7

The Euler–Lagrange Equations

In classical field theory, the dynamics of the physical system is revealed by analyzing the Euler–Lagrange (EL) equations corresponding to the classical action principle. These EL equations are the physical equations (like the Maxwell or Einstein equations). They have the mathematical structure of partial differential equations. Likewise, for the causal action principle and causal variational principles, the EL equations describe the dynamics. However, they are no longer differential equations but have a quite different form. In this chapter, we shall derive the *EL equations* and discuss their general structure.

7.1 The Euler–Lagrange Equations

Let ρ be a minimizer of the causal variational principle in the non-compact setting (more precisely, a minimizer with respect to variations of finite volume; see Section 6.3). We now derive the EL equations, following the method in the compact setting [74, Lemma 3.4]. We again define spacetime as the support of ρ,

$$M := \operatorname{supp} \rho \subset \mathcal{F} . \tag{7.1}$$

In words, the EL equations state that the causal action is minimal under first variations of the measure. In order to make mathematical sense of the variations, we need the following assumptions:

(i) The measure ρ is *locally finite* (meaning that any $x \in \mathcal{F}$ has an open neighborhood U with $\rho(U) < \infty$).

(ii) The function $\mathcal{L}(x, .)$ is ρ-integrable for any $x \in \mathcal{F}$, and the function

$$x \mapsto \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho(y), \tag{7.2}$$

is a bounded continuous function on $\mathcal{F}$.

These technical assumptions are satisfied in most applications and are sufficiently general for the purpose of this book (we note that the continuity assumption in (ii) could be relaxed to lower semi-continuity; the details are worked out in [66]). We introduce the function

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y)\, \mathrm{d}\rho(y) - \mathfrak{s} \;:\; \mathcal{F} \to \mathbb{R} , \tag{7.3}$$

where $\mathfrak{s} \in \mathbb{R}$ is a parameter whose value will be specified later.

Theorem 7.1.1 *(The Euler–Lagrange equations) Let ρ be a minimizer of the causal action with respect to variations of finite volume and assume that ρ satisfies the conditions* (i) *and* (ii) *given earlier. Then*

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell \,. \tag{7.4}$$

Proof Given $x_0 \in \operatorname{supp}\rho$, we choose an open neighborhood U with $0 < \rho(U) < \infty$. For any $y \in \mathcal{F}$, we consider the family of measures $(\tilde{\rho}_\tau)_{\tau\in[0,1)}$ given by

$$\tilde{\rho}_\tau = \chi_{M\setminus U}\,\rho + (1-\tau)\,\chi_U\,\rho + \tau\,\rho(U)\,\delta_y, \tag{7.5}$$

(where δ_y is the Dirac measure supported at y). Then

$$\tilde{\rho}_\tau - \rho = -\tau\,\chi_U\,\rho + \tau\,\rho(U)\,\delta_y = \tau\big(\rho(U)\,\delta_y - \chi_U\,\rho\big)\,. \tag{7.6}$$

Using this formula, one readily verifies that $\tilde{\rho}_\tau$ is a variation of finite volume satisfying the volume constraint. Hence

$$\begin{aligned} 0 \leq \big(\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\rho)\big) &= 2\tau\left(\rho(U)\left(\ell(y)+\mathfrak{s}\right) - \int_U \left(\ell(x)+\mathfrak{s}\right)\mathrm{d}\rho(x)\right) + \mathcal{O}\big(\tau^2\big) \\ &= 2\tau\left(\rho(U)\,\ell(y)) - \int_U \ell(x)\mathrm{d}\rho(x)\right) + \mathcal{O}\big(\tau^2\big), \end{aligned} \tag{7.7}$$

(here we may carry out the integrals in arbitrary order using Tonelli's theorem for nonnegative integrands). Since this holds for any $\tau \in [0,1)$, the linear term must be nonnegative, and thus

$$\ell(y) \geq \frac{1}{\rho(U)} \int_U \ell(x)\,\mathrm{d}\rho(x)\,. \tag{7.8}$$

Now assume that (7.4) is false. Then, there is $x_0 \in \operatorname{supp}\rho$ and $y \in \mathcal{F}$ such that $\ell(x_0) > \ell(y)$. Continuity of ℓ implies that there is an open neighborhood U of x_0 such that $\ell(x) > \ell(y)$ for all $x \in U$. But this contradicts (7.8). ☐

It is indeed no loss of generality to restrict attention to first variations within the special class (7.6); for details, see Exercise 7.1.

