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RESEARCH ARTICLE

The interplay between textual procedures and material operations from the viewpoint of Chinese mathematical texts

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Argument
For more than 2,000 years, counting rods were the main tool used in Chinese mathematics. However, direct evidence for their use is lacking. The current evidence is primarily derived from two sources: procedural texts in ancient mathematical writings and counting diagrams drawn with rod signs in thirteenth-century writings. This study analyzes the procedural texts in two ancient Chinese mathematical books: 1) *The Nine Chapters on Mathematical Procedures*, completed by approximately 100 BCE or 100 CE, and 2) the *Mathematical Canon by Master Sun*, completed by approximately 400 CE. This article argues that the differences between the texts insufficiently explain the fundamental differences in the operations that could be performed with mathematical rods. Further, by examining two mathematical books from the thirteenth century, namely the *Mathematical Book in Nine Chapters* written by Qin Jiushao in 1247 and *Fast Methods on Various Categories of Multiplication and Division of Areas of Fields* written by Yang Hui in 1275, this article argues that the relationships between counting diagrams and their accompanying text vary depending on the author, thereby highlighting authors' different epistemological perspectives. Examining the historical context is essential for understanding the relationship between procedural texts and material operations and for developing new methods to investigate the use of counting rods.

Keywords: *The Nine Chapters on Mathematical Procedures*; *Mathematical Canon by Master Sun*; Qin Jiushao; Yang Hui; mathematical rods; mathematical diagrams

1. Introduction

The Nine Chapters on Mathematical Procedures (*Jiuzhang Suanshu* [九章筭術], ca. 100 BCE or 100 CE,[1] hereafter *The Nine Chapters*) is seen as the most important classical mathematical text from ancient China.[2] Nearly a hundred procedures (written in Chinese) were recorded in *The Nine Chapters*. Historians of mathematics assume that all these procedures were carried out with

[1]There is a debate on when *The Nine Chapters* were completed. One school of thought says it was completed around 100 BCE, i.e., during the Western Han dynasty; another says it was completed around 100 CE, i.e., during the Eastern Han dynasty. Since this issue is not the topic of the article, I do not discuss the details here.

[2]As far as we know, *The Nine Chapters* had the position of the classical text no later than 179 CE, when it was first recorded in the sentences carved in the official copper vessel published by the central government of imperial China, i.e., the Han dynasty (202 BCE–220 CE).

one particular instrument, namely, counting rods (*suan chou* [筭籌]).[3] From antiquity, counting rods served as the primary mathematical tool for Chinese scholars until the abacus rose to popularity in the sixteenth century.[4] However, *The Nine Chapters* does not contain much information about how counting rods were used.[5]

During the fifth and sixth centuries, China was divided into two regions, one controlled by the Southern dynasties (420–589 CE) and one controlled by the Northern dynasties (439–581 CE). Three mathematical books from northern China, the *Mathematical Canon by Master Sun* (*Sunzi Suanjing* [孫子筭經], ca. 400 CE, hereafter *Master Sun*), the *Mathematical Canon by Zhang Qiujian* (*Zhang Qiujian Suanjing* [張丘建筭經], ca. 431–450 CE, hereafter *Zhang Qiujian*), and the *Mathematical Canon by Xiahou Yang* (*Xiahou Yang Suanjing* [夏侯陽筭經], hereafter *Xiahou Yang*)[6] recorded fundamental mathematical knowledge about the uses of counting rods, such as for subtraction, multiplication, division, and fraction operations.[7] In 656 CE, Li Chunfeng (李淳風, 602–670 CE), an official scholar, with his colleagues, edited, sub-commented, and compiled the four already mentioned texts (i.e., *The Nine Chapters*, *Master Sun*, *Zhang Qiujian*, and *Xiahou Yang*), along with six others, to create the *Ten Mathematical Classics* (*shi bu suan jing* [十部筭經]).[8] The *Ten Mathematical Classics*, together with another two elementary books related to number systems, served as textbooks in the School of Mathematics of the Imperial University of the Tang dynasty (618–907 CE).[9]

Recent studies have shown that since the fifth century, when official scholars, such as Huang Kan (皇侃, 488–545 CE), Kong Yingda (孔穎達, 574–648 CE), and Jia Gongyan (賈公彥, fl. 650–655 CE), commented and sub-commented on Confucian classics, counting rods were no longer widely used.[10] However, in the domain of mathematical texts, counting rods continued to serve as the main instrument for mathematical computations.[11] Moreover, official scholars of the Tang dynasty were required to take counting sacks (*suan dai* [筭袋], which were used to store counting rods) when they traveled to the imperial court. This shows that counting rods remained a symbol of mathematics in imperial China. For the modern study of the history of mathematics, the aforementioned four books constitute a major source of information. Historians of mathematics

[3]For an introduction to the numerals used in counting rods, see Li Y. 1955a; Needham 1959, 9–10; Li Y. and Du 1987, 6–11; and Martzloff 2006, 210–211.

[4]Many books written in the sixteenth century – such as *Suanfa Zongtong* (筭法統宗 *General Source of Mathematical Methods*, 1592) – included images of the abacus, suggesting that people used the abacus to compute. On the other hand, we do not have a record of any mathematical book written before the sixteenth century that contains such an image. These facts are the main evidence of the primary use of counting rods until the sixteenth century, when the abacus became more popular.

[5]*The Nine Chapters* does not record how addition, subtraction, multiplication, division, and fraction operations were carried out with counting rods. It does not contain enough information for us to deduce today how counting rods were placed and operated in different procedures. In *The Nine Chapters*, there is a procedure for square root extraction and a procedure for cubic root extraction using counting rods. However, the information regarding the operations was insufficient, as we will see in this article.

[6]The original text of *Xiahou Yang* was lost – only about 600 characters were handed down through the written tradition. The extant main text of *Xiahou Yang* handed down through the written tradition was in fact written by Han Yan 韓延 in the eighth century. See Qian 1963, 551–553.

[7]I have argued elsewhere (Zhu 2021) that the mathematical books and practices were different in the two parts of China. The three mathematical books written in northern China focused on writing down the fundamental mathematical knowledge (including the usage of counting rods) and new types of problems that were not presented in *The Nine Chapters*. Hence, the three northern books retained some pieces of evidence about how counting rods were used. By contrast, those scholars specializing in mathematics in southern China, such as He Chengtian (何承天, 370–447 CE), Zu Chongzhi (祖沖之, 429–500 CE), and Zu Gengzhi (祖暅之, 456–536 CE), continued to study and develop procedures based on *The Nine Chapters* and Liu Hui's 劉徽 commentary (written in 263 CE).

[8]For Li Chunfeng et al.'s work on the *Ten Mathematical Classics*, see Zhu 2019a.

[9]Regarding mathematics education in the early Tang dynasty, see Volkov 2012.

[10]The reasons Confucian scholars did not use counting rods are complex, involving many aspects. For details on this issue, see Zhu 2016, 2022, 2023.

[11]In fact, in the seventh century, there were two cultures of computations. One used counting rods in the domain of mathematical texts; the other did not rely on counting rods in the domain of Confucian texts. See Chemla and Zhu 2022.

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have utilized these texts as a foundation for investigating how mathematical procedures were conducted using rods, highlighting the importance of counting rods in Chinese mathematics.[12] Nevertheless, these previous studies have only partially revealed the interplay between textual procedures and material operations.

In the eleventh century, the central government of the Song dynasty (960–1279) continued to support the School of Mathematics at the Imperial University. The four books were also used as textbooks in the School of Mathematics. However, the capital of the Song dynasty (the city of Kai Feng [開封] in present-day China) was captured by the Jin dynasty army (1115–1234) in 1127. Thereafter, the Song dynasty relocated to the south of China (the capital was Hang Zhou [杭州] in present-day China), dividing China into two parts. As a result, numerous mathematical books were scattered and lost, and the School of Mathematics was never rebuilt during the reign of the southern Song dynasty. During his collection and printing of mathematical books in 1213, Bao Huanzhi (鮑澣之) in the southern region discovered *The Nine Chapters*, *Master Sun*, *Zhang Qiujian*, and other books, but failed to find the *Xia Houyang*.[13] In the thirteenth century, scholars specialized in mathematics, such as Qin Jiushao (秦九韶, 1208–ca. 1268) and Yang Hui (楊輝, fl. 1261–1275) in the south, as well as Li Ye (李冶, 1192–1279) in the north, all wrote down counting diagrams (*suan tu* [筭图]) in their mathematical writings (see figure 3 and tables 2 and 3), which were related to how counting rods were used on the computational surface. Notably, Yang Hui's treatises (written from 1261 to 1275) reveal that he read *The Nine Chapters* and *Master Sun*. By contrast, Qin Jiushao's mathematical knowledge was mainly derived from calendarists. His own preface in his *Mathematical Book in Nine Chapters* (*Shushu Jiuzhang* [數書九章], hereafter *Mathematical Book*) from 1247 shows that he also read *The Nine Chapters*. However, it seems that Qin did not know *Master Sun*.[14] These thirteenth-century scholars followed the tradition of using counting rods and added counting diagrams in their writings.

