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Moduli space theory for complete, constant Q-curvature metrics on finitely punctured spheres

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We study constant Q-curvature metrics conformal to the the round metric on the sphere with finitely many point singularities. We show that the moduli space of solutions with finitely many punctures in fixed positions, equipped with the Gromov–Hausdorff topology, has the local structure of a real algebraic variety with formal dimension equal to the number of the punctures. If a nondegeneracy hypothesis holds, we show that a neighbourhood in the moduli spaces is actually a smooth, real-analytic manifold of the expected dimension. We also construct a geometrically natural set of parameters, construct a symplectic structure on this parameter space and show that in the smooth case a small neighbourhood of the moduli space embeds as a Lagrangian submanifold in the parameter space. We remark that our construction of the symplectic structure is quite different from the one in the scalar curvature setting, due to the fact that the associated partial differential equation is fourth-order rather than second-order.

Keywords: Q-curvature; moduli spaces; Paneitz–Branson operator

2020 *Mathematics Subject Classification:* 53C18, 58D17, 58D27

1. Introduction

In this manuscript, we study a fourth-order analog of the singular Yamabe problem on finitely punctured spheres, as formulated by Schoen and Yau [13], Mazzeo, Pollack, and Uhlenbeck [10] and others. Our main result characterizes the local structure of the moduli space of solutions in the case of the round metric on a finitely punctured sphere.

In general, if (M, g) is a Riemannian manifold of dimension $n \geq 5$ one defines the Q-curvature of g as

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|\operatorname{Ric}_g|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2, \tag{1}$$

where R_g and Ric_g are the scalar and Ricci curvatures and Δ_g is the Laplace-Beltrami operator. A short computation demonstrates that $Q_{\overset{\circ}{g}} = \frac{n(n^2-4)}{8}$, where $\overset{\circ}{g}$ is the usual round metric on the sphere $\mathbf{S}^n$. A longer computation shows that the task of finding a conformal metric $\widetilde{g} = U^{\frac{4}{n-4}}g$ with Q-curvature equal to $\frac{n(n^2-4)}{8}$ is equivalent to solving the nonlinear partial differential equation

$$\mathcal{H}_g(U) := P_g(U) - \frac{n(n-4)(n^2-4)}{16}U^{\frac{n+4}{n-4}} = 0, \tag{2}$$

where

$$\begin{aligned} P_g(U) := (-\Delta_g)^2(U) + \operatorname{div}\left(\frac{4}{n-2}\operatorname{Ric}_g(\nabla U, \cdot) - \frac{(n-2)^2+4}{2(n-1)(n-2)}R_g\nabla U\right) \\ + \frac{n-4}{2}Q_g U, \end{aligned} \tag{3}$$

is the Paneitz-Branson operator. This operator is conformally covariant, in that

$$P_{U^{\frac{4}{n-4}}g}(\phi) = U^{-\frac{n+4}{n-4}}P_g(U\phi). \tag{4}$$

Substituting $\phi = 1$ into (4) we obtain the transformation law for Q-curvature under a conformal change of metric, which is

$$Q_{U^{\frac{4}{n-4}}g} = \frac{2}{n-4}U^{-\frac{n+4}{n-4}}P_g(U). \tag{5}$$

In the case that the background metric is $\overset{\circ}{g}$ the Paneitz operator factors as

$$P_{\overset{\circ}{g}} = \left(-\Delta_{\overset{\circ}{g}} + \frac{(n-4)(n+2)}{4}\right)\left(-\Delta_{\overset{\circ}{g}} + \frac{n(n-2)}{4}\right). \tag{6}$$

This factorization in some sense simplifies the analysis of (2), making the study of the space of solutions in this setting seem more approachable. However, the conformal invariance of (4) combined with the noncompactness of the conformal group of the round metric leads to the blow-up of sequences of solutions. For this reason we study the following singular problem: given a closed subset $\Lambda \subset \mathbf{S}^n$, describe all the conformal metrics $g = U^{\frac{4}{n-4}}\overset{\circ}{g}$ that are complete on $\mathbf{S}^n\backslash\Lambda$ and have $Q_g = \dfrac{n(n^2-4)}{8}$. We can reformulate this geometric problem as the following infinite boundary value problem:

$$U : \mathbf{S}^n \backslash \Lambda \to (0, \infty), \qquad \mathcal{H}_{\mathring{g}}(U) = 0, \qquad \liminf_{p \to \Lambda} U(p) = \infty. \tag{7}$$

We concentrate on the case that $\Lambda = \{p_1, \ldots, p_k\}$ is a finite set of points and define the marked moduli space

$$\mathcal{M}_\Lambda = \left\{ g \in [\mathring{g}] : Q_g = \frac{n(n^2-4)}{8} \text{ and } g \text{ is complete on } \Lambda \right\}$$

and the unmarked moduli space

$$\mathcal{M}_k = \left\{ g \in [\mathring{g}] : Q_g = \frac{n(n^2-4)}{8}, \, g \text{ is complete on } \Lambda, \, \#\Lambda = k \right\}.$$

Here $[\mathring{g}]$ is the set of all conformal metric to $\mathring{g}$. We equip both moduli spaces with the Gromov–Hausdorff topology. The difference between the two is that in the marked moduli space we fix the singular points, whereas in the unmarked moduli space we allow them to vary, so long as they remain k distinct points.

Let $\Lambda = \{p_1, \ldots, p_k\} \subset \mathbf{S}^n$ and let $g \in \mathcal{M}_\Lambda$. Then g admits a definite asymptotic structure near each singular point p_i, and is asymptotic to one of the Delaunay metrics described below in § 2. These Delaunay metrics are, after an appropriate change of variables, periodic and uniquely described by their necksizes $\varepsilon \in (0, \bar{\varepsilon}]$, where the maximal necksize $\bar{\varepsilon}$ depends only on the dimension n, which allows one to assign asymptotic data to $g \in \mathcal{M}_k$, including the asymptotic necksize ε_i at the singular point p_i. We ask the following question: how well does this asymptotic data determine a metric $g \in \mathcal{M}_\Lambda$? Our results below form a first step in answering this question. More precisely, we show that under some conditions one can use the asymptotic data to parameterize a small neighbourhood of moduli space near $g \in \mathcal{M}_\Lambda$.

Our main theorem is the following result.

Theorem 1. *For each finite subset $\Lambda \subset \mathbf{S}^n$ with $\#\Lambda = k \geq 3$ the moduli space $\mathcal{M}_\Lambda$ is locally a real analytic variety of formal dimension k.*

Our proof follows the road map developed by Kusner, Mazzeo, Pollack and Uhlenbeck in [8] and [10], combining the implicit function theorem and the Lyaponov-Schmidt process. The key technical part of our analysis is a fine understanding the linearized operator L_g of the operator $\mathcal{H}_g$ defined in (2), which we describe in detail in § 2.1.

As is usually the case, the analysis allows us to make a more precise statement if L_g is injective or surjective when acting on an appropriate function space.

Definition 2. *A metric $g \in \mathcal{M}_\Lambda$ is nondegenerate if $w \in L^2$ and $L_g(w) = 0$ implies $w \equiv 0$.*

Theorem 3. *If $g \in \mathcal{M}_\Lambda$ is nondegenerate then there exists an open neighbourhood $\mathcal{U} \subset \mathcal{M}_\Lambda$ of* g *that is a real analytic manifold of dimension k.*

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As a part of proving these two local regularity theorems we construct a $2k$-dimensional parameter space $\mathcal{W}_g$ to parameterize all metrics $\mathcal{M}_\Lambda$ nearby a given g. Once we construct $\mathcal{W}_g$ and use it to understand the local regularity of the moduli space, we construct a symplectic structure on the geometric parameter space and show that, in the nondegenerate case, a small neighbourhood of g in $\mathcal{M}_\Lambda$ embeds in this parameter space $\mathcal{W}_g$ as a Lagrangian submanifold.