We always choose the parameter $\mathfrak{s}$ such that the infimum of ℓ in (7.4) is zero. Then, the EL equations read

$$\ell|_{\operatorname{supp}\rho} \equiv \inf_{\mathcal{F}} \ell = 0\,. \tag{7.9}$$

The parameter $\mathfrak{s}$ can be understood as the "action per volume" (see Exercise 7.2). We finally point out that solutions of the EL equations do not need to be minimizers of the causal action principle. Similar to the situation for local maxima or saddle points in the finite-dimensional setting, there may be variations for which $\mathcal{S}$ is stationary, but whose second or higher variations are negative.

7.2 The Restricted Euler–Lagrange Equations in the Smooth Setting

The EL equations (7.9) make a statement on the function ℓ even at points on $\mathcal{F}$ which are far away from spacetime M (see the left of Figure 7.1). In this way, the

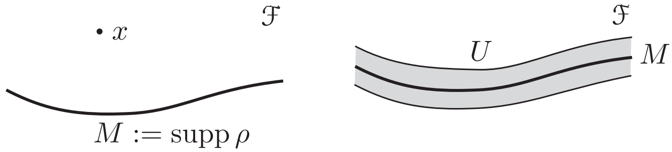

Figure 7.1 Evaluation of ℓ away from and near M.

EL equations contain much more information than conventional physical equations formulated in spacetime. At present, it is unclear how this additional information can be used or interpreted. One way of understanding this situation is to take the point of view that all information on the physical system must be obtained by performing observations or measurements in spacetime, which means that the information contained in ℓ away from M is inaccessible for fundamental reasons. Here, we shall not take sides or discuss whether or to which extent this point of view is correct. Instead, we simply note that it seems preferable and physically sensible to restrict attention to the function ℓ in an arbitrarily small neighborhood U of M in $\mathcal{F}$ (see the right of Figure 7.1). In practice, this means that we shall evaluate ℓ as well as its derivatives only on M. In this way, the causal action principle gives rise to an interaction described by equations in spacetime.

This concept leads us to the so-called *restricted EL equations*, which we now introduce. For technical simplicity, we again restrict attention to the *smooth setting* (for a more general derivation, see [62, Section 4]). This means that we assume that the Lagrangian is smooth (see (6.10) and the discussion thereafter). To avoid confusion, we point out that this assumption does *not* entail that spacetime $M := \operatorname{supp} \rho$ is a smooth manifold. Nevertheless, we can speak of a smooth function or a smooth vector field on M, meaning that the function (or vector field) has a smooth extension to $\mathcal{F}$.[1] Moreover, for technical simplicity, we also assume that the function ℓ defined by (7.3) is smooth on $\mathcal{F}$. Under these assumptions, the minimality of ℓ implies that the derivative of ℓ vanishes on M. We thus obtain the equations

$$\ell|_M \equiv 0 \qquad \text{and} \qquad D\ell|_M \equiv 0, \tag{7.10}$$

(where $D\ell(p) : T_p\mathcal{F} \to \mathbb{R}$ is the derivative). In order to combine these two equations in a compact form, it is convenient to consider a pair $\mathfrak{u} := (a, u)$ consisting of a real-valued function a on M and a vector field u on $T\mathcal{F}$ along M and to denote the combination of multiplication and directional derivative by

$$\nabla_{\mathfrak{u}} \ell(x) := a(x)\, \ell(x) + \big(D_u \ell\big)(x)\,. \tag{7.11}$$

[1] We remark that the question on whether a function or vector field on M can be extended smoothly to $\mathcal{F}$ is rather subtle. The needed conditions are made precise by Whitney's extension theorem (see, e.g., the more recent account in [34]). Here, we do not enter the details of these conditions but use them as implicit assumptions entering our notion of smoothness. We remark that these conditions are fulfilled whenever $M := \operatorname{supp} \rho$ carries a manifold structure.