Chemla (2010) has convincingly argued that diagrams from the third century are material objects, whereas those from the thirteenth century were written on the surface of the page. Chemla (2018) has also argued that geometrical diagrams were used to ensure the correctness of certain procedures in one of Yang Hui's treatises. While her study did not extend to discuss counting diagrams, I concur with her assertion in a broader sense. I have further claimed (Zhu 2020a) that the use of counting diagrams in the thirteenth century, in particular in Qin Jiushao's treatise, represents an intermediate phase in textualization and symbolization in Chinese mathematics. Therefore, for the modern study of the history of mathematics, counting diagrams in thirteenth-century mathematical writings comprise another major source for investigating counting rods operations, a topic which still requires comprehensive research.[15]

[12]Based on *Master Sun* and *The Nine Chapters*, Qian (1964, 7–13, 46–56), Li Y. and Du (1987) explain the operations of addition, subtraction, multiplication, division, and square and cubic root extractions carried out with counting rods. Wu (1987) summarizes the features of Chinese mathematics as "constructive" and "mechanical." That is to say, the textual procedures are constructive, and the operations carried out with counting rods are mechanical. Li J. (1993, 38) argues that a primary characteristic of Chinese mathematics is that "mathematical reasoning resides in counting rod computations" (*yu li yu suan*寓理於算). In fact, his claim was that one can only understand mathematical reasoning by doing computations. Chemla (2005) discovered a difference between the textual procedure and the material operation when calculating the area of a circle. I have studied how the textual procedures were written subject to the operations carried out with counting rods (Zhu 2009, 2010, 2020b).

[13]In fact, Bao Huanzhi found another book written by Han Yan in the eighth century. He misrecognized it as *Xia Houyang* and printed it.

[14]The first problem about the Chinese Remainder Theorem was recorded in the *Master Sun*. Qin Jiushao's treatise also contributed a lot to the Chinese Remainder Theorem. However, he never mentions *Master Sun*. For this issue, see Zhu 2011, 2017.

[15]The rod numerals were more widely discussed in the thirteenth-century writings. This is the main evidence regarding how counting rods were used to represent numbers. Moreover, since the abacus is still used today, and since it is believed that the abacus was derived from counting rods, the operational similarities between counting rods and the abacus were also used to study counting rods operations. For example, see Mikami 1913, 14, 27; Qian 1932, 257; Li Y. and Du 1987, 12.

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More specifically, further research is necessary for the following reasons. First, some technical details regarding material operations have not been carefully analyzed.[16] Second, counting diagrams in thirteenth-century texts and their relation to procedural texts and material operations have not been fully investigated.[17] Third, the operational similarity between counting rods and the abacus needs to be reconsidered in the context of the Ming dynasty (1368–1644).[18] Moreover, modern historians of mathematics usually consider counting rods as a tool for representing numbers (referred to as "rod numerals") and performing calculations (i.e., addition, subtraction, multiplication, and division). Consequently, they often compare counting rods to the abacus and modern Hindu-Arabic numerals, without revealing the additional functions of counting rods (e.g. Needham 1959, 5–17, 70–72; Lam and Ang 2004, 43–41, 54–56; Martzloff 2006, 185–190, 210–211). Modern historians of mathematics often use modern notations to interpret ancient procedures, overlooking the possibility that this approach might differ from the actual historical operations with counting rods.[19] In summary, all previous studies have relied on an unwritten common hypothesis, namely, that the operations carried out with counting rods and the procedural texts have a basic one-to-one correspondence, thereby allowing historians to investigate counting rods operations based on the relevant sources.

However, counting rods were not only used to represent numbers and do calculations; they also had other functions, such as determining positions, as will be discussed in this article. Furthermore, textual procedures were sometimes designed and written down based on the operations of counting rods.[20] In addition, procedural texts did not always record all the details of counting rod operations.[21] Indeed, even the translation of "*suan chou*" 筭筹 as "counting rods" implies that rods were used for counting. However, "*suan*" should be understood as relating to mathematics; hence, "mathematical rods" may be a more correct translation for *suan chou*, which also implies that the rods were a part of the mathematical knowledge system, not just a simple tool.[22] More specifically, sufficient direct and detailed evidence of how the counting rods were used for computations is lacking. Therefore, this article argues that this unwritten common

[16]For example, no mathematical writings record how one moved three rods and four rods, and made the addition 3+4. Therefore, it is hard to study these operational details. In fact, in all modern publications, there is no discussion of these details. One might think these details were too trivial to write down. However, similar technical details on the abacus were written down during the Ming dynasty (1368–1644).

[17]For example, Qin Jiushao's *Mathematical Book in Nine Chapters* contains many counting diagrams (see figure 3), which used rod signs to represent mathematical operations carried out with counting rods. These counting diagrams are extremely useful for our understanding on how counting rods were used. However, these diagrams have not been fully studied. See Zhu 2020a.

[18]For example, recent studies have shown a relationship between how the abacus replaced counting rods and the introduction of Western written calculations; this should be examined further. See Zhu 2018 and Jami 2019. For a complete discussion on the use of the Chinese abacus, see Guo et al. 2010, 568–572; Li Y. 1955b; and Needham 1959, 74–80.

[19]This problem appears in the following works: Needham 1959; Qian 1963; Li Y. and Du 1987; Lam and Ang 2004; Martzloff 2006; Guo et al. 2010.

[20]For example, in ancient texts, the term "additionally placing [counting rods]" (副置*fu zhi*) usually appears. The operation corresponding to this term is just to copy and place some additional rods to record the same result. The aim of the operation is to preserve the previous numbers represented by the counting rods, that is to ensure that these numbers are not negated in later rounds. However, since all numbers with modern notations are written down on paper, they will not all disappear. For details, see Zhu 2010, 2010.

[21]For example, numbers represented by counting rods have two forms, namely vertical signs (A) and horizontal signs (B), as shown in table 1. A number should have a form as … … BABA. However, the procedural texts never tell the readers whether the form needs to be changed or not when a number is moved. For example, if a number is moved to the left for one position, should the form be changed to … … ABAB or not? This kind information is very important – without it, we cannot recover any ancient operation. However, in a diagram in his treatise, Qin Jiushao writes down the changed form when the number is moved. For the details, see Zhu 2020a, 366).

[22]The character *suan*筭 has an original meaning of mathematical rods. When the eighteenth-century Chinese scholars compiled and edited the ancient writings, they did not know how to use counting rods, and hence changed the character筭 to another character *suan*算, which means to count or to compute. This also caused misunderstandings regarding mathematical rods.

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hypothesis may not always hold true – the relationship between the textual and material practices could vary and, in some cases, could be very different. Therefore, reconsidering this hypothesis is necessary.

The aim of this article is twofold. First, it seeks to analyze the complex relationships between texts and operations, which heavily depend on their historical context. Second, it aims to contribute methodologically by providing a new perspective to investigate visual and material cultures in the history of mathematics, an area that holds significant interest in the history of science.[23] To achieve this aim, I compare *The Nine Chapters* and *Master Sun* to analyze the relationship between procedural texts and material operations. I then compare the counting diagrams in Qin Jiushao's and Yang Hui's writings to analyze the relationship between textual diagrams and material operations. In each of these sources, I focus on a single procedure: root extraction, a procedure which held a special position in mathematics of the period. On the one hand, root extraction was usually viewed as a type of division in ancient Chinese mathematics.[24] Hence, it can be considered as belonging to one of the basic four operations (i.e., addition, subtraction, multiplication, and division). On the other hand, it was developed to solve linear equations with one unknown value and higher degrees in the thirteenth century. This could explain why Qin and Yang used the counting diagrams for this procedure as examples of operations using counting rods.

2. The relationship between operations carried out with counting rods and textual procedures: *The Nine Chapters* and *Master Sun*

The earliest record of counting rods being used to execute the square root extraction procedure comes from *The Nine Chapters*. This procedure was called *kai fang* (開方 literally, "to open/establish a square"). From a modern viewpoint, this procedure relied on equality $(a+b)^2 = a^2+2ab+b^2$, whereas its geometrical meaning could be understood as computing the length of the sides of a square with a given area.[25] The *Master Sun* was completed later than *The Nine Chapters*. The former contains two problems involving a square root extraction procedure using counting rods. Some scholars have argued that the procedure in *Master Sun* improved upon the one in *The Nine Chapters*.[26] The two procedures were not the same.

Generally, the procedures described in each book can be viewed as a series of steps that compute each digit in the results, from the highest to lowest order (e.g., hundreds, tens, and units) one by one. For example, in *The Nine Chapters*, the author extracts the square root of 55,225, which has an integer result of 235, and obtained "2," "3," and "5" in successive order. In *Master Sun*, the author extracted the square root of 234,567, which has an integer result of 484, and successively obtained "4," "8," and "4." Scholars believe that there are two main operational differences between the procedures depicted in *The Nine Chapters* and those in *Master Sun*: first, the methods necessary to determine the positions in the first step, and second, the methods necessary to determine positions in the following steps. Both relate to the movements of a counting rod (called *jie suan* [借筭], literally "a borrowed counting rod").[27] The differences can be compared using the example of the radicand (i.e., dividend) 234,567 in *Master Sun*.

[23]For example, the Ninth Conference of the European Society for the History of Science (held from August 31 to September 3, 2020) was on "Visual, Material and Sensory Cultures of Sciences."

[24]For the connection between division and square and cubic root extraction in ancient China, see Lam and Ang 2004, 103–105.