We remark here that our development of the symplectic form in § 6 is quite different from the development of the symplectic form in Section 7 of [10], mostly due to the fact that the associated PDE in our case is fourth-order and the PDE associate to scalar curvature is second-order. In the scalar curvature case one sees directly after integrating by parts twice that the integrand in the definition of the symplectic structure is the Wronksian of a certain second order ODE. This fact immediately implies the limit defining the symplectic structure exists and that it defines a nondegenerate bilinear form. In our case we must analyze the integrand more carefully to its connection to the Hamiltonian associated to the Delaunay solutions (see the proof of Theorem 16 in § 6 below). We believe this more direct connection between the symplectic form and the Hamiltonian of the Delaunay solutions is of independent interest.

Our results form a natural progression of the current understanding of constant Q-curvature metrics. Previously, C. S. Lin [9] showed that all smooth metrics with constant Q-curvature in the conformal class of the round metric must be the image of $\mathring{g}$ under a Möbius transformation. In the language we established above, $\mathcal{M}_0 = SO(n+1,1)$, which is the Möbius group of global conformal transformations of the sphere. In the same paper Lin proved there are no solutions with a single puncture, *i.e.* $\mathcal{M}_1 = \varnothing$ and that (after a conformal motion) any solution with two punctures is rotationally invariant. Afterwards Frank and König [5] characterized all the two ended solutions, showing $\mathcal{M}_{\{p,q\}} \simeq (0, \overline{\varepsilon}]$ for each $p \neq q$, where $\overline{\varepsilon}$ is a finite, positive number depending only on the dimension n. We describe these solutions in some detail below. In general, explicit gluing constructions demonstrate that the moduli space $\mathcal{M}_k$ is nonempty, provided $k \geq 2$. Baraket and Rebhi [4] constructed solutions with an even number of punctures by gluing cylinders together, using small necks as a bridge. Andrade, Wei and Ye [3] construct many examples in the conformal class of the sphere and the authors of this paper together with Andrade and do Ò [1] construct many other examples in the inhomogeneous setting, using a different gluing technique. Together with Andrade and do Ò [2], the second author described a geometric characterization of compact subsets of the moduli space. We remark also that the question of compactness of the space of solutions is completely resolved in the case that the background manifold is compact. If (M, g) is not conformally equivalent to the round sphere, then the set of solutions is compact precisely if $n \leq 24$ [6] and is not compact if $n \geq 25$ [15]. If the background metric is the round sphere then the set of solutions is never compact, due to the noncompactness of the group of Möbius transformations.

The rest of the paper proceeds as follows. In § 2 we discuss some analytic preliminaries, such as the Delaunay solutions, the local asymptotics of a singular Yamabe metric near a puncture and the appropriate functions spaces. In § 3 we analyze the mapping properties of the linearized operator L_g in various weighted function spaces and introduce the deficiency space $\mathcal{W}_g$, a $2k$-dimensional vector space that

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will serve as a parameter space to describe the asymptotic geometry of nearby metrics in $\mathcal{M}_\Lambda$. We prove Theorem 3 in Theorem 1 in § 4 and complete the proof of Theorem 1 in § 5. Finally, in Theorem 1 in § 6 we discuss a symplectic structure on the natural parameter space of $\mathcal{M}_\Lambda$ and prove that, near smooth points, the moduli space $\mathcal{M}_\Lambda$ is a Lagrangian submanifold of this parameter space.

2. Preliminaries

2.1. The choice of a gauge

The choice of a gauge in formulating the moduli problem is equivalent to choosing the background metric in a conformal class.

While we have thus far phrased this problem in the sphere, it will often be useful to rewrite in Euclidean space after stereographic projection and to transfer our analysis between the two settings. Let $\mathrm{Pr} : \mathbf{R}^n \to \mathbf{S}^n \backslash \{N\}$ be (the inverse of) stereographic projection mapping Euclidean space to the sphere minus a pole. It is now a standard exercise to verify that

$$\mathring{g} = u_{\mathrm{sph}}^{\frac{4}{n-4}} \delta, \qquad u_{\mathrm{sph}}(x) = \left(\frac{1+|x|^2}{2} \right)^{\frac{4-n}{2}},$$

where δ is the Euclidean metric. Using this transformation we can identify $g = U^{\frac{4}{n-4}} \mathring{g}$ with $g = u^{\frac{4}{n-4}} \delta$ where $u = U u_{\mathrm{sph}}$. We also denote the preimage of the singular set by $\widetilde{\Lambda} = \mathrm{Pr}^{-1}(\Lambda)$. Without loss of generality we let the north pole N be a smooth point of g, so that the conformal factor u decays at infinity. More precisely,

$$\limsup_{|x| \to \infty} |x|^{\frac{n-4}{2}} u(x) < \infty.$$

REMARK 4. Hereafter we adopt the convention that capital letters will denote conformal factors relative the round metric and lower case letters will denote conformal factors relative to the Euclidean metric. The two are always related as described above, *e.g.* $u = U u_{\mathrm{sph}}$.

Furthermore, we can also rephrase the condition that a metric lies in the moduli space $\mathcal{M}_\Lambda$ in the language of PDEs. Recall that $g \in [\mathring{g}]$ precisely when $g = U^{\frac{4}{n-4}} \mathring{g}$, so that the condition $Q_g = \frac{n(n^2-4)}{8}$ becomes

$$\mathcal{H}_{\mathring{g}}(U) = P_{\mathring{g}}(U) - \frac{n(n-4)(n^2-4)}{16} U^{\frac{n+4}{n-4}} = 0.$$

Thus

$$\mathcal{M}_\Lambda = \left\{ U : \mathbf{S}^n \backslash \Lambda \to (0,\infty) : \mathcal{H}_{\mathring{g}}(U) = 0 \text{ and } \liminf_{p \to \Lambda} U(p) = \infty \right\}.$$

In the spherical setting, the linearized operator has the form

$$L_g(v) = \left. \frac{d}{dt} \right|_{t=0} \mathcal{H}_{\mathring{g}}(U + tv) = P_{\mathring{g}}(v) - \frac{n(n+4)(n^2-4)}{16} U^{\frac{8}{n-4}} v, \tag{8}$$

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where the Paneitz operator $P_{\overset{\circ}{g}}$ can be factor as in (6).

The operators in question have an even simpler appearance in the Euclidean setting. This time we use the fact that $g = u^{\frac{4}{n-4}}\delta$, so $g \in \mathcal{M}_g$ is now equivalent to the PDE

$$\mathcal{H}_\delta(u) = (-\Delta_0)^2 u - \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}} = 0,$$

which in turn implies

$$\mathcal{M}_\Lambda = \Big\{ u : \mathbf{R}^n \backslash \widetilde{\Lambda} \to (0,\infty) : \mathcal{H}_\delta(u) = 0, \liminf_{x\to\widetilde{\Lambda}} u(x) = \infty \text{ and } \limsup_{|x|\to\infty} |x|^{\frac{n-4}{2}} u(x) < \infty \Big\}.$$

In the Euclidean setting the linearized operator has the form

$$L_g(v) = (-\Delta_0)^2 v - \frac{n(n+4)(n^2-4)}{16} u^{\frac{8}{n-4}} v. \tag{9}$$

In either setting, we refer to a function satisfying the PDE $L_g(v) = 0$ as a Jacobi field.

2.2. Delaunay metrics

The Delaunay metrics are all the constant Q-curvature metrics on a twice-punctured sphere and, as we will see later, play an important role in understanding the behavior of singular constant Q-curvature metrics with isolated singularities.