The pair $\mathfrak{u} = (a, u)$ is referred to as a *jet*. This jet is a vector in a corresponding *jet space* $\mathfrak{J}$ defined by

$$\mathfrak{u} = (a, u) \in \mathfrak{J} := C^\infty(M, \mathbb{R}) \oplus \Gamma^\infty(M, T\mathcal{F}), \tag{7.12}$$

where $C^\infty(M, \mathbb{R})$ and $\Gamma^\infty(M, T\mathcal{F})$ denote the space of real-valued functions and vector fields on M, respectively, which admit a smooth extension to $\mathcal{F}$. Then, the equations (7.10) imply that $\nabla_\mathfrak{u} \ell(x)$ vanishes for all $x \in M$,

$$\nabla_\mathfrak{u} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}. \tag{7.13}$$

These are the so-called *restricted EL equations*. For brevity, a solution of the restricted EL equations is also referred to as a *critical measure*. We remark that, in the literature, the restricted EL equations are sometimes also referred to as the *weak* EL equations. Here, we prefer the notion "restricted" in order to avoid potential confusion with weak solutions of these equations (as constructed in [22]; see also Chapter 14).

7.3 Symmetries and Symmetric Criticality

In many applications, variational principles have an underlying symmetry (e.g., spherical symmetry or time independence). Typically, it simplifies the variational problem to vary within the class of functions that respect this symmetry. Having found a minimizer within this restricted class, the question arises whether it is also a minimizer of the full variational problem. The general answer to this question is no, simply because the absolute minimizer does not necessarily respect the symmetry of the variational principle. For causal variational principles, the situation is similar if we only replace "function" by "measure." As a simple example, we saw in Section 6.1 for the causal variational principle on the sphere that, although the variational principle is spherically symmetric, minimizing measures are typically weighted counting measures, thus breaking spherical symmetry.

Nevertheless, one can hope that minimizers within the class of symmetric functions are critical points of the full variational problem. This statement, referred to as the *principle of symmetric criticality*, has been formulated and proven under general assumptions in [126]. In this section, we explain how the principle of symmetric criticality can be stated and proved in the setting of causal variational principles. As we shall see, the proof is quite simple and rather different from that in the classical calculus of variations. We begin by explaining the basic idea in the simplest possible situation, where we consider the *compact setting* and also assume that the *symmetry group is compact*. Afterward, we explain how to treat a *non-compact symmetry group*.

As in Section 6.2, we let $\mathcal{F}$ be a compact manifold. Moreover, we again assume that the Lagrangian $\mathcal{L}$ is continuous (6.11), symmetric and strictly positive on the diagonal (see the assumptions (i) and (ii) in Definition 6.2.1). In order to describe the symmetry, we let $\mathcal{G}$ be a *compact Lie group*, which should act as a group of diffeomorphisms on $\mathcal{F}$ (for basics on Lie groups see, e.g., [118, Chapter 7]). More precisely, we assume the group action $\Phi : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$ to be a continuous mapping with the properties that $\Phi_g := \Phi(g, .)$ is a diffeomorphism of $\mathcal{F}$ for any $g \in \mathcal{G}$ and that

$$\Phi_g \circ \Phi_h = \Phi_{gh} \qquad \text{for all } g, h \in \mathcal{G} \,. \tag{7.14}$$

Moreover, the symmetry is expressed by the condition that the Lagrangian be invariant under the group action, that is,

$$\mathcal{L}(\Phi_g x, \Phi_g y) = \mathcal{L}(x, y) \qquad \text{for all } x, y \in \mathcal{F} \text{ and } g \in \mathcal{G} \,. \tag{7.15}$$