[25]For the details of this procedure, see Mikami 1913, 29–32; Needham 1959, 65–68; Berezkina 1980, 207–223; Li Y. and Du 1987, 50–53; Chemla 1987, 1994; Martzloff 2006, 221–224.

[26]See Wang L. and Needham 1955, 390; Xu 1986, 1987; Chemla 1987, 1994. Ji (1999, 37–44) and Lam and Ang (2004, 93) followed this argument.

[27]*Jie suan* literally should be translated as a borrowed (*jie*) counting rod (*suan*). It is used to determine the positions of the quotients. The word "borrowed" is used here because it was taken from the collection of counting rods (i.e., a place to collect

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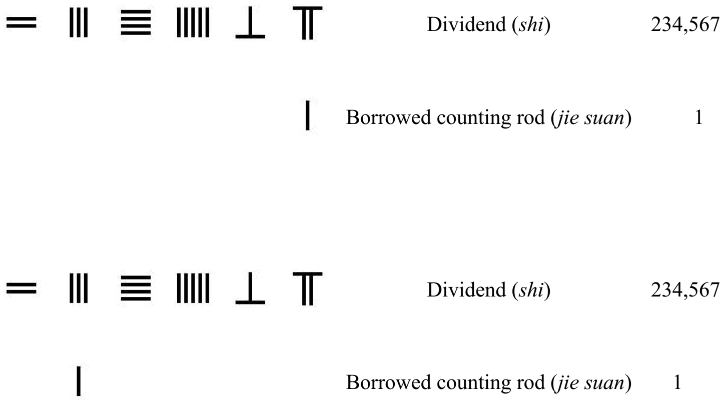

Figure 1. Movement of the borrowed counting rod in the first round of the square root extraction procedure.

The Nine Chapters and *Master Sun* include different operations for the first step. *The Nine Chapters* states: "The procedure for extracting the square root is: Put down the area (*ji*積) as the dividend (*shi*實).[28] Borrow a counting rod and move it, skipping every other position (*deng*等)."[29] If the radicand is 234,567, one uses the counting rods to represent this value on the computational surface (figure 1). This is what the phrase "put down the area as the dividend" means. The square root must be less than 1,000 because $234{,}567 < 1000^2$. One must move the borrowed rods from the ones place to the ten-thousands place, i.e., from the place under the number 7 to the place under the number 3. This is what the instruction "borrow a counting rod and move it, skipping every other position" indicates. Although historians of mathematics have different explanations of the term "*deng*," they all agree that the borrowed counting rods should be moved to the ten-thousands place.[30] Figure 1 shows the operation that determines the positions in the first step according to *The Nine Chapters*.

counting rods) at the beginning, and should be returned to the collection after it finished its function. In the second round of computation (as we will see), the rod would be borrowed again. The borrowed rod was called *xia fa* (the lower divisor) in the *Master Sun*, because according to the *Master Sun* procedure, one need not return it to the collection of counting rods.

[28]Since the square root extraction was viewed as a kind of division, the radicand was also called the dividend.

[29]The original text is: 開方術曰：置積爲實。步之，超一等。(*Jiuzhang suanjing* 1980, 59a). There are several translations for *The Nine Chapters*. For example, see Chemla and Guo 2004 for a French translation and Guo, Dauben, and Xu 2013 for an English one.

[30]Most scholars believe "*deng*" to be similar to "*wei*" (position), and one *deng* to mean one position. Therefore, 超一等*chao yi deng*, i.e. "skips every other position," means skip the tens position, the thousands position, etc. The following all support this viewpoint: Mikami 1913, 13; Wang L. and Needham 1955, 354; Qian 1964, 47; Guo 1992, 29; Chemla and Guo 2004, 363; Guo, Dauben, and Xu 2013, 381–383. However, I think in the case of square root extraction, one *deng* means the positions the quotient holds, and hence one has different *deng* in different steps. For example, in the first round of extracting 23,467, the *deng* is 100; in the second round, the *deng* is 10; and in the third round, the *deng* is 1. My point is indeed the same as that of Li Y. (1937, 64) and Li J. (1993, 389–392). My main evidence comes from the meaning of *deng* in Chinese, which usually means "rank." Li Jimin (1993, 392) also mentioned an ancient Chinese scholar Kong Yingda's commentary 億之數有大小二法，其小數以十爲等……其大數以萬爲等 "there are two systems about numbers of hundred millions. One small system uses a tens (place) as (one) *deng* ... another big system uses a ten thousands (place) as (one) *deng*." The similarity between division and root extraction carried out with counting rods is another reason. Square root extraction was viewed a kind of division; and this division had a specific feature that the quotient is equal to the divisor. Hence, according to the layout of division, the divisor should be moved until its unit digit is in the same column as the digit of the highest position of the quotient. The *deng* is in fact the magnitude of the quotient/divisor. See the following diagram.

			●	○	○	○	Quotient
●	●	●	●	●	●	●	Dividend
●	○	○	○				Divisor

Nevertheless, even though the explanations are different, the operations carried out with the counting rods are the same.

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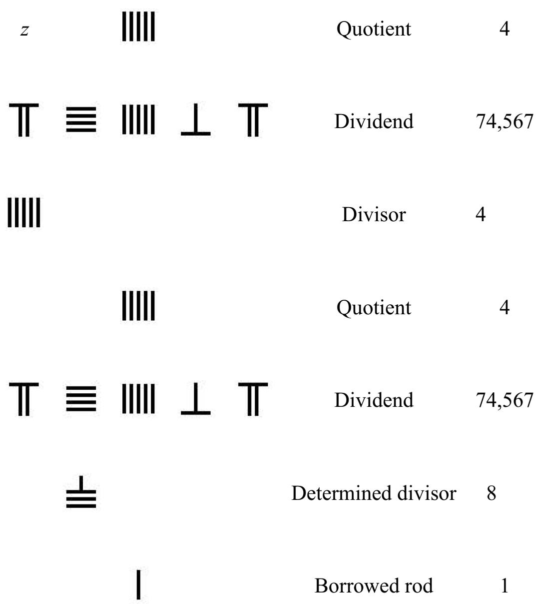

Figure 2. Movement of the borrowed counting rod in the second round of the square root extraction procedure.

Master Sun offers different instructions. It states:

> The procedure is: put down the area (*ji*積), 234,567 *bu*,[31] as the dividend (*shi*實). Next, borrow a counting rod as the lower divisor (*xia fa*下法) and move it, skipping every other place (*wei*位) to reach the hundreds and stop (至百而止).[32]

There are three differences from *The Nine Chapters*: 1) *Master Sun* provides detailed quantities (i.e., 234,567 *bu*) whereas *The Nine Chapters* does not; 2) some technical terms are different, such as *deng* 等 (position) in *The Nine Chapters* and *wei* 位 (place) in *Master Sun*,[33] and *jie suan* 借筭 (borrowed rod) in *The Nine Chapters* and *xia fa* 下法 (lower divisor) in *Master Sun*; and 3) *The Nine Chapters* states: "skip every other position," whereas *Master Sun* states "skip every other place (*wei*位) to reach the hundreds and stop." Lam and Ang (2004, 95) argue that "there is an error in the wording of the last sentence. The word 'hundreds' should be replaced by 'ten thousands.'" As a result, they claim that the material operation behind the text is also similar to what is shown in figure 1. Ji Zhigang (1999, 39) disagrees with Lam and Ang, arguing that

[31]A *bu*步is a measuring unit for length. In this case, it is used to measure area. One can understand it as (square) *bu*.

[32]The original text is: 術曰: 置積二十三萬四千五百六十七步爲實。次借一筭爲下法，步之，超一位，置百而止。(*Sunzi suanjing* 1980, 22b–23a).

[33]Karine Chemla correctly indicates in her report that in different editions of *The Nine Chapters*, there also exists *wei*位, not *deng*等. However, the earliest and the second earliest extant editions of *The Nine Chapters* (i.e. the 1213 edition and the early fourteenth-century edition) both present the term *deng*. The editions that record *wei* were all completed after the eighteenth century. Moreover, in all modern editions of *The Nine Chapters* (including Karine Chemla's French translation), scholars believe the earliest character to be used was *deng*. I think it is clear that in the original text of *The Nine Chapters* that the character was *deng*. During and after the eighteenth century, when scholars in the Qing dynasty (1644–1911) made critical editions of *The Nine Chapters*, they changed the character from *deng* to *wei* based on their misunderstanding of the procedure carried out with counting rods. In the Qing dynasty, scholars used the abacus and did not understand either how to use counting rods or how they had been used earlier. However, in all extant editions of *Master Sun*, the term *wei* is found. Therefore, I argue that there is a technical difference between the terms *deng* and *wei*.

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Master Sun's real meaning is "when the quotient is in the hundreds place, the lower divisor (i.e., the borrowed rod) stops." For Ji, the result is between 400 and 500; thus, the borrowed rod should also be moved to the ten-thousands place. Thus, Ji also interprets the material operation as shown in figure 1.[34] Therefore, although scholars have different interpretations of the sentence in *Master Sun*, they all agree with the operations carried out with counting rods shown in figure 1.

