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ARTICLE

Hypergraphs without complete partite subgraphs

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Abstract

Fix integers $r \geq 2$ and $1 \leq s_1 \leq \cdots \leq s_{r-1} \leq t$ and set $s = \prod_{i=1}^{r-1} s_i$. Let $K = K(s_1, \ldots, s_{r-1}, t)$ denote the complete r-partite r-uniform hypergraph with parts of size $s_1, \ldots, s_{r-1}, t$. We prove that the Zarankiewicz number $z(n, K) = n^{r-1/s-o(1)}$ provided $t > 3^{s+o(s)}$. Previously this was known only for $t > ((r-1)(s-1))!$ due to Pohoata and Zakharov. Our novel approach, which uses Behrend's construction of sets with no 3-term arithmetic progression, also applies for small values of s_i, for example, it gives $z(n, K(2, 2, 7)) = n^{11/4-o(1)}$ where the exponent 11/4 is optimal, whereas previously this was only known with 7 replaced by 721.

Keywords: Hypergraphs; Zarankiewicz problem; induced matchings

2020 MSC Codes: Primary: 05C35; Secondary: 05D99, 05C65

1. Introduction

Write $K = K(s_1, \ldots, s_r)$ for the complete r-partite r-uniform hypergraph (henceforth r-graph) with parts of size $s_1 \leq s_2 \leq \cdots \leq s_r$. More precisely, the vertex set of K comprises disjoint sets $S_1, \ldots, S_r$, where $|S_i| = s_i$ for $1 \leq i \leq r$, and the edge set of K is

$$\{\{x_i, \ldots, x_r\} : (x_1, \ldots, x_r) \in S_1 \times \cdots \times S_r\}.$$

Given K as above, write $\mathrm{ex}(n, K)$ for the maximum number of edges in an n-vertex r-graph that contains no copy of K as a subhypergraph. Similarly, write $z(n, K)$ for the maximum number of edges in an r-partite r-graph H with parts $X_1, \ldots, X_r$, each of size n, such that there is no copy of $K(s_1, \ldots, s_r)$ in H with $S_i \subset X_i$ for all $1 \leq i \leq r$ (there could be copies of K in H, where for some i, $S_i \not\subset X_i$). Determining $\mathrm{ex}(n, K) = \mathrm{ex}_r(n, K)$ is usually called the Turán problem, while determining $z(n, K) = z_r(n, K)$ is called the Zarankiewicz problem (we will omit the subscript r if it is obvious from context). These are fundamental questions in combinatorics with applications in analysis [1, 7], number theory [16], group theory [13], geometry [10], and computer science [3].

A basic result in extremal hypergraph theory, due to Erdős [9], is the upper bound

$$\mathrm{ex}(n, K(s_1, \ldots, s_r)) = O(n^{r-1/s}), \tag{1}$$

where $s = s_1 s_2 \cdots s_{r-1}$ (and, as before $s_1 \leq s_2 \leq \cdots \leq s_{r-1} \leq s_r$). Here $s_1, \ldots, s_r$ are fixed and the asymptotic notation is taken as $n \to \infty$. As r is fixed, and $z(n, K(s_1, \ldots, s_r)) \leq \mathrm{ex}(rn, K(s_1, \ldots, s_r))$, the same upper bound as (1) holds for $z(n, K(s_1, \ldots, s_r))$.

A major problem in extremal (hyper)graph theory is to obtain corresponding lower bounds to (1) (or prove that no such lower bounds exist). In fact, it was conjectured in [15] that the exponent $r - 1/s$ in (1) is optimal. This question has been studied for graphs since the 1930s, and results of Erdős-Rényi and Brown [5] gave optimal (in the exponent) lower bounds for $K(2, t)$ and $K(3, t)$. The first breakthrough for arbitrary s_1 occurred in the mid 1990's by Kollar-Ronyai-Szabo [11] and then Alon-Ronyai-Szabo [2], who proved that $\mathrm{ex}(n, K(s_1, s_2)) = \Omega(n^{2-1/s_1})$ as long as $s_2 > (s_1 - 1)!$. More recently, in another significant advance, Bukh [6] has proved the same lower bound as long as $s_2 > 9^{s_1+o(s_1)}$.