Consider a metric $g = U^{\frac{4}{n-4}}\overset{\circ}{g}$ on $\mathbf{S}^n \backslash \{p,q\}$ where p and q are distinct. After a rotation and a dilation, we can assume $p = N$ is the north pole and $q = S$ is the south pole. As in the previous section, we transfer now $\mathbf{R}^n \backslash \{0\}$ using stereographic projection and let $u = U u_{\mathrm{sph}}$. Using (4) we see that $u : \mathbf{R}^n \backslash \{0\} \to (0,\infty)$ satisfies

$$\mathcal{H}_\delta(u) = 0. \tag{10}$$

Frank and König [5] classified all the solutions of (10), and we describe them here. First we perform the Emden-Fowler change of coordinates, defining

$$\mathfrak{F} : \mathcal{C}^\infty(\mathbf{B}_r(0)\backslash\{0\}) \to \mathcal{C}^\infty((-\log r, \infty) \times \mathbf{S}^{n-1}), \qquad \mathfrak{F}(u)(t,\theta) = e^{\frac{4-n}{2}t} u(e^{-t}\theta). \tag{11}$$

We can of course invert $\mathfrak{F}$, obtaining

$$\mathfrak{F}^{-1}(v)(x) = |x|^{\frac{4-n}{2}} v(-\log|x|, \theta).$$

While the prefactor of $e^{\frac{4-n}{2}t}$ might look a little strange at first, a short computation shows it is geometrically necessary. Letting

$$\Upsilon : \mathbf{R} \times \mathbf{S}^{n-1} \to \mathbf{R}^n \backslash \{0\}, \qquad \Upsilon(t,\theta) = e^{-t}\theta,$$

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we see

$$\Upsilon^*(\delta) = e^{-2t} g_{\rm cyl}.$$

where $g_{\rm cyl} = dt^2 + d\theta^2$ is the cylindrical metric. If we now consider a conformal metric $g = u^{\frac{4}{n-4}}\delta$, we see that

$$\Upsilon^*(g)(t,\theta) = \mathfrak{F}(u)(t,\theta)^{\frac{4}{n-4}} g_{\rm cyl}.$$

After the Emden-Fowler change of coordinates, using (4), (10) becomes

$$\mathcal{H}_{\rm cyl}(v) = P_{\rm cyl}(v) - \frac{n(n-4)(n^2-4)}{16} v^{\frac{n+4}{n-4}} = 0, \tag{12}$$

where $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0,\infty)$ and

$$\begin{aligned} P_{\rm cyl} &= (-\Delta_{\rm cyl})^2 - \frac{n(n-4)}{2}\Delta_{\rm cyl} - 4\partial_t^2 + \frac{n^2(n-4)^2}{16} \qquad (13)\\ &= \partial_t^4 + \Delta_{\mathbf{S}^{n-1}}^2 + 2\Delta_{\mathbf{S}^{n-1}}\partial_t^2 - \frac{n(n-4)+8}{2}\partial_t^2 - \frac{n(n-4)}{2}\Delta_{\mathbf{S}^{n-1}} \\ &+ \frac{n^2(n-4)^2}{16} \end{aligned}$$

is the Paneitz operator of the cylindrical metric. Note that $\Delta_{\rm cyl} = \partial_t^2 + \Delta_{\mathbf{S}^{n-1}}$. C. S. Lin [9] used a moving planes argument to prove that solutions of (10) are rotationally invariant, reducing (12) to the ODE

$$\ddddot{v} - \frac{n(n-4)+8}{2}\ddot{v} + \frac{n^2(n-4)^2}{16} v - \frac{n(n-4)(n^2-4)}{16} v^{\frac{n+4}{n-4}} = 0. \tag{14}$$

Notice that one can find a first integral for this ODE defined as

$$\mathcal{H}_\varepsilon = -\dot{v}_\varepsilon \dddot{v}_\varepsilon + \frac{1}{2}\ddot{v}_\varepsilon^2 + \frac{n(n-4)+8}{4}\dot{v}_\varepsilon^2 - \frac{n^2(n-4)^2}{32} v_\varepsilon^2 + \frac{(n-4)^2(n^2-4)}{32} v_\varepsilon^{\frac{2n}{n-4}}. \tag{15}$$

We denote the nonzero constant solution of (14) by

$$\overline{\varepsilon} = \left(\frac{n(n-4)}{n^2-4}\right)^{\frac{n-4}{8}}.$$

THEOREM 5 (Frank and König [5]). *For each $\varepsilon \in (0,\overline{\varepsilon}]$ there exists a unique $v_\varepsilon : \mathbf{R} \to (0,\infty)$ solving the ODE (14) attaining its minimal value of* ε *at* t = 0*. All these solutions are periodic. Furthermore, let $v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0,\infty)$ be a smooth solution of the PDE (12). Then either $v(t,\theta) = (\cosh(t+T))^{\frac{4-n}{2}}$ for some $T \in \mathbf{R}$ or there exist $\varepsilon \in (0,\overline{\varepsilon}]$ and $T \in \mathbf{R}$ such that $v(t,\theta) = v_\varepsilon(t+T)$.*

Later in this paper we will use the fact that the set of Delaunay solutions is ordered by the Hamiltonian energy $\mathcal{H}$. In other words, $\mathcal{H}$ is a strictly decreasing function of the necksize ε.

We can now write the Delaunay metric in Euclidean coordinates by reversing the coordinate transformation (11), letting

$$u_\varepsilon(x) = \mathfrak{F}^{-1}(v)(x) = |x|^{\frac{4-n}{2}} v_\varepsilon(-\log|x|), \qquad g_\varepsilon = u_\varepsilon^{\frac{4}{n-4}} \delta = v_\varepsilon^{\frac{4}{n-4}} g_{\mathrm{cyl}}. \tag{16}$$

The geometric formulation of the Frank-König classification now reads: if $g = U^{\frac{4}{n-4}} \mathring{g}$ is a constant Q-curvature metric on $\mathbf{S}^n \backslash \{p, q\}$ then, after a global conformal transformation, either g extends to smoothly to the round metric or g is singular at both p and q and is the image of a Delaunay metric g_ε after said conformal transformation.

2.3. Local asymptotics

A metric $g = U^{\frac{4}{n-4}} \mathring{g} \in \mathcal{M}_k$ with constant Q-curvature and finitely many singular points has a definite asymptotic structure near each singular point. Let $p_i \in \Lambda$ be a singular point of g and choose stereographic coordinates x centered at p_i. With respect to these coordinates we have $g = u^{\frac{4}{n-4}} \delta = (U u_{\mathrm{sph}})^{\frac{4}{n-4}} \delta$ there exist $\varepsilon \in (0, \bar{\varepsilon}]$, $R > 0$, $a \in \mathbf{R}^n$ and $\beta > 1$ so that

$$\begin{aligned} u(x) = R^{\frac{n-4}{2}} u_\varepsilon(Rx) + |x|^{\frac{4-n}{2}} \Bigg(\langle x, a \rangle \left(\frac{n-4}{2} v_\varepsilon(-\log(R|x|)) - \dot{v}_\varepsilon(-\log(R|x|)) \right) \\ + \mathcal{O}(|x|^\beta) \Bigg). \end{aligned} \tag{17}$$

This expansion combines the local asymptotic expansions in [7] and in [12]. As is usually the case, the asymptotic expansion (17) is more tractable in Emden-Fowler coordinates. The transformed function $v = \mathfrak{F}(u)$ satisfies the equation (12) on the half-infinite cylinder $(T_0, \infty) \times \mathbf{S}^{n-1}$ and the asymptotic expansion now reads

$$v(t, \theta) = v_\varepsilon(t + T) + e^{-t} \langle a, \theta \rangle \left(\frac{n-4}{2} v_\varepsilon(t+T) - \dot{v}_\varepsilon(t+T) \right) + \mathcal{O}(e^{-\beta t}), \tag{18}$$

where $T = -\log R$.

These asymptotic expansions (17) and (18) allow us to define an asymptotes map

$$\mathcal{A} : \mathcal{M}_\Lambda \to (0, \bar{\varepsilon}]^k \times \mathbf{R}^k, \qquad \mathcal{A}(g) = (\varepsilon_1, \ldots, \varepsilon_k, T_1, \ldots, T_k), \tag{19}$$

where $g = u^{\frac{4}{n-4}} \delta$ and

$$u(x) \simeq \mathfrak{F}^{-1}(v_{\varepsilon_i}(-\log|x - p_i| + T_i)) \qquad \text{near } p_i. \tag{20}$$

We will see later on, in the proofs of Theorems 1 and 3, that the asymptotes maps provides us with local coordinates for the moduli space in the nondegenerate setting.