A new general conclusion can be drawn: Although the procedural texts of square root extraction for the first step in *The Nine Chapters* and in *Master Sun* are different, the results of the operations carried out with counting rods are the same. For example, 4, 400, and 40,000 are written differently in Chinese texts, but they look the same when counting rods are used to represent them (||||) because zero is shown by leaving an empty space on the computational surface.[35] Also, for example, when three numbers, e.g., 2, 3, and 4, are placed in three rows from top to bottom on the surface, one does the calculations as follows: $2 \times 4 + 3$. This can be understood as a multiplication with an addition. However, in another case, three numbers could be a result of $11 \div 4$ (i.e., $2 + 3/4$). Hence, $2 \times 4 + 3$ can be also understood as an inverse operation of the division. Furthermore, the operation can also be understood as a transformation of the fraction (i.e., $2 + 3/4 = (2 \times 4+3)/4$), called *tong fen na zi*通分内子 (communicate the fraction and add the numerator). In summary, the same operation (i.e., $2 \times 4 + 3$) carried out with counting rods could have three different mathematical meanings and thus correspond to three different procedural texts. Therefore, although the procedural texts in *The Nine Chapters* and *Master Sun* are different, the operations carried out with counting rods could be similar. More precisely, for similar operations, they could have different mathematical meanings in the two texts.[36]

Analyzing how positions are determined in the second round in each book will make this conclusion clear. The first digit of the quotient is 4; 400 times 400 is 16,000, and 234,567 minus 160,000 is 74,567.[37] *The Nine Chapters* continues as follows: "Once the division is done, double the divisor, which makes the determined divisor (*ding fa*定法). For the next step in the square root

[34]Karine Chemla puts forward a different interpretation in her report where the number of each move (*bu*) corresponds to adding a digit to the quotient. For the square root extraction, this is the same as in the division, but each move jumps over a column (i.e., 步之，超一位，至百而止 move it, skipping every other place to reach the hundreds and stop). I think this interpretation fits perfectly with the earliest edition of *Master Sun*. However, according to this interpretation, the material operation can also be shown in figure 1.

[35]As Karine Chemla indicates, it depends on whether the computational surface had marks or not. However, we know that in the Han dynasty (202 BCE–220 CE) the counting rods were operated on the ground; and in the Song dynasty (960–1279), the counting rods were operated on desks. No extant sources mention marks (used to determine positions) in the ground or on desks.

[36]The differences in the procedural texts of *The Nine Chapters* and *Master Sun* can be generally explained by the historical contexts in which the two books were completed. *The Nine Chapters* usually explains a general procedure, while the aim of *Master Sun* was to offer the elementary knowledge that *The Nine Chapters* did not write down. See Zhu 2021.

[37]Since the aim of the article is not to analyze the whole procedure of square root extraction using counting rods, I do not explain every step. Here, I provide my translation on the whole procedure:

> The procedure for extracting square root is: Put down the area (*ji*積) as the dividend (*shi*實). Borrow a counting rod, and move it, skipping every other position. Estimate (*yi*議) what should be obtained. Multiply it with the borrowed counting rod as the divisor (*fa*法), and divide. Once the division is done, double the divisor, which makes the determined divisor (*ding fa*定法). For the next step in the square root process, move the divisor back. Again, take a borrowed counting rod and move it as at the start. Then multiply the second estimate by the borrowed counting rod. Copy (*fu*副) the result and add it to the determined divisor, then divide. Add the copied result to the determined divisor. Counting the square root process, move the divisor back as before."

The original text is:

> 開方術曰：置積爲實。借一筭，步之，超一等。議所得，以一乘所借一筭爲法，而以除。除已，倍法爲定法。其復除，折法而下。復置借筭，步之如初。以復議一乘之，所得副以加定法，以除。以所得副從定法。復除，折下如前。(*Jiuzhang suanjing* 1980, 58b–59a).

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process, move the (determined) divisor back (*zhe*折). Again, take a borrowed counting rod and move it as at the start."[38] This operation is shown in two steps, demonstrated in figure 2: First, double 4 makes 8 (which is called *ding fa*);[39] 8 is then moved to the thousands place; second, a counting rod is borrowed again and moved from the ones place to the hundreds place (because the second digit of the quotient will be in the tens place, the borrowed rod should be moved to the hundreds place). The rod is moved as it was moved at the beginning; the *deng* is 10 in this round and the rod is skipped past the tens place and stopped at the hundreds place.

Master Sun again offers a different perspective: "Once the division is done, double the square divisor (*fang fa*方法). Move backward (*tui*退) (the square divisor to the right) by one place and the lower divisor by two places."[40] Compared to *The Nine Chapters*, *Master Sun* also uses different terms. *fang fa*方法 (square divisor) in *Master Sun* corresponds to *ding fa*定法 (determined divisor) in *The Nine Chapters*; similarly, *tui*退 (move backward) in *Master Sun* corresponds to *zhe* 折 (move back) in *The Nine Chapters*. However, these terms designate the same quantities or operations using counting rods. Further, both texts refer to the same operation: Double 4 and move the resulting 8 to the thousands place. Wang L. and Joseph (1955), Xu (1986), Chemla (1987), Guo et al. (2010, 244), and others claim that in *Master Sun* (and also in *Zhang Qiujian*), the lower divisor (i.e., the borrowed rod) is moved from the ten-thousands place to the hundreds place (i.e., it is moved two places), whereas in *The Nine Chapters*, the borrowed rod is moved from the ones place to the hundreds place. I agree with the statement that this demonstrates an operational difference between *Master Sun* and *The Nine Chapters*. Xu and Chemla further note another improvement in *Master Sun*, namely, the addition of new rows for different divisors (i.e., square divisor, rectangle divisor, and corner divisor). Therefore, Chemla (1987, 308) stresses that "these algorithms bring a positional notation for equations into play and show the evolution of this notation in the computations: this is not the case in the *Nine Chapters*." However, she also argues that "nothing has changed in the actual computations; they are just described in a new way" (ibid., 307). I agree with all her statements in this regard.

Nevertheless, this article argues that, although the change or evolution that Chemla argues for does happen when we look at and compare the procedural texts in *The Nine Chapters* and *Master Sun*, comparing the two procedures carried out with counting rods reveals that in some steps (e.g., as shown in figures 1 and 2) the two material operations are similar or almost identical. More precisely, in the case of square root extraction, not all the changes are reflected in the operations. In summary, there are more differences between the procedural texts in *The Nine Chapters* and *Master Sun* than in both texts' results of the material operations. This phenomenon can be explained by Liu Hui, who offered a geometrical basis for square root extraction in his 263 CE commentary on *The Nine Chapters*; he understood square root extraction as computing the length of the sides when the area of a square is given. *Master Sun* used the same material instrument to execute the procedure as that employed by Liu Hui.

Furthermore, the difference between the procedural texts is less pronounced than that between the material operations in other cases. For example, in *The Nine Chapters*, *Master Sun*, and other mathematical writings, the term *kai fang* (to open/establish a square) was used to name the square root extraction procedure. However, in the scholars' commentary and sub-commentary in canonical Confucian literature, the term *kai fang* also means to square a number, which is in contrast to the discussion above.[41] Moreover, in these mathematical writings, because the root

[38]The original text is: 除已，倍法爲定法。其復除，折法而下。復置借筭，步之如初。(*Jiuzhang suanjing* 1980, 59a).

[39]The square root extraction procedure relies on the equality: $(a + b)^2 = a^2 + 2ab + b^2$. Since a = 400 has been obtained, it needs to be doubled (2a) for the next computation, i.e., 2ab + b = (2a+b) x b.

[40]The original text is: 除訖，倍方法。（方法)一退，下法再退。(*Sunzi suanjing* 1980, 23a). In this article, I also do not explain *Master Sun*'s methods step by step. For the whole translation of its procedure, see Lam and Ang 2004, 206–207.

[41]For example, when Kong Yingda et al. sub-commented on the *Records of Rites*, they wrote "this section discusses the method of *kai fang* ... 25 *bu*, one multiplies by *kai fang*, totally making 625 *bu*." The original text is: 此節論開方之法 ... 二十五步，開方乘之，總積得六百二十五步。(*Liji Zhengyi* 1980, 1347–1348).

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extraction was viewed as a type of division (as established above), the term *kai fang chu zhi* (開方除之; to divide by square root extraction) was usually used. However, in Confucian literature, the term *kai fang cheng zhi* (開方乘之; multiply by squaring) was used, showing that squaring was regarded as a type of multiplication.[42] Hence, although the same term (i.e., *kai fang*) is used, their mathematical meanings and operations can be different. In summary, the two procedural texts might have been almost identical; however, their corresponding material operations were completely different. The reason is that these texts belonged to different domains, both of which produced mathematical practices.[43]

This analysis reveals that the differences between procedural texts and between material operations do not exhibit a one-to-one correspondence. Nevertheless, the procedures written down in Chinese characters and the operations carried out with counting rods represent the two layers of Chinese mathematics. These two layers offer a new interpretation of the technical term *suan shu* (筭術) that was usually used in the name of mathematical books, i.e., textual procedures (*shu*) carried out with counting rods (*suan*). This fact makes using only the difference in procedural texts to draw a simple conclusion about material operations difficult. More specifically, upon discovering that the procedural texts in different writings are dissimilar, it cannot be simply inferred that the material operations informing the texts are different nor that the operations are the same for no reason other than the fact that texts use the same terms. Indeed, it is necessary to draw heavily on historical texts to understand the differences in mathematical writings, which is the core issue in the study of the history of mathematics.