For $r \geq 3$, the first nontrivial constructions that were superior to the bound given by the probabilistic deletion method were provided in the cases $s_1 = \cdots = s_{r-2} = 1$ and $K(2, 2, 3)$ by the current author [15] and, soon after for $K(2, 2, 2)$ by Katz-Krop-Maggioni [12] (see also [8] for recent results on the r-uniform case $K(2, \ldots, 2)$ that are superior to the probabilistic deletion bound but not optimal in the exponent). Later, optimal bounds for both $\mathrm{ex}(n, K)$ and $z(n, K)$ were provided by Ma, Yuan, Zhang [14] (and independently by Verstraëte) by extending the method of Bukh, however, the threshold for s_r for which the bound holds was not even explicitly calculated. More recently, lower bounds matching the exponent $r - 1/s$ from (1) have been proved for $s_r > ((r - 1)(s - 1))!$ by Pohoata and Zakharov [17]. Here we improve this lower bound on s_r substantially in the Zarankiewicz case, from factorial to exponential at the expense of a small $o(1)$ error parameter in the exponent. The following is our main result.

Theorem 1. *Fix $r \geq 3$, and positive integers $s_1, \ldots, s_{r-1}, t$. Then as $n \to \infty$,*

$$z_r(n, K(s_1, \ldots, s_{r-1}, t)) > n^{1-o(1)} \cdot z_{r-1}(n, K(s_1, \ldots, s_{r-3}, s_{r-2}s_{r-1}, t)).$$

Applying Theorem 1 repeatedly (or doing induction on r) yields

$$z_r(n, K(s_1, \ldots, s_{r-1}, t)) > n^{r-2-o(1)} \cdot z_2(n, K(s, t))$$

where $s = s_1 \cdots s_{r-1}$. Bukh [6] proved that $z(n, K(s, t)) = \Omega(n^{2-1/s})$ provided $t > 3^{s+o(s)}$ and this yields the following corollary.

Corollary 2. *Fix $r \geq 2$, and integers $1 \leq s_1 \leq \cdots \leq s_{r-1} < t$ where $t > 3^{s+o(s)}$ and $s = s_1 \cdots s_{r-1}$. Then as $n \to \infty$,*

$$z_r(n, K(s_1, \ldots, s_{r-1}, t)) = n^{r-1/s-o(1)}.$$

We remark that Theorem 1 can also be applied for small values of s_i. For example, using the result of Alon-Rónyai-Szabó [2] that $z(n, K(4, 7)) = \Omega(n^{7/4})$, it gives

$$z(n, K(2, 2, 7)) > n^{1-o(1)}\, z(n, K(4, 7)) > n^{1-o(1)}\, n^{7/4} = n^{11/4-o(1)},$$

where the exponent 11/4 is tight. For contrast, the previous best result due to Pohoata and Zakharov [17] yields only $z(n, K(2, 2, 721)) > \Omega(n^{11/4})$. If, as is widely believed, $z(n, K(4, 4)) = \Omega(n^{7/4})$, then this would imply via Theorem 1, that $z(n, K(2, 2, 4)) = n^{11/4-o(1)}$.

2. Proof

Write $e(H) = |E(H)|$ for a hypergraph H. To prove Theorem 1, we need the following well-known consequence of Behrend's construction [4] of a subset of $[n]$ with no 3-term arithmetic progression (see, e.g., [18]). There exists a bipartite graph G with parts of size n and $n^{2-o(1)}$ edges whose edge set is a union of n induced matchings. More precisely, there are pairwise disjoint matchings $M_1, \ldots, M_n$ such that $E(G) = \cup_{i=1}^{n} M_i$ and for all i, j the edge set $M_i \cup M_j$ contains no path with three edges. Additionally, $e(G) = \sum_i |M_i| = n^{2-o(1)}$.