We denote the set of normalized regular Borel measures on $\mathcal{F}$ by $\mathfrak{M}$. Taking the push-forward of Φ, we obtain a group action on $\mathfrak{M}$ (for the definition of the push-forward measure, see again Section 2.3). We denote the measures that are invariant under this group action by $\mathfrak{M}^{\mathcal{G}}$, that is,

$$\mathfrak{M}^{\mathcal{G}} := \big\{ \rho \in \mathfrak{M} \,\big|\, (\Phi_g)_* \rho = \rho \quad \text{for all } g \in \mathcal{G} \big\} \,. \tag{7.16}$$

We also refer to the measures in $\mathfrak{M}^{\mathcal{G}}$ as being *equivariant* (for more details on equivariant causal variational principles, see [13, Section 4]).

Theorem 7.3.1 *Let ρ be a minimizer of the causal action under variations within the class $\mathfrak{M}^{\mathcal{G}}$ of equivariant measures. Then, ρ is a critical point of the full variational principle in the sense that the EL equations* (7.9) *hold.*

We point out for clarity that minimizers under variations in $\mathfrak{M}^{\mathcal{G}}$ will, in general, *not* be minimizers under variations in $\mathfrak{M}$. The reason is that the minimizers in $\mathfrak{M}$ are typically not invariant under the action of the group $\mathcal{G}$. A concrete example of this phenomenon is given in Exercise 7.4.

Proof of Theorem 7.3.1. We denote the orbits of the group action by $\langle x \rangle := \Phi_{\mathcal{G}} x$ with $x \in \mathcal{F}$. Since $\mathcal{G}$ is compact, so are the orbits. On $\mathcal{G}$, there is a uniquely defined normalized measure that is invariant under the group action by left multiplication, the so-called *normalized Haar measure* μ (for details on the Haar measure see, e.g., [118, Chapter 16]). A particular class of equivariant measures is obtained by taking the push-forward of μ by the mapping $\Phi(., x)$. More precisely, for given $x \in \mathcal{F}$, we define the (also normalized) Borel measure $\delta_{\langle x \rangle}$ on $\mathcal{F}$ by

$$\delta_{\langle x \rangle}(\Omega) := \mu\big(\{ g \in \mathcal{G} \,|\, \Phi(g, x) \in \Omega \} \big), \tag{7.17}$$

for any Borel set $\Omega \subset \mathcal{F}$. The subscript $\langle x \rangle$ indicates that, being equivariant, this measure depends only on the orbit.

Given $y \in \mathcal{F}$, we now consider the variation $(\tilde{\rho}_\tau)_{\tau \in [0,1)}$ within the class of equivariant measures defined by

$$\tilde{\rho}_\tau = (1 - \tau)\, \rho + \tau\, \delta_{\langle y \rangle} \,. \tag{7.18}$$

Note that, as a convex combination of two normalized measures, also $\tilde{\rho}_\tau$ is normalized. Using that ρ is a minimizer within this class, we can proceed similarly to the proof of Theorem 7.1.1 to obtain

$$\int_{\mathcal{F}} \ell(x)\, d\delta_{\langle y \rangle}(x) \geq \int_{\mathcal{F}} \ell(x)\, d\rho(x) \,. \tag{7.19}$$

Moreover, it follows by symmetry that the function ℓ is constant on the orbits because

$$\begin{aligned}\ell\big(\Phi_g y\big) &= \int_{\mathcal{F}} \mathcal{L}\big(\Phi_g y, x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} = \int_{\mathcal{F}} \mathcal{L}\big(y, \Phi_{g^{-1}} x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} \\ &= \int_{\mathcal{F}} \mathcal{L}\big(y, x\big)\, \mathrm{d}\rho(x) - \mathfrak{s} = \ell(y)\,, \end{aligned} \tag{7.20}$$

where in the first line we used the symmetry of $\mathcal{L}$, and in the second line, we used that ρ is equivariant. Hence, integrating over the orbit, we obtain

$$\ell(y) = \int_{\mathcal{F}} \ell(x)\ \mathrm{d}\delta_{\langle y \rangle}\,. \tag{7.21}$$