3. The relationship between operations carried out with counting rods and counting diagrams: Qin Jiushao and Yang Hui

From a modern viewpoint, the thirteenth century was a period of rapid development in Chinese mathematics. Chinese scholars who specialized in mathematics made several advances, such as searching for the root of any algebraic equation (see, e.g., Qian 1966). In *The Nine Chapters*, *Master Sun*, and other earlier texts, the procedures were limited to the extraction of square and cubic roots with positive coefficients. These procedures were expanded to solve quadratic and cubic equations with one unknown value in the seventh century. In the eleventh century, Jia Xian (賈憲) discovered the Chinese version of the coefficients of $(a + b)^n$. Subsequently, the procedure for computing the root of an equation with any high power was created. Soon after this development, Chinese scholars developed methods of numerical solutions for algebraic equations of higher degrees. Several scholars in both northern and southern China expanded these methods to include negative coefficients. Although scholars specializing in mathematics in these two regions focused on different aspects of mathematics, both used numerals derived from counting rods, and wrote down counting diagrams in their treatises (figure 3).

During the thirteenth century, China was politically divided into two polities: the Song dynasty in the south and the Jin dynasty in the north. Both Qin Jiushao and Yang Hui were lower officials of the southern Song dynasty and lived near the capital, Lin'an City (臨安, present-day Hangzhou in Zhejiang province, China). Qin (1208–ca. 1268) completed his *Mathematical Book* in 1247.[44]

[42]In addition, the term *fang zhi*方之 (to make a square) described the operation of square root extraction in Confucian texts, and Confucian scholars only carried out their computations with Chinese writings. For example, in his sub-commentary on the *Rites of Zhou*, Jia Gongyan writes "(the rectangle is) one *chi* wide and one *zhang* two *chi* long; make a square of it." The original text is: 廣一尺，長□二尺，方之。(*Zhouli zhushu* 1980, 910). Jia understood the square root extraction as looking for the length of the sides of an unknown square, whose area is equal to the given rectangle (whose width is the unit and area is the radicand). See Zhu 2016.

[43]Another example is the writings regarding mathematical astronomy in ancient China. The procedural texts in calendric writings are similar to mathematical writings, and the instruments they used are also counting rods. However, the operations for the counting rods differed between the mathematical and calendric domains. See Zhu 2020a.

[44]For an overall introduction to Qin Jiushao and his mathematical treatise, see Libbrecht 1973.

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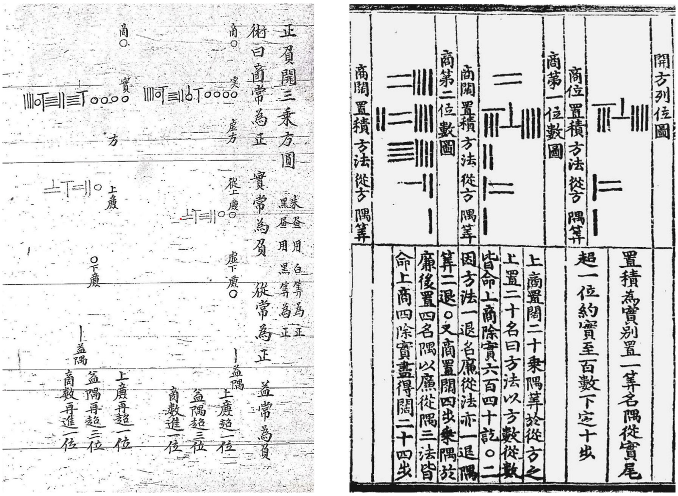


Qin Jiushao (1616, *juan* 5) Yang Hui (1275, 18a)

Figure 3. Qin Jiushao's and Yang Hui's diagrams for computing the root of an equation.

Yang completed five mathematical treatises between 1261 and 1275.[45] The functions of the counting diagrams[46] by the two scholars are mostly different. Qin's diagrams used lines to illustrate the steps of the procedures carried out with counting rods.[47] However, Qin never used counting diagrams to record the details of operations of addition, subtraction, multiplication, and division. Instead, he used different lines connecting numbers to represent these four operations.[48] Qin, in his preface of the *Mathematical Book*, explained why he used counting diagrams: "I set up procedures and recorded detailed solutions and sometimes elucidated them

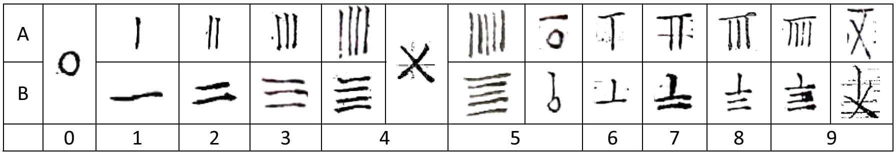
[45]For an overall introduction to Yang Hui, see Yan 1966 and Lam 1977.

[46]Qin Jiushao used 圖*tu* (diagram) or 筭圖 *suan tu* (counting diagram) while Yang Hui used 圖 *tu* (diagram). It is clear that counting diagrams (as shown figure 3) are diagrams containing written-rod numerals, and presenting how counting rods were placed or operated.

[47]The original manuscript written by Qin Jiushao has been lost. The closest extant text to it is the copy in the *Great Compendium of the Yongle Reign* (*Yongle Dadian*永樂大典), which was a huge encyclopedia compiled during the Yongle reign (1403–1424) of the Ming dynasty (1368–1644). Qin's *Mathematical Book in Nine Chapters*, containing eighty-one problems and Qin's own preface, was transcribed separately into different volumes of the *Great Compendium of the Yongle Regin* during 1405–1408. As this compendium has been scattered without a trace since late nineteenth century and only about four percent of it exists today, only three problems from the *Mathematical Book in Nine Chapters* as it was presented in the *Great Compendium of the Yongle Regin* are preserved. Fortunately, one of the three problems contains lines within the counting diagrams. This strongly indicates that the original text of Qin's treatise had lines – otherwise we have to imagine that someone lived between 1247 and 1405 and was specialized in mathematical practice with counting rods but was never mentioned in any historical source. The master copy used by the compendium was preserved in the Imperial Library of the Ming dynasty at that time. It has also been lost. The master copy was, however, transcribed by the bibliophile Zhao Qimei趙琦美 (1563–1624) in 1616, and that is Zhao Qimei's handwritten copy. It is currently in the National Library of China in Beijing. In the eighteenth century, based in the *Yongle Dadian* edition of the *Mathematical Book in Nine Chapters*, Qin's treatise was edited and transcribed into *Complete Works on the Four Treatises* (*Siku Quanshu*四庫全書), and this is the *Siku Quanshu* edition. However, all lines within counting diagrams were removed since scholars in the Qing dynasty (1644–1911) did not understand Qin Jiushao's mathematical practice with counting rods. In 1842, Song Jingchang宋景昌 published the first-printed edition in the *Yijiatang Collectanea*, and this is the *Yijiatang* edition. This edition at least partly relied on Zhao Qimei's handwritten copy and the *Siku Quanshu* edition. However, no line existed within the counting diagram either. For an introduction on the textual history of Qin's treatise, see Zheng and Zhu 2010. For Qin's uses of lines within counting diagrams in Zhao Qimei's handwritten copy of Qin's treatise, see Zhu 2017, 2020a. For an analysis of the first printed edition of Qin's treatise, see Zhu and Zheng 2024.

[48]For a complete analysis of Qin's lines, see Zhu 2020a, 357–370.

Table 1. Written numeral system in Zhao Qimei's handwritten copy of Qin's treatise



[that is, the detailed solutions] using [counting] diagrams."[49] The functions of counting diagrams in Qin's treatise vary, depending on different problems (see Zhu 2020a, 349–354). Yang's diagrams, however, sometimes show the steps of these four operations. Interestingly, both Qin and Yang recorded the counting diagrams for root extraction (figure 3 and its translation in figures 4 and 5). This fact confirms the special role of root extraction and provides a basis for comparing the two.

Yang's diagram (figure 3) came from his *Fast Methods on Various Categories of Multiplication and Division of Areas of Fields* (田畝乘除比類捷法; *Tianmu chengchu bilei jiefa*, 1275). This diagram was presented in a mathematical problem: "the area of a rectangular field is 864 *bu*, while its width is 12 *bu* less than its length. What is its width?" Just before this, Yang wrote that he quoted this type of problem from Liu Yi's (劉益; ca. eleventh century) *Discussion on Ancients Roots and Sources* (*Yigu Genyuan* [議古根源]), which he commented on in detail using diagrams and detailed solutions (*xiang zhu tu cao* [詳注圖草]).[50] Accordingly, the procedures in Yang's text were likely drawn from Liu Yi's work, whereas the geometrical and computational diagrams, including the surrounding texts, were all written by Yang Hui. However, it is still difficult to distinguish what was written by Liu and what was added by Yang.[51] Because the aim of this article is to analyze the nature of the diagrams, the conclusion should not be affected by questions of authorship.

To begin an analysis of the difference between Qin and Yang, table 1 shows the writing numeral system from Qin's treatise, which was also used in Yang's writings (the left diagram of figure 3). This system contains two parts: A and B. System A was used with places of units, hundreds, ten thousands, and so on, whereas system B was used with places of tens, thousands, hundred thousands, and so on. Obviously, this numeral system was derived from the use of counting rods. Because the symbols ○ and ✕ were used, the system exhibited some variations compared with counting rods.[52] One difference between Qin and Yang is the representation of positive and negative numbers. As Qin states, he used red and black colors to differentiate two opposite numbers in text and correspondingly used white and black rods in operations.[53] In

[49]The original text is: 立術具草，兼以圖發之。(Qin Jiushao 2010, 98).