2.4. Weighted function spaces

We perform most of our analysis below on weighted Sobolev spaces. We first define these weighted spaces on a half-infinite cylinder, and then transfer the definition to

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a punctured ball (and thereafter to a finitely punctured sphere) using the Emden-Fowler change of coordinates.

DEFINITION 6. *Let $\delta \in \mathbf{R}$ and let $v \in L^2_{\mathrm{loc}}((0,\infty) \times \mathbf{S}^{n-1})$. We say $v \in L^2_\delta((0,\infty) \times \mathbf{S}^{n-1})$ if*

$$\|v\|^2_{L^2_\delta} = \int_0^\infty \int_{\mathbf{S}^{n-1}} e^{-2\delta t}|v(t,\theta)|^2 d\theta dt < \infty.$$

One can similarly define the Sobolev spaces $W^{k,2}_\delta((0,\infty) \times \mathbf{S}^{n-1})$ for any natural number k.

Observe that if $|v(t,\theta)| \leq Ce^{\widetilde{\delta} t}$ for each $\widetilde{\delta} < \delta$ and $t > 0$, then $v \in L^2_\delta((0,\infty) \times \mathbf{S}^{n-1})$. Next we undo the Emden-Fowler change of coordinates, letting $u = \mathfrak{F}^{-1}(v)$ to see

$$\begin{aligned}\int_{\mathbf{S}^{n-1}} \int_{t_1}^{t_2} e^{-2\delta t}|v(t,\theta)|^2 dt d\theta &= -\int_{\mathbf{S}^{n-1}} \int_{e^{-t_1}}^{e^{-t_2}} r^{2\delta+n-5}|u(r\theta)|^2 dr d\theta \\ &= \int_{r_2 \leq |x| \leq r_1} |x|^{2\delta-4}|u(x)|^2 d\mu_0(x),\end{aligned}$$

where $r_1 = e^{-t_1}$ and $r_2 = e^{-t_2}$. Here $d\mu_0$ is the Euclidean volume element. Thus we have the following definition.

DEFINITION 7. *Let $\delta \in \mathbf{R}$, let $\mathrm{r} > 0$ and let $u \in L^2_{\mathrm{loc}}(\mathbf{B}_r(0) \backslash \{0\})$. We say $u \in L^2_\delta(\mathbf{B}_r(0) \backslash \{0\})$ if*

$$\|u\|^2_{L^2_\delta} = \int_{\mathbf{B}_r(0) \backslash \{0\}} |x|^{2\delta-4}|u(x)|^2 d\mu_0 < \infty.$$

More generally we let $\widetilde{\Lambda} \subset \mathbf{R}^n$ be a finite set and $u \in L^2_{\mathrm{loc}}(\mathbf{R}^n \backslash \widetilde{\Lambda})$. We say $u \in L^2_\delta(\mathbf{R}^n \backslash \widetilde{\Lambda})$ if

$$\|u\|_{L^2_\delta} = \int_{\mathbf{R}^n \backslash \widetilde{\Lambda}} (\mathrm{dist}(x,\widetilde{\Lambda}))^{2\delta-4}|u(x)|^2 d\mu_0 < \infty.$$

Once again, we see that if $|u(x)| \leq C(\mathrm{dist}(x,\widetilde{\Lambda}))^{2-\widetilde{\delta}}$ near each singularity, for each $\widetilde{\delta} < \delta + \frac{n}{2}$, and $|u(x)| \leq C|x|^{\frac{4-n}{2}-\lambda}$ for $|x|$ sufficiently large and any $\lambda > \delta$, then $u \in L^2_\delta(\mathbf{R}^n \backslash \widetilde{\Lambda})$.

3. Linear analysis

3.1. The linearization about a Delaunay solution

Here we study the linearized operator about a Delaunay solution, which we denote as L_ε, and some of its mapping properties.

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Following Section 5.2 of [1] we write

$$L_\varepsilon = (-\Delta_0)^2 - \frac{n(n+4)(n^2-4)}{16} u_\varepsilon^{\frac{8}{n-4}}$$

and promptly transform to Emden-Fowler coordinates using (11), obtaining the operator $\mathcal{L}_\varepsilon$ defined by

$$\mathcal{L}_\varepsilon(w)(t,\theta) = e^{\frac{4-n}{2}t} L_\varepsilon(\mathfrak{F}^{-1}(w)) \circ \Upsilon(t,\theta) = \mathfrak{F}(L_\varepsilon(\mathfrak{F}^{-1}(w)))(t,\theta). \tag{21}$$

Some computation give us

$$\begin{aligned} \mathcal{L}_\varepsilon &= P_{\mathrm{cyl}} - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^{\frac{8}{n-4}} \\ &= \partial_t^4 + \Delta_{\mathbf{S}^{n-1}}^2 + 2\Delta_{\mathbf{S}^{n-1}}\partial_t^2 - \frac{n(n-4)}{2}\Delta_{\mathbf{S}^{n-1}} \\ &\qquad - \frac{n(n-4)+8}{2}\partial_t^2 + \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^{\frac{8}{n-4}}. \end{aligned} \tag{22}$$

Here P_{cyl} is given by (13).

We isolate two specific Jacobi fields of a Delaunay solution: the Jacobi field $w_0^+(\varepsilon)$ generating translations along the axis and the Jacobi field $w_0^-(\varepsilon)$ generating changes to the necksize. In Emden-Fowler coordinates these are given by

$$w_0^+(\varepsilon) = \dot{v}_\varepsilon, \qquad w_0^-(\varepsilon) = \frac{d}{d\varepsilon} v_\varepsilon. \tag{23}$$

Differentating the relation $v_\varepsilon(t+T_\varepsilon) = v_\varepsilon(t)$ it is straight-forward to verify that w_0^+ is bounded and periodic while w_0^- grows linearly. The formulation of $w_0^\pm$ above is well-formed in the case that $\varepsilon < \overline{\varepsilon}$, but both Jacobi fields vanish in the cylindrical case. If $\varepsilon = \overline{\varepsilon}$ we define

$$w_0^+ = \sin(\sqrt{\mu}t), \qquad w_0^- = \cos(\sqrt{\mu}t), \qquad \mu = \frac{\sqrt{n^4 - 64n + 64} - (n^2 - 4n + 8)}{4}. \tag{24}$$

The analysis in Proposition 1 of [4] shows these two Jacobi fields play the role of varying the necksize and translation parameter on the cylinder.

One can find the following results and their proofs in Section 3.6 of [12].

We first write a Jacobi field in Fourier series. Recall that the jth eigenvalue of $-\Delta_{\mathbf{S}^{n-1}}$ is $\lambda_j = j(n-1+j)$ and it has multiplicity

$$m_j = \binom{n-1+j}{j} + \binom{n-3+j}{j-2},$$

and so we can expand w in Fourier series as

$$w(t,\theta) = \sum_{j=0}^{\infty} \sum_{l=1}^{m_j} w_{j,l}(t) E_{j,l}(\theta),$$

where $\{E_{j,1},\ldots,E_{j,m_j}\}$ is an orthonormal basis of the eigenspace of $-\Delta_{\mathbf{S}^{n-1}}$ with eigenvalue λ_j. Thus the restriction of the operator $\mathcal{L}_\varepsilon$ to the eigenspace

$$\mathrm{Span}\{E_{j,1},\ldots,E_{j,m_j}\}$$

is the ordinary differential operator

$$\begin{aligned}\mathcal{L}_{\varepsilon,j} = \frac{d^4}{dt^4} - \frac{n(n-4)+8+4\lambda_j}{2}\frac{d^2}{dt^2} + \frac{n^2(n-4)^2}{16} + \frac{n(n-4)}{2}\lambda_j + \lambda_j^2\\ - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^{\frac{8}{n-4}}.\end{aligned}$$

Lemma 8. *For each $j \geq 1$ we have $0 \notin \mathrm{spec}(\mathcal{L}_{\varepsilon,j})$*

The two functions $w_0^{\pm}(\varepsilon)$ described above both lie in the kernel of $\mathcal{L}_{\varepsilon,0}$, and so $0 \in \mathrm{spec}(\mathcal{L}_{\varepsilon,0})$ for each $\varepsilon \in (0,\bar{\varepsilon}]$.