Combining this identity with (7.19), we conclude that

$$\ell(y) \geq \int_{\mathcal{F}} \ell(x)\, \mathrm{d}\rho(x) \qquad \text{for all } y \in \mathcal{F}\,. \tag{7.22}$$

Now we can argue exactly as in the proof of Theorem 7.1.1 to obtain the result. □

We next consider the case that the symmetry group $\mathcal{G}$ is a *non-compact Lie group.* A typical example is the translation group, giving rise to the homogeneous causal action principle as considered in [85]. We again assume that $\mathcal{G}$ acts on $\mathcal{F}$ as a group of diffeomorphisms $\Phi : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$. We can again single out the equivariant measures $\mathfrak{M}^{\mathcal{G}}$ by (7.16). Moreover, on $\mathcal{G}$, one can introduce a left-invariant measure μ (again referred to as the Haar measure). However, in contrast to the case of a compact Lie group, now the measure μ has infinite total volume. As a consequence, it cannot be normalized, and moreover, it is unique only up to a positive prefactor. It is a basic difficulty that for any nonzero equivariant measure ρ, the integrals in the causal action (6.8) diverge because the integral over the group elements g describing the symmetry (7.15) diverge. In simple terms, this group integral gives an infinite prefactor. This suggests that the problem could be cured simply by leaving out this integral. We now explain how this can be done. For simplicity, we restrict attention to the case that $\mathcal{G}$ *acts freely* (in the sense that $gx = x$ with $g \in \mathcal{G}$ implies that $g = e$ is the neutral element). Then, for any $x \in \mathcal{F}$, the mapping $g \mapsto \Phi(g, x)$ is a continuous injective mapping from $\mathcal{G}$ to $\mathcal{F}$. In other words, each orbit is homeomorphic to $\mathcal{G}$. Again denoting the space of orbits by $\mathcal{F}/\mathcal{G}$, we can thus identify $\mathcal{F} \simeq (\mathcal{F}/\mathcal{G}) \times \mathcal{G}$. Moreover, using this identification, the equivariant measure can be written as

$$\rho = \rho_{\mathcal{F}/\mathcal{G}} \times \mu\,, \tag{7.23}$$

where $\rho_{\mathcal{F}/\mathcal{G}}$ is a measure on the orbits. Now we replace the action (6.8) by

$$\mathcal{S}(\rho) = \int_{\mathcal{F}/\mathcal{G}} \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x) \int_{\mathcal{F}} \mathrm{d}\rho(y)\, \mathcal{L}(x, y) \qquad \text{with } \rho \in \mathfrak{M}^{\mathcal{G}}\,. \tag{7.24}$$

The *equivariant causal variational principle* is to minimize this action under variations in $\mathfrak{M}^{\mathcal{G}}$, leaving the total volume of $\rho_{\mathcal{F}/\mathcal{G}}$ fixed. If $\mathcal{F}/\mathcal{G}$ is compact, we can normalize this total volume by demanding that

$$\rho_{\mathcal{F}/\mathcal{G}}(\mathcal{F}/\mathcal{G}) = 1 \,. \tag{7.25}$$

If $\mathcal{F}/\mathcal{G}$ is non-compact, the volume constraint can be treated similarly, as explained for causal variational principles in the non-compact setting in Section 6.3. For more details on this procedure and the resulting existence theory, we refer to [13, Section 4] and [85].

The justification for considering the equivariant causal variational principle (7.24) is that it gives a method for constructing critical points of the full variational principle. The basis is the following result, which applies in the case that $\mathcal{F}/\mathcal{G}$ is compact.