[50]Yang Hui's full text (1275, 18a) is as follows: "Mr. Liu from Zhongshan says in his preface: mathematical procedures carried out with counting rods, which are used in different parts of mathematics, however all applied to computing the squares of rectangle fields. Therefore, *Discussions on Ancient Roots and Sources* set up one-hundred problems about 'deducing pieces' (*yan duan*), which means deducing the pieces of computational steps (*pian duan*) [carried out with counting rods]. When one knows computational steps, one will know their root and source. After one knows the root and source, their heart will not be ignorant. Now I select several problems, and detailedly comment on them using diagrams and detailed solutions, so as to clarify these ideas for younger generations. The other problems can naturally also be introduced, extended, and understood with analogy; hence I needn't exhaust my words." 中山劉先生序謂：筭之術，入則諸門，出則直田。《議古根源》故立演段百問，蓋欲演筭之片段了。知片段則能窮根源。既知根源，而於心無蒙昧矣。今姑摘數問，詳注圖草，以明後學。其餘自可引而申之，觸類而長，不待盡述也.

[51]For a discussion of Liu Yi and Yang Hui, also see Li D. 1999, 8–11; Chemla 2018; and Pollet 2020.

[52]Li Di (1991) systematically studied the symbols used in mathematical writing during the Song and Yuan periods.

[53]Qin Jiushao (2010, 155) says: "One just uses white counting rods where red strokes are written; one just uses black counting rods where black strokes are written." The original text is 朱畫用白筭爲正，黑畫用黑筭爲正. This sentence only appears in Zhao Qimei's handwritten copy (i.e., figure 3) and does not exist in the *Siku Quanshu* edition nor the first printed edition. Qin Jiushao (2010, 187) also says: "dividend and *yi* divisor are both negative, and black strokes are drawn; quotient and *cong* divisor are both positive, and red strokes are drawn." The original text is 實與益皆負畫黑，商與從皆正畫朱. This

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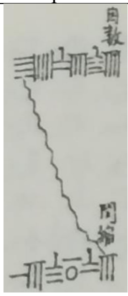
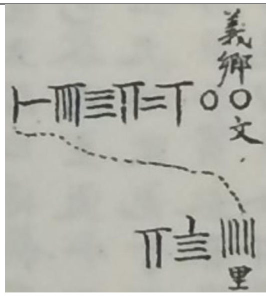
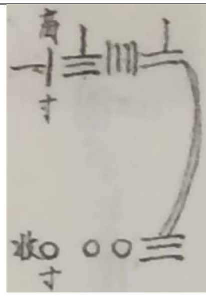
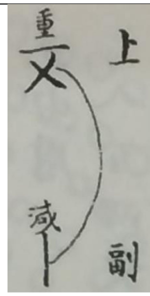
Table 2. Main representations of multiplication, division, addition, and subtraction in Zhao Qimei's handwritten copy of Qin's treatise

Multiplication	Division	Addition	Subtraction
			
Problem 12	Problem 44	Problem 49	Problem 66
457999× 188578	119472600÷ 795	11.847+0.003	9-1
Qin (2010: 137)	Qin (2010: 222)	Qin (2010: 242)	Qin (2010: 285)

contrast to Qin, Yang added an oblique rod to the numerals to represent negative numbers. For example,  means −5. Liu Hui mentioned two methods to represent positive and negative numbers in his commentary on *The Nine Chapters* in 263.[54] Qin's and Yang's methods exactly followed those of Liu Hui's. My analysis of instrumental operations reveals that the different symbols for two opposite numbers cannot prove that Chinese scholars used different counting rods; however, this demonstrates that Qin and Yang could follow different parts of Liu Hui's commentary.

Another difference between Qin and Yang is their use of lines. In Qin's treatise, he used different types of lines to represent different operations. The linear system is characterized by a wavy line connecting two numbers, which usually indicates multiplication, a dotted line, which usually indicates division, a double full line, which usually indicates addition, and a full line, which usually indicates subtraction (table 2).[55] In contrast, Yang used lines to connect the various

---

sentence appears in Zhao Qimei's handwritten copy and the *Siku Quanshu* edition, but it disappears from the first printed edition. However, Qin's principle is only executed in *kai fang*開方 procedure (a procedure to solve the algebraic equation with any high degree), where he uses red colors to represent positive numbers and black colors to represent negative numbers; in *fang cheng*方程 procedure (a procedure used to solve systems of linear equations), he uses black to represent positive numbers and red to represent negative numbers. See Zhu 2020a, 355–357.

[54]In his commentary on the eighth chapter (i.e., *fang cheng*) of *The Nine Chapters on Mathematical Procedures* in 263, Liu Hui says: "The positive counting rods are red; the negative counting rods are black. Otherwise, they can be differentiated by slanted or upright [counting rods]." The original text is 正筭赤，負筭黑，否則以邪正爲異. See Chemla and Guo 2004, 624 and Guo, Dauben, and Xu 2013, 928–931.

[55]Based on a complete analysis of Zhao Qimei's copy of Qin's treatise, there are a total of 726 meaningful lines used in the counting diagrams, unevenly distributed between the forty-two problems. Seven-hundred out of 726 lines are used for indicating four operations: multiplication (388 lines), division (163 lines), addition (98 lines), and subtraction (51 lines). There are overall four types of lines: single full line, single dotted line, single wavy line, and double full line. And there are overall eight types of connections of two numbers. A further observation is revealing: For multiplication, 358 lines out of 388 are wavy; for division, 140 lines out of 163 are dotted; for addition in 98 lines, 90 are double full; and for subtraction in 51 lines, 50 lines are full. Given that the correlation between the types of lines and operations is approximately ninety percent, we can safely infer that these four types of lines represent four operations in Zhao Qimei's copy of Qin's treatise. In summary, we can conclude the following. 1) A wavy line connecting the first digit of a long number to the last digit of another long number represents the operation of multiplication. 2) A dotted line connecting the first digit of a long number to the last digit of another long number represents the operation of division. 3) A double full line connecting the last digit of a long number to the last digit of another long number represents the operation of addition. 4) A full line connecting the last digit of a long number to the last digit of another long number represents the operation of subtraction. For the details of the analysis, see Zhu 2020a, 357–364.

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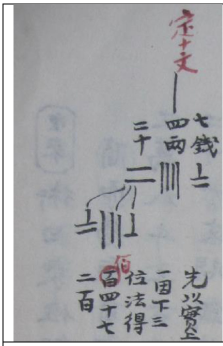
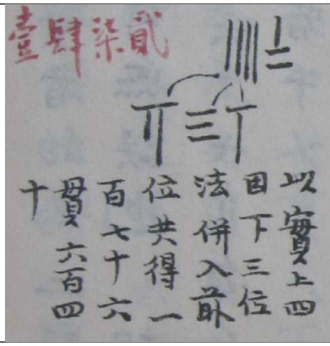
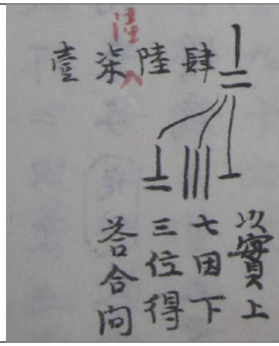
Table 3. Yang Hui's usage of written numerals and lines to write down a multiplication (Yang Hui 1274, 14a)

		
247×736	47×736	7×736

operands, as shown in table 3.[56] Specifically, different meanings of lines reflect Qin's and Yang's different epistemological focus in mathematics. Qin regarded operations and procedures consisting of operations as important, whereas Yang specifically valued the operands and details of computations.

The key difference between Qin and Yang lies in their positions regarding the relationship between the counting diagrams and their related texts. In modern terms, Qin Jiushao's diagram, shown in figure 3 (translation in figure 4), is the first step toward solving the following equation:

$$-x^4 + 763200x^2 - 40642560000 = 0$$

Before this diagram, Qin presents a problem with an answer, a procedure, and a detailed solution.[57] The problem arises in the computation of the area of a field with four sides. The procedure is non-specific, only referencing an algebraic equation with a fourth degree.[58] The detailed solution shows how the computations may be based on this procedure. Qin's diagram (figure 3) contains two parts: a counting diagram and the accompanying text. The counting diagram (figure 4) shows how these computations were carried out with counting rods. In figure 4, the two sentences in the lower right section explain how the rods were used according to the color in the text. The sentences in the lower left section explain how the lowest rod was moved. Hence, the text accompanying the counting diagram was used to explain the instrumental operation.

Yang Hui's writing consists of three diagrams (see the right-hand side of figure 3 and its translation in figure 5). In modern terms, Yang describes the procedure used to solve the following equation:

$$x \times (x + 12) = 864$$

[56]The problem as shown in table 3 is: There are 24 *liang*兩 7 *qian*錢 silver (*liang* and *qian* are weight units in the Song dynasty, 1 *liang* = 10 *qian*), and each *liang* silver costs 360 *guan*貫 7 *wen*文 (*guan* and *wen* are monetary units in the Song dynasty, 1 *guan* = 1000 *wen*). One asks how much money does the silver cost? Yang Hui used the written-rod numerals and various Chinese characters to show the process of this multiplication, that is: 24 *liang* 7 *qian* silver multiplied by 7 *guan* 360 *wen* per *liang* makes 181 *guan* 792 *wen* (24.7 *liang* × 7.36 *guan* /*liang* = 181.792 *guan*). For a detailed analysis of this procedure, see Zhu 2020c, 99–100.