For proof of the next proposition see [12, Proposition 28].

Proposition 9. *There exists a discrete set of real numbers*

$$\Gamma_\varepsilon = \{\ldots, -\gamma_2(\varepsilon) < -\gamma_1(\varepsilon) < 0 < \gamma_1(\varepsilon) < \gamma_2(\varepsilon), \ldots\}, \tag{25}$$

with $\gamma_j(\varepsilon) \to \infty$ as $j \to \infty$ such that the operator

$$\mathcal{L}_\varepsilon : W^{4,2}_\delta((0,\infty)\times\mathbf{S}^{n-1}) \to L^2_\delta((0,\infty)\times\mathbf{S}^{n-1})$$

is Fredholm provided $\delta \notin \Gamma_\varepsilon$. In particular, for any $\delta \in (0,\gamma_1(\varepsilon))$

$$\mathcal{L}_\varepsilon : W^{4,2}_{-\delta}((0,\infty)\times\mathbf{S}^{n-1}) \to L^2_{-\delta}((0,\infty)\times\mathbf{S}^{n-1})$$

is injective and

$$\mathcal{L}_\varepsilon : W^{4,2}_\delta((0,\infty)\times\mathbf{S}^{n-1}) \to L^2_\delta((0,\infty)\times\mathbf{S}^{n-1})$$

is surjective.

One calls $\gamma_j(\varepsilon)$ the jth indicial root of the Jacobi operator $\mathcal{L}_\varepsilon$ and Γ_ε the set of indicial roots associated to the Delaunay solution v_ε.

Proposition 10. *Let $\phi : (0,\infty)\times\mathbf{S}^{n-1} \to \mathbf{R}$ be a smooth, compactly supported function and let $\mathcal{L}_\varepsilon(v) = \phi$. Then v satisfies the asymptotic expansion $v(t,\theta) \simeq \sum_{j=0}^\infty v_j(t,\theta)$ as $t \to +\infty$ where each $\mathrm{v_j}$ is a Jacobi field,* i.e. *$L_\varepsilon(v_j) = 0$, and $\mathrm{v_j}$ decays like a polynomial times $e^{-\gamma_j t}$, where $\gamma_j > 0$ is the* j*th indicial root.*

Corollary 11. (Linear Decomposition Lemma I) *Let $\delta \in (0,\gamma_1(\varepsilon))$, let $v \in W^{4,2}_\delta((0,\infty)\times\mathbf{S}^{n-1})$ and let $\phi \in \mathcal{C}^\infty((0,\infty)\times\mathbf{S}^{n-1}) \cap L^2_{-\delta}((0,\infty)\times\mathbf{S}^{n-1})$ be such that $\mathcal{L}_\varepsilon(v) = \phi$. Then there exist $z \in W^{4,2}_{-\delta}((0,\infty)\times\mathbf{S}^{n-1})$ and $w \in \mathrm{Span}(w_0^+(\varepsilon), w_0^-(\varepsilon))$ such that $v = z + w$.*

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For reasons that will become apparent later in the paper, we call $\mathcal{W}_\varepsilon = \mathrm{Span}(w_0^+(\varepsilon), w_0^-(\varepsilon))$ the deficiency space associated to the Delaunay metric with necksize ε.

3.2. The linearization about a singular Yamabe metric

We transfer the mapping properties of the linearization about a Delaunay solution to study the mapping properties of L_g, where $g \in \mathcal{M}_\Lambda$ is a conformally flat, singular, constant Q-curvature metric with k prescribed singularities. We denote the asymptotic necksize of the puncture p_j by ε_j, and define the indicial set

$$\Gamma_g = \bigcup_{i=1}^k \Gamma_{\varepsilon_i}.$$

It follows directly from Proposition 9 that

$$L_g : W_\delta^{4,2}(\mathbf{R}^n \backslash \widetilde{\Lambda}) \to L_{\delta-4}^2(\mathbf{R}^n \backslash \widetilde{\Lambda})$$

is Fredholm if and only if $\delta \notin \Gamma_g$.

Definition 12. *Let $g \in \mathcal{M}_\Lambda$ and choose $r_0 > 0$ sufficiently small such that $\mathbf{B}_{2r_0}(p_i) \cap \mathbf{B}_{2r_0}(p_j) = \varnothing$ for each distinct pair of punctures. We define the deficiency space $\mathcal{W}_g$ by*

$$\mathcal{W}_g = \mathrm{Span}\{\chi \mathfrak{F}^{-1}(w_0^+(\varepsilon_i)), \chi \mathfrak{F}^{-1}(w_0^-(\varepsilon_i)) : i = 1, \ldots, k\},$$

where χ is a fixed cut-off function such that

$$\chi(x) = \begin{cases} 1 & |x| < r_0 \\ 0 & |x| > 3r_0/2 \end{cases}, \qquad \|\nabla^k \chi\|_{\mathcal{C}^0} \leq c r^{-k}.$$

Proposition 13. (Linear decomposition lemma II) *Let $0 < \delta < \min_{1 \leq i \leq k} \gamma_1(\varepsilon_i)$ and let $u \in W_\delta^{4,2}(\mathbf{R}^n \backslash \widetilde{\Lambda})$ and $\phi \in L_{-\delta-4}^2(\mathbf{R}^n \backslash \widetilde{\Lambda})$ satisfying $L_g(u) = \phi$. Then there exist $w \in \mathcal{W}_g$ and $v \in W_{-\delta}^{4,2}(\mathbf{R}^n \backslash \widetilde{\Lambda})$ such that $u = w + v$.*

We now define the bounded null space. Once again we fix a number δ such that $0 < \delta < \min_{1 \leq i \leq k} \gamma_1(\varepsilon_i)$. Each element of the bounded null space is, strictly speaking, an equivalence class of functions, that is

$$\mathcal{B}_g = \frac{\ker(L_g : W_\delta^{4,2} \to W_{\delta-4}^{0,2})}{\ker(L_g : W_{-\delta}^{4,2} \to W_{-\delta-4}^{0,2})}.$$

Using the Hilbert space structure of $W_\delta^{k,2}$ we can identify

$$\mathcal{B}_g \simeq \{\ker(L_g : W_\delta^{4,2} \to W_{\delta-4}^{0,2})\} \cap \{\ker(L_g : W_{-\delta}^{4,2} \to W_{-\delta-4}^{0,2})\}^\perp.$$

Combining this characterization with the linear decomposition lemma we see that one can identify any $v = w + \phi$ for any $v \in \mathcal{B}$, where $w \in \mathcal{W}_g$ and $\phi \in W_{-\delta}^{4,2}(\mathbf{S}^n \backslash \Lambda)$ decays at each puncture.

Applying Melrose's relative-index calculus we show the following dimension count.

THEOREM 14. $\dim(\mathcal{B}_g) = k$.

The proof below is more or less the same as the proof of Theorem 4.24 in [10].

