathcal{G}(m_0)} \int_\Omega mu\,dx \tag{67}$$

and

$$\inf_{m \in \mathcal{G}(m_0)} \int_\Omega mu\,dx$$

admit solution.

Finally, we recall the following result proved in [10, Theorem 5].

Proposition 17. *Let* $u \in L^1(\Omega)$ *and* $m_0 \in L^\infty(\Omega)$. *If problem (67) has a unique solution* $\mathrm{m_M}$, *then there exists an increasing function* ψ *such that* $m_M = \psi \circ u$ *a.e. in* Ω.

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