[57]For the detailed solution to this equation, see Qian 1966, 48–51; Libbrecht 1973, 180–191; Wang S. 1992, 208–221; Berezkina 1980, 232–236; Zharov 1986.

[58]Qin's method is to divide the four-sided field into two triangles, computing the area of every triangle, and adding them together. According to Qin's method, the area of the triangle is related to an equation with second degree. When Qin added them, this finally caused an equation with fourth degree.

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<table>
<tr><td colspan="3">Quotient
0</td><td colspan="3">Quotient
0</td><td>Procedure says:</td><td colspan="2" rowspan="2">Diagram for a fourth root extraction with positive and negative numbers[2]</td></tr>
<tr><td colspan="3">Dividend
-40642560000</td><td colspan="3">Dividend
-40642560000[3]</td><td>The quotient is always positive.</td></tr>
<tr><td colspan="3">Square (divisor)</td><td colspan="3">Empty square (divisor)[4]</td><td>The dividend is always negative.</td><td rowspan="5">One uses black counting rods where black strokes are written.</td><td rowspan="5">One uses white counting rods where red stokes are written.</td></tr>
<tr><td colspan="3">763200
Higher rectangle (divisor)</td><td colspan="3">Cong (added) above rectangle (divisor)[5]
763200</td><td>When the character “cong” (added) is used, the number (after the character) is always positive.</td></tr>
<tr><td colspan="3">0
Lower rectangle (divisor)</td><td colspan="3">Empty lower rectangle (divisor)[6]
0</td><td rowspan="3">When the character “yi” (increased) is used, the number (after the character) is always negative.</td></tr>
<tr><td colspan="3">-1
Yi (increased) corner (divisor)</td><td colspan="3">-1
Yi (increased) corner (divisor)[7]</td></tr>
<tr><td>Number of the quotient further moves forward one place.</td><td>Yi (increased) corner (divisor) further skips three places.</td><td>Higher rectangle (divisor) further skips every other place.</td><td>Number of the quotient moves forward one place.</td><td>Yi (increased) corner (divisor) skips three places.</td><td>Higher rectangle (divisor) skips every other place.</td></tr>
</table>

Figure 4. Translation of Qin Jiushao's diagram (left-side of figure 3).There is no table in the original text. I add the table in order to make the layout of the diagram clear. As mentioned above, Qin's method was developed from the square root extraction procedure, and relies on the equality $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. The six sentences in the bottom of the table show how to move the numbers represented by counting rods. The *yi* (increased) corner (divisor) (*yi yu*益隅) is similar to the borrowed counting rods as analyzed above. The borrowed rod skips one place every time in *Master Sun*, when the equation has only second degree. Since the equation has fourth degrees here, this divisor should correspondingly skip three places every time. The two sentences in the right part of the table give the principle to deal with two opposite numbers. However, since they only appear in Zhao Qimei's handwritten copy of Qin's treatise, no other scholar has quoted them before Zheng Cheng and myself. [2] As I have mentioned, *kai fang* procedure (root extraction in modern terms) was used to solve algebraic equations with higher degrees. Hence, Qin used "fourth root extraction" to indicate the equation has the fourth degree. Qin's mention of "positive and negative numbers" means he would use these two opposite numbers in the process of solving the equation. [3] In Zhao Qimei's handwritten copy (figure 3), we see Qin use black and red colors to show two opposite numbers, i.e., red for positive numbers, and black for negative numbers. However, this feature only appears in Zhao Qimei's copy. This is why I translate Qin's numbers into negatives here. [4] The empty square (*xu fang*) divisor is the coefficient of x in the whole equation, $-x^4 + 763200x^2 - 40642560000$. Since the character *xu*虚 (empty) is written, the number is 0. [5] The added above rectangle (*cong shang lian*) divisor is the coefficient of x^2 in the whole equation. Since the character *cong*從 (added) is written, the number is positive. [6] The empty lower rectangle (*xu xia lian*) divisor is the coefficient of x^3 in the whole equation. Since the character *xu*虚 (empty) is written, the number is 0. [7] The increased corner (*yi yu*) divisor is the coefficient of x^4 in the whole equation. Since the character *yi*益(increased) is written, the number is negative.

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<table>
<tr><td>Position of quotient</td><td>24</td><td rowspan="5">Diagram for the second digit of the quotient</td><td>Position of quotient</td><td>2</td><td rowspan="5">Diagram for the first digit of the quotient</td><td>Position of quotient</td><td></td><td rowspan="5">Diagram for positions of square root extraction[1]</td></tr>
<tr><td>Put down the area</td><td>224</td><td>Put down the area</td><td>864</td><td>Put down the area</td><td>864</td></tr>
<tr><td>Square divisor</td><td>44</td><td>Square divisor</td><td>2</td><td>Square divisor[2]</td><td></td></tr>
<tr><td>Cong (added) rectangle (divisor)</td><td>12</td><td>Cong (added) rectangle (divisor)</td><td>12</td><td>Cong (added) rectangle (divisor)[3]</td><td>12</td></tr>
<tr><td>Corner rod</td><td>1</td><td>Corner rod</td><td>1</td><td>Corner rod[4]</td><td>1</td></tr>
<tr><td></td><td colspan="4">Put down the above quotient, width 20 (bu), multiplying the corner rod. Put down 20 above the cong (added) rectangle (divisor), called the square divisor. Multiply the above quotient (20) by both numbers of the square (divisor) and cong (added) rectangle (divisor) and remove 640 (what has been obtained) from the dividend.[5]
Multiply the square divisor by 2. Move backwards (the square divisor, 4, to the right) by one place, called side (divisor).[6] Also move backwards the added rectangle (divisor) by one place. Move backwards the corner rod by two places.[7]
Further, put down the quotient, a width of 4 bu, multiplying the corner (rod). Put down a 4 just after the side (divisor), called the corner (divisor). Multiply the above quotient, 4, by the three divisors (side divisor, added rectangle divisor, and corner divisor), and remove (what has been obtained) from the dividend, exhausting (the dividend). Obtain the width 24 bu.[8]</td><td></td><td colspan="2">Put down the area as the dividend. Separately, put down a counting rod, called “corner (rod).” Move it from the tail of the dividend, skipping every other place. To divide, move the corner (rod) under the hundreds place, and determine the tens place bu (of the quotient).</td><td></td></tr>
</table>

Figure 5. Translation of Yang Hui’s diagram (right-side of figure 3). [1]As I have mentioned, *kai fang* procedure (root extraction in modern terms) was used to solve algebraic equations with higher degrees. Hence, Yang used “square root extraction” to indicate the equation has the second degree. [2]The square divisor (*fang fa*) is equation to quotient in the first round of the procedure. The layout of Yang Hui’s procedure can be understood as an extension of *The Nine Chapters*, that is from the up to bottom: x, 864, x, 12, 1, which means the equation is x (x + 12) = 864. [3] The *cong* (added) rectangle divisor is the coefficient of x in the whole equation, $x^2 + 12x = 864$. [4] In Yang Hui’s procedure, the corner rod is the same as the borrowed rod in *The Nine Chapters*. It was used to determine positions. [5] In this round, Yang obtains 20 as the first quotient. Hence, 20 x (20+12) = 640. 864 - 640 = 224. [6] For the same term *lian fa*廉法, Qin Jiushao and Yanghui have different mathematical meanings. Qin named all coefficients of the equation as *lian fa*, i.e., rectangle divisors. Yang followed *Master Sun*, calling the double square divisor *lian fa* (i.e., side divisor). In this problem, Yang called the coefficient of x *cong fang*從方 (i.e., added rectangle divisor), which indeed is equal to Qin’s *lian fa*. Chemla (2018, 62) translates *cong fang* into “what joins the square.” However, Yang Hui (1275, 18b) mentions *ping fang yi duan*平方一段 (a piece of the flat square) and *cong fang yi duan*從方一段 (a piece of the added rectangle). Hence, it is clear that *cong fang* refers to the rectangle that is added to (i.e., *cong*) the square. [7] Since Yang also relies on the equality $(a + b)^2 = a^2 thinsp; + thinsp; 2ab + b^2$, the 20 (i.e., a) should be doubled for the next computation. [8] In this round, Yang obtains 4 (i.e., b) as the second quotient. Hence, 4^2 + 2 x 20 x 4 + 4 x 12 (i.e., b^2 + 2ab + b x 12) = 4 x (4 + 40 + 12) = 224. This exactly exhausts the remaining dividend. The three divisors are 4 (corner divisor), 40 (side divisor), and 12 (added rectangle divisor).