Proof. We compute the relative index of L_g acting on the appropriate weighted function spaces. Recall that the index of

$$L_g : W^{4,2}_{\delta}(\mathbf{R}^n \backslash \widetilde{\Lambda}) \to W^{0,2}_{\delta-4}(\mathbf{R}^n \backslash \widetilde{\Lambda})$$

is

$$\operatorname{ind}(\delta) = \dim(\ker(L_g)) - \dim(\operatorname{coker}(L_g)).$$

Integration by parts shows that the L^2-adjoint of L_g acting on $W^{4,2}_{\delta}$ is L_g acting on $W^{4,2}_{-\delta}$, and so it follows

$$\operatorname{ind}(-\delta) = -\operatorname{ind}(\delta)$$

provided $\delta \notin \Gamma_g$. (In this case, reversing the sign of the weight δ exchanges the kernel and the cokernel.) Next recall that, provided $\delta_1, \delta_2 \notin \Gamma_g$, the relative index is defined as

$$\text{rel-ind}(\delta_1, \delta_2) = \operatorname{ind}(\delta_1) - \operatorname{ind}(\delta_2).$$

We use duality once more (*i.e.* the operator L_g is formally self-adjoint in L^2) to see so long as $0 < \delta < \min_{1 \le i \le k} \gamma_1(\varepsilon_i)$ we have

$$\text{rel-ind}(\delta, -\delta) = \operatorname{ind}(\delta) - \operatorname{ind}(-\delta) = 2 \operatorname{ind}(\delta) \tag{26}$$

$$= 2\left(\dim(\ker(L_g|_{W^{4,2}_{\delta}})) - \dim(\ker(L_g|_{W^{4,2}_{-\delta}}))\right) = 2 \dim \mathcal{B}_g.$$

Thus it suffices to show that $\text{rel-ind}(\delta, -\delta) = 2k$ for an appropriate choice of δ. We choose $0 < \delta < \min_{1 \le i \le k} \gamma_1(\varepsilon_i)$.

We compute this relative index theorem using the Melrose's relative index theorem (see Theorem 6.5 of [11]). We first decompose

$$\mathbf{S}^n \backslash \{p_1, \ldots, p_k\} = \Omega^c \cup \left(\bigcup_{i=1}^{k} \mathbf{B}_r(p_i) \backslash \{p_i\} \right).$$

We can now write L_g as the sum of restrictions

$$L_g = L_g|_{\Omega^c} + \sum_{i=1}^{k} L_g|_{\mathbf{B}_r(p_i) \backslash \{p_i\}}$$

and compute the relative index of each restriction separately. It might appear that we first have to take the boundary data of the restrictions into account, but since

$$\partial \Omega^c = \bigcup_{i=1}^{k} \partial \mathbf{B}_r(p_i),$$

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except for the opposite orientations, the boundary contributions in the relative indices will cancel out. Thus it suffices to use the Dirichlet boundary data on the k spheres $\{\partial \mathbf{B}_r(p_i)\}$.

The operator L_g is elliptic and has index zero on Ω^c, so now we're left with computing the relative index of the restriction $L_g|_{\mathbf{B}_r(p_i)\backslash\{p_i\}}$ for some $i \in \{1, \ldots, k\}$. Next we observe that, because the relative index is a topological invariant, we can deform the metric of g to be **exactly** Delaunay in a small neighborhood of each puncture p_j. After transforming to cylindrical coordinates using the Emden-Fowler change of coordinates $\mathfrak{F}$, we finally arrive at the problem of computing the relative index of

$$\mathcal{L}_{\varepsilon_i} : W_\delta^{4,2}((0,\infty) \times \mathbf{S}^{n-1}) \to W_\delta^{0,2}((0,\infty) \times \mathbf{S}^{n-1}).$$

This is where we use Melrose's machinery, as developed in Chapters 4,5 and 6 of [11]. To do so we introduce the Fourier-Laplace transform

$$\widehat{v}(t,\zeta,\theta) = \mathcal{F}_{\varepsilon_i}(v)(t,\zeta,\theta) = \sum_{m=-\infty}^{\infty} e^{-im\zeta} v(t + mT_{\varepsilon_i}, \theta) \tag{27}$$

and the twisted operator

$$\widetilde{\mathcal{L}_{\varepsilon_i}} : W^{4,2}(\mathbf{S}^1_{T_{\varepsilon_i}} \times \mathbf{S}^{n-1}) \to W^{0,2}(\mathbf{S}^1_{T_{\varepsilon_i}} \times \mathbf{S}^{n-1})$$

defined by

$$\widetilde{\mathcal{L}_{\varepsilon_i}}(\zeta)(\widehat{v}) = e^{i\zeta}\mathcal{F}_{\varepsilon_i} \circ \mathcal{L}_{\varepsilon_i} \circ \mathcal{F}_{\varepsilon_i}^{-1}(e^{-i\zeta}\widehat{v}). \tag{28}$$

We make several observations before continuing. First observe that $\zeta \in \mathbf{C}$ is a parameter in the Fourier-Laplace transform, and the sum in (27) converges precisely when ζ is in the half-space $\{\zeta : \mathrm{Im}(\zeta) < -\delta T_{\varepsilon_i}\}$. Next observe that $\widetilde{\mathcal{L}_{\varepsilon_i}}$ is now a family of operators defined between the **fixed** function spaces $W^{4,2}(\mathbf{S}^1_{T_{\varepsilon_i}} \times \mathbf{S}^{n-1})$ and $W^{0,2}(\mathbf{S}^1_{T_{\varepsilon_i}} \times \mathbf{S}^{n-1})$, that depends holomorphically on the complex parameter ζ. This allows us to use the analytic Fredholm theorem (see Section 5.2 of [11]) to conclude that $\widetilde{\mathcal{L}_{\varepsilon_i}}$ is Fredholm so long as ζ avoids a discrete set in the complex plan, which in turn allows us to define a right-inverse $\widetilde{\mathcal{G}_{\varepsilon_i}}(\zeta)$ for $\widetilde{\mathcal{L}_{\varepsilon_i}}$. This right inverse $\widetilde{\mathcal{G}_{\varepsilon_i}}$ has a meromorphic extenstion to $\mathbf{C}$ with poles at $\widetilde{\Gamma}_{\varepsilon_i}$. In fact, the indicial roots Γ_{ε_i} are precisely the imaginary parts of the points in $\widetilde{\Gamma}_{\varepsilon_i}$.

Melrose's relative index theorem states in this context that the relative index is given by a contour integral of the resolvent $(\widetilde{\mathcal{L}_{\varepsilon_i}} - \zeta)^{-1}$ about a contour surrounding the pole corresponding to the weight 0, as described in the proof of Proposition 26 of [12] and the proof of Proposition 4.15 of [10]. This contour integral counts the number of tempered, non-decaying Jacobi fields with subexponential growth on a Delaunay end. However, we already know there are only two such Jacobi fields, namely $w_0^+(\varepsilon_i)$ and $w_0^-(\varepsilon_i)$. We conclude that

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$$\begin{aligned}
2\dim(\mathcal{B}_g) &= \sum_{i=1}^{k} \mathrm{ind}(\mathcal{L}_{\varepsilon_i} : W^{2,2}_{\delta}((0,\infty)\times \mathbf{S}^{n-1}) \to W^{-2,2}_{\delta}((0,\infty)\times \mathbf{S}^{n-1})) \\
&\quad - \sum_{i=1}^{k} \mathrm{ind}(\mathcal{L}_{\varepsilon_i} : W^{2,2}_{-\delta}((0,\infty)\times \mathbf{S}^{n-1}) \to W^{-2,2}_{-\delta}((0,\infty)\times \mathbf{S}^{n-1})) \\
&= \sum_{i=1}^{k} 2 = 2k,
\end{aligned}$$

as we claimed. □

4. Local structure in the nondegenerate case

In this section, we prove local regularity of the moduli space near nondegenerate points, as stated in Theorem 3. We first recall the statement of the theorem, namely that if $g \in \mathcal{M}_\Lambda$ is nondegenerate then there exists an open neighborhood $\mathcal{U} \subset \mathcal{M}_\Lambda$ of g that is a smooth k-dimensional manifold.