Proof of Theorem 1. Let H' be an $(r-1)$-partite $(r-1)$-graph with parts $X_1, \ldots, X_{r-3}, [n]$ and Y each of size n with $e(H') = z_{r-1}(n, K')$ that contains no copy of the complete $(r-1)$-partite $(r-1)$-graph $K' = K(s_1, \ldots, s_{r-3}, s_{r-2}s_{r-1}, t)$. Here we only assume that there are no copies of K' where the ith part of size s_i is a subset of X_i for $1 \leq i \leq r-3$, the $r-2$ part of size $s_{r-2}s_{r-1}$ is a subset of $[n]$, and the rth part of size t is a subset of Y. Write $d(j)$ for the degree in H' of vertex $j \in [n]$, so $e(H') = \sum_j d(j)$. By relabeling we may assume that $d(1) \geq d(2) \geq \cdots \geq d(n)$.

Let G be a bipartite graph with parts A and B, each of size n comprising n induced matchings $M_1, \ldots, M_n$, with $e(G) = \sum_i |M_i| = n^{2-o(1)}$. Moreover, we may assume that $|M_1| \geq |M_2| \geq \cdots \geq |M_n|$.

Now define the r-partite r-graph H as follows: the parts of H are $X_1, \ldots, X_{r-3}, A, B, Y$, each of size n. For each $j \in [n]$, let

$$E_j = \{\{x_1, \ldots, x_{r-3}, a, b, y\} : \{x_1, \ldots, x_{r-3}, j, y\} \in E(H'), (a, b) \in A \times B, \{a, b\} \in M_j\}$$

and let $E(H) = \cup_{j=1}^n E_j$. Observe that $e(H) = \sum_j d(j)|M_j|$. In words, we have replaced vertex j that lies in edge $\{x_1, \ldots, x_{r-3}, j, y\}$ of H' by all possible pairs ab of M_j to create $|M_j|$ edges of H. Now Chebyshev's sum inequality and $e(G) = n^{2-o(1)}$ yield

$$\frac{1}{n}\sum_{j=1}^{n} d(j)|M_j| \geq \frac{1}{n^2}\sum_{j=1}^{n} d(j)\sum_{j=1}^{n} |M_j| = \frac{1}{n^2}e(H')e(G) = e(H')n^{-o(1)}.$$

Hence $e(H) = \sum_j d(j)|M_j| = n^{1-o(1)}e(H')$ as required.

Now suppose there is a copy L of $K = K(s_1, \ldots, s_{r-1}, t)$ in H where the part of size s_i lies in X_i for $1 \leq i \leq r-3$, the part A' of size s_{r-2} lies in A, the B' part of size s_{r-1} lies in B, and the part of size t lies in Y. Then all $s_{r-2}s_{r-1}$ pairs in $\binom{V(L)}{2}$ within $A' \times B'$ must come from different matchings M_i as the matchings are induced. Indeed, if there is an i such that ab and $a'b'$ are distinct edges of M_i, where $a, a' \in A'$ and $b, b' \in B'$, then ab' cannot lie in any edge of H, as M' is an induced matching, but ab' must lie in many edges of L, contradiction. The number of these matchings M_i is therefore $|A'||B'| = s_{r-2}s_{r-1}$ and each such matching M_j corresponds to a vertex j of $[n]$. This means that we have a forbidden copy of K' in H', contradiction. □

Remarks.

- One shortcoming of our approach is that it applies only to the Zarankiewicz problem and not the Turán problem. It would be interesting to rectify this.
- For some r-partite r-graphs H one can define an appropriate $(r-1)$-partite $(r-1)$-graph H' such that $z_r(n, H) > n^{1-o(1)}z_{r-1}(n, H')$. This may give some further new results for hypergraphs.
- For the proof, we do not need that the matchings M_i are induced, we only needed that for any two edges ab and $a'b'$ of M_i, either ab' or $a'b$ is not in any other matching. But this relaxed property doesn't seem to help improve the $n^{o(1)}$ error term.

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