Theorem 7.3.2 (Symmetric criticality for causal variational principles) *Let $\mathcal{G}$ be a non-compact Lie group acting freely on $\mathcal{F}$ as a group of diffeomorphisms. Assume that $\mathcal{F}/\mathcal{G}$ is compact. Let ρ be a minimizer of the equivariant causal action principle, which is normalized on the orbits* (7.25). *Then, ρ is a critical point of the full variational principle in the sense that the EL equations* (7.9) *hold.*

Proof The measure (7.17) is normalized on $\mathcal{F}/\mathcal{G}$. Therefore, the variation (7.18) satisfies the volume constraint (7.25). Computing the first variation of the action, in analogy to (7.18), we now obtain

$$\int_{\mathcal{F}/G} \ell(x)\; \mathrm{d}\delta_{\langle y \rangle}(x) \geq \int_{\mathcal{F}/\mathcal{G}} \ell(x)\, \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x), \tag{7.26}$$

(note that the integrands are constant on the orbits). Carrying out the integral on the left-hand side, we conclude that

$$\ell(y) = \int_{\mathcal{F}/\mathcal{G}} \ell(x)\, \mathrm{d}\rho_{\mathcal{F}/\mathcal{G}}(x) \,, \tag{7.27}$$

giving the claim. □

In the case that $\mathcal{F}/\mathcal{G}$ is not compact, it is not clear if minimizers exist. One strategy for constructing minimizers is to exhaust $\mathcal{F}/\mathcal{G}$ by compact sets, similarly to what is done in [66] for causal variational principles in the non-compact setting (see also Section 12.8). If an equivariant minimizer ρ exists, we know by symmetry that ℓ is constant on the orbits, and moreover, the corresponding EL equations imply that ℓ is minimal on the orbits in the support of ρ. Combining these facts, we immediately obtain the EL equations (7.4). In this way, we conclude that symmetric criticality always holds for causal variational principles.

7.4 Exercises

Exercise 7.1 *(More general first variations)* In the proof of Theorem 7.1.1, we restricted attention to very specific variations (7.5). In this exercise, we verify that the resulting EL equations (7.4) guarantee that the action is also minimal under more general variations. To this end, let μ be a normalized measure on $\mathcal{F}$, for technical simplicity with compact support. Consider variations of the form

$$\tilde{\rho}_\tau = \chi_{M \setminus U}\, \rho + (1-\tau)\, \chi_U\, \rho + \tau\, \rho(U)\, \mu. \tag{7.28}$$

Show that (7.4) implies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{S}\big(\tilde{\rho}_\tau\big)\Big|_{\tau=0} \geq 0\,. \tag{7.29}$$

Exercise 7.2 Assume that ρ is a minimizer of a causal variational principle with finite total volume. Show that the parameter $\mathfrak{s}$ in (7.3) takes the value

$$\mathfrak{s} = \frac{\mathcal{S}(\rho)}{\rho(\mathcal{F})}\,. \tag{7.30}$$

Exercise 7.3 *(Non-smooth EL equations)* We return to the example of the counting measure on the octahedron as considered in Exercise 6.3.

(a) Compute the function $\ell(x)$. Show that the EL equations (7.4) are satisfied.
(b) Show that the function ℓ is *not* differentiable at any point x of the octahedron. Therefore, it is not possible to formulate the restricted EL equations (7.13).

This example illustrates why in the research papers [61, 57] one carefully keeps track of differentiability properties by introducing suitable jet spaces.

Exercise 7.4 *(Symmetric criticality on the sphere)* We consider the causal variational principle on the sphere as introduced in Section 6.1.

(a) Show that the symmetric measure on the sphere

$$\mathrm{d}\mu(\vartheta, \varphi) = \frac{1}{4\pi}\, \mathrm{d}\varphi\, \sin\vartheta\, \mathrm{d}\vartheta, \tag{7.31}$$

is critical in the sense that it satisfies the EL equations (7.4).
(b) Use the minimizer with singular support constructed in Exercise 6.3 to argue that minimizers within the class of symmetric measures are, in general, *not* minimizers within the class of measures without symmetries. More details on this effect of symmetry breaking can be found in [43, 74, 80].