Proof. We begin by prescribing the singular set $\Lambda = \{p_1, p_2, \ldots, p_k\}$ and choosing a nondegenerate metric $g \in \mathcal{M}_\Lambda$. Using the Euclidean gauge, we write g as $g = u^{\frac{4}{n-4}}\delta$, where

$$u : \mathbf{R}^n \backslash \widetilde{\Lambda} \to (0,\infty), \quad (-\Delta_0)^2 u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}}, \quad \liminf_{x\to\widetilde{\Lambda}} u(x) = \infty.$$

Nondegeneracy of g states that the linearized operator

$$L_g = (-\Delta_0)^2 - \frac{n(n+4)(n^2-4)}{16} u^{\frac{8}{n-4}}$$

acting on $W^{4,2}(\mathbf{R}^n \backslash \{\mathrm{Pr}^{-1}(p_1), \ldots, \mathrm{Pr}^{-1}(p_k)\})$ has no kernel. By the linear decomposition lemma, this is equivalent to the condition that

$$\ker(L_g : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \to W^{0,2}_{-\delta}) = \mathcal{B}_g, \tag{29}$$

whenever $\delta > 0$ is sufficiently small. The bounded null space $\mathcal{B}_g$ always lies in the kernel of (29), but in the degenerate case the kernel will also contain a finite-dimensional space of decaying Jacobi fields.

Intuitively, we would like to describe the metrics in $\mathcal{M}_\Lambda$ near g as

$$\begin{aligned}
\mathcal{U} = \bigg\{ g_v = (u+v)^{\frac{4}{n-4}}\delta : \mathcal{H}_\delta(u+v) = (-\Delta_0)^2(u+v) - \frac{n(n-4)(n^2-4)}{16} \\
\times (u+v)^{\frac{n+4}{n-4}} = 0 \bigg\},
\end{aligned}$$

where v is small with respect to an appropriate norm. If we only allow v to decay, the linearized operator does not have any kernel by our hypothesis, and so it would be an exercise in futility to construct a solution set this way. Furthermore, we

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should allow the nearby metrics to have slightly different asymptotic data, which we cannot encode with a decaying perturbing function v. On the other hand, if we allow perturbing functions v with any order of growth (or even non-decay), it is difficult to analyze the zero-set of the operator $\mathcal{H}$, and in particular it is impossible to relate the kernel of the linearization to this zero-set. We remedy this problem by deforming the asymptotic data according to an element of the deficiency space $\mathcal{W}_g$, as described below.

We denote the asymptotic necksize of g at the puncture p_i by ε_i. Choose δ such that

$$0 < \delta < \min_{1 \le i \le k} \gamma_1(\varepsilon_i).$$

We can identify conformally-related, constant Q-curvature metrics in a neighborhood of g with

$$\mathcal{Z} = \{(v,w) \in \mathcal{V}_1 \oplus \mathcal{V}_2 \subset W^{4,2}_{-\delta} \oplus \mathcal{W}_g : \mathcal{H}(v,w) = 0\}, \tag{30}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are small neighborhoods of the origin. To make sense of this, we should describe the mapping

$$\mathcal{H} : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \to W^{0,2}_{-\delta} \tag{31}$$

in some detail. By the expansion (17) (or, equivalently (18)) there exist parameters $\varepsilon_i \in (0, \overline{\varepsilon})$, $T_i \in \mathbf{R}$ and a decaying function $z \in W^{4,2}_{-\delta}(\mathbf{B}_r(0))$ such that

$$u(x - p_i) = \mathfrak{F}^{-1}\left(v_{\varepsilon_i}(-\log|\cdot - p_i| + T_i)\right) + z(x - p_i).$$

Now let $v \in W^{4,2}_{-\delta}$ and let $w \in \mathcal{W}_g$. By definition,

$$w = \sum_{i=1}^{k} (a_i^+ w_0^+(\varepsilon_i) + a_i^- w_0^-(\varepsilon_i)),$$

where $a_i^{\pm} \in \mathbf{R}$. We define the metric $\widetilde{g} = \widetilde{u}^{\frac{4}{n-4}}\delta$, where

$$\widetilde{u}(x) = \begin{cases} u(x) + v(x) & |x - p_i| > 2r_0 \\ v(x) + (1 - \chi(x))u(x) + & \\ \chi(x)(\mathfrak{F}^{-1}(v_{\varepsilon_i + a_i^-}(-\log|\cdot - p_i| + T_i + a_i^+)) & r_0 < |x - p_i| < 2r_0 \\ +z(x - p_i)) & \\ \mathfrak{F}^{-1}(v_{\varepsilon_i + a_i^-}(-\log|\cdot - p_i| + T_i + a_i^+) & 0 < |x - p_i| < r_0, \\ +z(x - p_i) + v(x - p_i) & \end{cases} \tag{32}$$

where r_0 and χ are as in Definition 12. Observe that the coefficients $\{a_i^+, a_i^-\}$ uniquely determine function $w \in \mathcal{W}_g$, so the dependence of $\widetilde{u}$ on w is given in how we deform the geometric asymptotic data of $g = u^{\frac{4}{n-4}}\delta$. The construction of $\widetilde{g}$ is well-defined so long as $\varepsilon_i < \overline{\varepsilon}$, but we must adjust it slightly if $\varepsilon_i = \overline{\varepsilon}$. In this case we replace

$$v_{\varepsilon_i + a_i^-}(-\log|\cdot - p_i| + T_i + a_i^+)$$

with

$$\widetilde{v}(t,\theta) = v_{\overline{\varepsilon}} + a_i^- \cos(\sqrt{\mu}(t + T_i + a_i^+) + \mathcal{O}(e^{-t}),$$

as constructed in Proposition 1 of [4].

Finally, we identify

$$\mathcal{H}(v,w) = \mathcal{H}_g(\widetilde{u}) = \widetilde{u}^{\frac{n+4}{n-4}} \left(P_{\widetilde{g}}(1) - \frac{n(n-4)(n^2-4)}{16} \right) \tag{33}$$

$$= \frac{(n-4)}{2} \widetilde{u}^{\frac{n+4}{n-4}} \left(Q_{\widetilde{g}} - \frac{n(n^2-4)}{8} \right).$$

We also observe that, by construction, $\mathcal{H}(0,0) = 0$.

With this definition, we see that the zero-set $\mathcal{Z}$ is exactly the set of constant Q-curvature metrics whose asymptotic data are close to that of g. Observe that we should not expect $\mathcal{H}(0,w) = 0$ for any nonzero element of the deficiency space $\mathcal{W}_g$. This is because we construct elements of $\mathcal{W}_g$ using a cut-off function χ to transfer deformations of the Delaunay asymptotes to the background metric g, and so the Q-curvature is non-constant in the transition region, where $\nabla \chi \neq 0$. However, the quantity $Q_{\widetilde{g}} - \frac{n(n^2-4)}{8}$ is small (assuming w is small) and compactly supported. Thus we expect to be able correct the Q-curvature with a decaying function $v \in W^{4,2}_{-\delta}$, exactly as described above. Additionally, the linearization of the operator $\mathcal{H}_g$ as applied to $W^{4,2}_{-\delta} \oplus \mathcal{W}_g$ is

$$L_g : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \to W^{0,2}_{-\delta-4}$$

and

$$\ker\left(L_g : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \to W^{0,2}_{-\delta-4}\right) = \mathcal{B}_g \oplus \ker\left(L_g : W^{4,2}_{-\delta} \to W^{0,2}_{-\delta-4}\right). \tag{34}$$

However, since g is nondegenerate the second summand on the right hand side of (34) is just to 0 function. Thus the kernel of L_g is precisely $\mathcal{B}_g$, which has dimension k, which is also its minimal possible dimension. Thus $\dim(\ker(L_{\widetilde{g}})) = k$ on an open neighborhood of g in $\mathcal{M}_\Lambda$, and so by the implicit function theorem an open neighborhood of $\mathcal{Z}$ containing g is a smooth, k-dimensional manifold. □

5. Local structure in the degenerate case

Our purpose is to discuss the local structure of the moduli space $\mathcal{M}_\Lambda$ without the hypothesis that the linearization has a trivial L^2-nullspace. In this context, we apply the Lyapunov-Schmidt argument as presented in [8]. The key idea goes back to Simon's proof of an infinite-dimensional version of the Łojasiewicz inequality, see Theorem 3 of [14].