Before the diagram, Yang offers a problem with an answer, a procedure, and two geometrical figures.[59] The problem is also related to the computation of the area of a rectangular field. Yang’s procedure is similar to the one in *The Nine Chapters* and is also used as an example to show how the operations are carried out with counting rods (like Qin Jiushao). According to Yang, the procedure was quoted from Liu Yi’s treatise. The two geometrical figures were used to explain that the procedure was correct.[60]

[59]For the whole translation of this problem, see Lam 1977, 113–114.
[60]For a detailed analysis of Yang Hui’s geometrical figures, and its use for proof, see Chemla 2018.

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Similar to Qin's writing, Yang's diagrams also contain two parts: a counting diagram (right-hand side of figure 3 and figure 5) and the accompanying text. Yang's counting diagrams are also employed to explain how computations are performed using counting rods. Both scholars add titles to the diagrams on the right. Despite some common features, there is an important difference. Yang's textual descriptions below the counting diagrams comprise the detailed solution for the computations (as shown in figure 5), differing from Qin's text in the same place (Qin only mentions the positional movement of different numbers, as shown in figure 4). Hence, the counting diagrams in Qin's writing provide more information about the material operations than the corresponding texts accompanying the diagrams; in Yang's writing, the counting diagrams provide less information than their accompanying texts.

This difference can be further understood by analyzing and comparing the whole structure of their problem. In Qin's text, this problem (figure 3) starts with a question, followed by an answer, a procedure (in Chinese), a detailed solution (with detailed numbers, also in Chinese), and counting diagrams (using written rod numerals, as shown in figure 3) to solve the equation. Therefore, the counting diagrams are independent of the detailed solution. Hence, Qin's texts accompanying the diagrams are used to explain part of their respective material operations, as shown in the diagrams. This exactly follows Qin's words "I set up procedures and recorded detailed solutions and sometimes elucidated them [that is, the detailed solutions] using [counting] diagrams" (立術具草，兼以圖發之). In Yang's text, the problem is accompanied by a question, an answer, and a procedure (all of them in Chinese), followed by two geometrical diagrams to clarify the proper procedure. Following the two geometrical diagrams are the counting diagrams, with texts included below them (as shown in figure 3). Hence, Yang's texts, found below his counting diagrams, present a detailed solution for the problem in question. The counting diagrams are used to explain their accompanying text. This coincides with Yang's statement: "I detailedly commented on them using diagrams and detailed solutions" (詳注圖草). Specifically, as seen in figure 3, Qin presents an independent counting diagram with explanatory text, whereas Yang presents a detailed solution with illustrated counting diagrams.

Although both Qin and Yang included counting diagrams accompanied by text, their reasons for doing so differed. For Qin, the most important element of his explanation was the counting diagram (i.e., *suan tu*), whereas the text shown below it clarifies the diagram. In contrast, for Yang, the most important elements were the texts shown below the diagrams, as they offer a detailed solution to the problem in question. The diagrams are to be used to help the reader understand the text. In other words, Qin focused on the counting diagrams, whereas the explanatory text was supplementary. In contrast, Yang focused on the detailed solution explained in the text, with the counting diagram serving as supplementary information. This difference could reflect different epistemological focuses on the relationships between procedural texts and material operations. In general, Qin emphasized material operations over procedural texts, whereas Yang did the opposite.

This conclusion can also be related to Qin's and Yang's mathematical achievements. Qin's treatise has eighty-one problems, and forty-five of them are supplemented with counting diagrams. These counting diagrams have different objectives in relation to the text. Most of Qin's mathematical achievements are shown through counting diagrams (see Zhu 2017, 2020a). Specifically, Qin's independent counting diagrams created new representations, leading him to new mathematical procedures. Following the traditional mathematical writing style, Yang's key points were always presented in his detailed textual explanations (*xiang jie* 詳解). Yang's counting diagrams were a part of his detailed solutions and used to explain his procedures. Comparing their other counting diagrams would shed further light on the differences between Qin and Yang.[61] Furthermore, the difference between Qin and Yang can be understood in the context of their writing. Qin's research was at the intersection of mathematics, calendrical computations, and the

[61]That is to say, most of Qin's counting diagrams focused on writing down procedures, while most of Yang's counting diagrams focused on the details of computations.

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Book of Changes (周易) (see Zhu 2017, 2019b), whereas Yang's research was mainly based on traditional mathematics. In summary, based on Qin Jiushao's and Yang Hui's writings, the relationship between procedural text and material operation could vary depending on the individual, and the role played by counting diagrams in their respective research efforts is key to analyzing this difference. Specifically, while Yang's counting diagrams illustrated some details of computations carried out with counting rods, Qin's counting diagrams represented a new way to write down procedures carried out with counting rods. Consequently, the objectives of textual procedures, counting diagrams, and counting rods operations differed between the two scholars.

4. Conclusion

This article has shown that the relationship between procedural texts and material operations depends on their context. Generally, there are two ways to study this relationship. The first approach involves analyzing the differences between procedural texts in ancient mathematical writings. The analysis of *The Nine Chapters* and *Master Sun* reveals that different procedural texts do not always correspond to different operations carried out with counting rods, as the operations could be more similar than the text shows.[62] Moreover, the same terms could correspond to different instrumental operations because they are used in multiple domains. As the most direct evidence about how counting rods were used comes from ancient mathematical writings without diagrams, the social and historical context in which these texts were written is key to studying the material operations.

In thirteenth-century China, scholars specializing in mathematics from both the south and north included counting diagrams in their mathematical writings. These diagrams allow for the analysis of the relationship between procedural texts and material operations. The analysis of Qin Jiushao's and Yang Hui's treatises reveals that despite differences between the counting diagrams presented in both texts, concluding that Qin and Yang carried out different operations with counting rods is not always possible. However, the difference between the numerals, lines, and the relationships between counting diagrams and their accompanying text reflects that the two scholars had different epistemological focuses on mathematics. Qin viewed the counting diagrams and the operations carried out with counting rods as more important, whereas Yang's focus remained on procedural texts. This point can be further confirmed by an analysis of the two scholars' mathematical achievements and the different domains in which they were involved. In this respect, the relationship between texts and operations can vary depending on the individual.

This study underscores the challenges inherent in studying material operations. Counting rods comprised the primary instrument used in Chinese mathematics for more than 2,000 years, with both textual procedures in ancient mathematical writings and thirteenth-century counting diagrams serving as indirect evidence of their use. Given this, it is imperative to reassess the earlier hypothesis that suggests a fundamental correspondence between procedural texts and counting rods operations and to develop new methods for examining these sources as well as their associated material operations. Written texts (including problems, textual procedures, and counting diagrams), along with material operations (including counting rods and other geometrical instruments), constitute interconnected yet distinct layers of mathematical practices that form the structure of Chinese mathematics. In this manner, ancient mathematical writings should be studied carefully to avoid making inferences not directly evident from the writings. In other words, this article suggests first literally interpreting the text and then connecting it with other aspects. Furthermore, importantly, any interpretation regarding material operations is

[62]Karine Chemla states (pers. comm.): "a text might do something in addition to prescribe operations, which, however, is a major reason why texts might differ even though they refer to the same operation." I certainly agree with her statement. However, in this article, I focus on another situation: that is, that while the material operations of counting rods are similar, they could have different mathematical meanings – this may be why they were described as different procedural texts. I think this is the nature of computations with counting rods, which contrasts with modern mathematics.

fundamentally fallible.[63] While our reconstruction of an operation may fully explain the corresponding text, it does not inherently guarantee the historical accuracy of the reconstruction. Discrepancies between written records and practical operations could differ among individuals, and there could be several ways for documenting the same operation or utilizing the same instrument.[64] Exploring the connections and interactions between these different practices presents new avenues for research. This article confirms a close relationship between historical sources and methodological issues in the study of the history of mathematics.

Finally, the research presented in this article holds philosophical significance. One philosopher argues that "instrumental practice can secure epistemic access to ideal objects of mathematics."[65] In this respect, this article not only expands the content of instrumental practice, but also offers an example of how different instrumental practices (i.e., procedural texts, operations of counting rods, and counting diagrams with texts) interact. More specifically, different layers of mathematical practices could offer different epistemic accesses to mathematical objects, whereas the interplay and relationships between different layers form an epistemic structure, which is also based on historical and cultural factors.[66]

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[63] One reviewer wondered: "if a text describes, very explicitly, operations with the counting instrument (counting rods), how can the connection between this text and the operations be denied?" For this question, I have to say that if this was the case, the connection really cannot be denied. However, no other known Chinese procedural text (written in Chinese) offers an explicit record of counting rod operations. The closest texts to what the reviewer imagined are exactly the counting diagrams written by thirteenth-century Chinese scholars. This is why I have analyzed and argued that these counting diagrams cannot fully reflect the counting rods operations either.

[64] For different uses of the same operation carried out with abacus, also see Chen 2013.

[65] Kvasz (2019) suggests a distinction between abstract and ideal objects, and says that mathematical objects are primarily ideal. He argues,

> To see the ideal objects of mathematics as an analogue of the ideal objects of physics (and thus to enable the development of mathematical epistemology to profit from the existence of the ideal objects of physics) means to focus on the representational tools, i.e., the material artifacts which mathematicians use to reduce phenomena such as quantity, shape, and structure to their linguistic representations, i.e., to numbers, geometric figures, algebraic equations, etc.

He further argues, "Thus by means of the instruments of symbolic and iconic representation we can study ideal mathematical objects."

[66] This in fact opens a new direction for the history of mathematics. For preliminary studies, see Zhu 2016, 2020b.

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