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THEOREM 15. *The space $\mathcal{M}_\Lambda$ is locally a finite dimensional real analytic variety.*

Proof. Once again, our problem can be reduced to the understanding of the zero set of

$$\mathcal{H} : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \to W^{0,2}_{-\delta},$$

where $\mathcal{H}$ is defined in (32) and (33). This time, however, the kernel of the linearization, which we denote as $K \equiv \ker\left(L_g : W^{4,2}_{-\delta} \to W^{0,2}_{-\delta-4}\right)$, is nontrivial in $W^{4,2}_{-\delta}$, and it can identify by duality with the cokernel of $L_g : W^{4,2}_{\delta} \to W^{0,2}_{\delta-4}$.

Following [8] we define

$$\widetilde{\mathcal{H}} : W^{4,2}_{-\delta} \oplus \mathcal{W}_g \oplus K \to W^{0,2}_{-\delta}, \qquad \widetilde{\mathcal{H}}(v, w, \phi) = \mathcal{H}(v, w) + \phi,$$

so that

$$\begin{aligned} \mathcal{Z} &= \{(v, w) \in \mathcal{V}_1 \oplus \mathcal{V}_2 : \mathcal{H}(v, w) = 0\} \\ &= \{(v, w, \phi) \in \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 : \widetilde{\mathcal{H}}(v, w, \phi) = \phi\}. \end{aligned}$$

where $\mathcal{V}_1 \subset W^{4,2}_{-\delta}$, $\mathcal{V}_2 \subset \mathcal{W}_g$ and $\mathcal{V}_3 \subset K$ are small neighborhoods of the origin in each respective Banach space.

We now see that

$$\mathcal{Z} \subset \widetilde{\mathcal{Z}} = \{(v, w, \phi) \in \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 : \widetilde{\mathcal{H}}(v, w, \phi) \in K\} \tag{35}$$

$$= \{(v, w, \phi) \in \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 : \Pi^\perp(\widetilde{\mathcal{H}}(v, w, \phi)) = 0\} = \ker(\Pi^\perp \circ \widetilde{\mathcal{H}}),$$

where $\Pi^\perp$ is the orthogonal projection of $W^{0,2}_{-\delta}$ onto $K^\perp$. The linearization of this operator is given by

$$\Pi^\perp \circ L_g : K^\perp \oplus \mathcal{W}_g \oplus K \to W^{0,2}_{-\delta},$$

which is now a surjective operator. Furthermore, we can characterize the kernel of the linearization as

$$\ker(\Pi^\perp \circ L_g) = \{(v, w, \phi) \in W^{4,2}_{-\delta} \oplus \mathcal{W}_g \oplus K : L_g(v + w) \in K\} \simeq K \oplus \mathcal{B}_g.$$

Thus by the implicit function theorem, there is a real-analytic function

$$\Psi : K \oplus \mathcal{B}_g \oplus K \to (W^{4,2}_{-\delta} \oplus \mathcal{W}_g)/(K \oplus \mathcal{B}_g), \qquad \Psi(v, w, \phi) = (\psi_1(v, w, \phi), \psi_2(v, w, \phi))$$

such that

$$\widetilde{\mathcal{Z}} = \{(\psi_1(v, w, \phi), \psi_2(v, w, \phi), \phi) : (v, w, \phi) \in K \oplus \mathcal{B}_g \oplus K\}.$$

Unraveling these definitions we see

$$\mathcal{Z} \simeq \{(v, w, \phi) \in K \oplus \mathcal{B}_g \oplus K : \mathcal{H}(v + \psi_1(v, w, \phi), w + \psi_2(v, w, \phi)) = 0\},$$

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which is indeed the zero set of an analytic function acting on a finite-dimensional vector space. This proves a small neighborhood $\mathcal{U} \subset \mathcal{M}_\Lambda$ containing g is indeed a real-analytic variety. $\square$

6. Symplectic structure

Here we discuss the asymptotes mapping from the marked moduli space $\mathcal{M}_\Lambda$ into a fixed configuration space $\mathbb{M}_\Lambda = (0, \overline{\varepsilon}]^k \times \mathbf{R}^k$, where each pair (ε_i, T_i) characterizes the Delaunay asymptote at the puncture p_i. We further show that if $g \in \mathcal{M}_\Lambda$ is nondegenerate, then this local mapping is a Lagrangian embedding with respect to the standard symplectic structure. One can construct a similar asymptotes map for the unmarked moduli space, and much of the properties we prove below carry through, but in this latter case, the configuration spaces are larger and constructing the mapping is more involved.

First we construct a symplectic form on $\mathcal{M}_\Lambda$. Let $g = U^{\frac{4}{n-4}} \overset{\circ}{g} \in \mathcal{M}_\Lambda$ and transfer g to $\mathbf{R}^n \backslash \widetilde{\Lambda}$ using stereographic projection, rewriting $g = u^{\frac{4}{n-4}} \delta$ with $u = U u_{\mathrm{sph}}$. For any sufficiently small $r > 0$ we define

$$\Omega_r = \mathbf{R}^n \backslash \left(\bigcup_{i=1}^k \mathbf{B}_r(p_i) \right)$$

and

$$\omega(v, w) = \lim_{r \searrow 0} \int_{\Omega_r} (v L_g(w) - w L_g(v)) d\mu_0. \tag{36}$$

Here $d\mu_0$ is the Euclidean volume element, $v, w \in \mathcal{W}_g$ lie in the deficiency space of g (See Definition 12) and L_g is the Jacobi operator of g, which is defined in (9).

Theorem 16. *The form ω defined in (36) is a symplectic form on the* 2k*-dimensional vector space $\mathcal{W}_g$.*

Proof. Our first order of business is to show that ω is well-defined, *i.e.* that the limit in (36) exists. By (9) observe that

$$v L_g(w) - w L_g(v) = v \Delta_0^2 w - w \Delta_0^2 v.$$

Next, we recall that the outer unit normal of Ω_r is $-\partial_r$ on each boundary sphere $\partial \mathbf{B}_r(p_i)$ and integrate by parts to see

$$\int_{\Omega_r} v \Delta_0^2 w d\mu_0 = \int_{\Omega_r} (\Delta_0 v)(\Delta_0 w) d\mu_0 - \sum_{i=1}^k \int_{\partial \mathbf{B}_r(p_i)} (v \partial_r \Delta_0 w - \partial_r v \Delta_0 w) d\sigma_0,$$

and so (36) becomes

$$\omega(v, w) = \lim_{r \searrow 0} \sum_{i=1}^k \int_{\partial \mathbf{B}_r(p_i)} (w \partial_r \Delta_0 v - v \partial_r \Delta_0 w + \partial_r v \Delta_0 w - \partial_r w \Delta_0 v) d\sigma_0. \tag{37}$$

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Next, we change variables using (11), letting $\widetilde{v} = \mathfrak{F}(v)$ and $\widetilde{w} = \mathfrak{F}(w)$. Under this change of variables

$$\partial_r v = -e^{\frac{n-2}{2}t}\left(\partial_t \widetilde{v} + \frac{n-4}{2}\widetilde{v}\right), \qquad \Delta_0 v = e^{\frac{n}{2}t}\left(\partial_t^2 \widetilde{v} - 2\partial_t \widetilde{v} - \frac{n(n-4)}{4}\widetilde{v} + \Delta_\theta \widetilde{v}\right), \tag{38}$$

and

$$\partial_r \Delta_0 v = -e^{\frac{n+2}{2}t}\left(\partial_t^3 \widetilde{v} + \frac{n-4}{2}\partial_t^2 \widetilde{v} - \frac{n^2}{4}\partial_t \widetilde{v} - \frac{n^2(n-4)}{8}\widetilde{v} + \Delta_\theta \partial_t \widetilde{v} + \frac{n}{2}\Delta_\theta \widetilde{v}\right). \tag{39}$$

Plugging (38) and (39) into (37) we obtain

$$\begin{aligned}
&\int_{\partial \mathbf{B}_r(p_i)} (w\partial_r \Delta_0 v - v\partial_r \Delta_0 w + \partial_r v \Delta_0 w - \partial_r w \Delta_0 v)\, d\sigma_0 \\
&\quad = \int_{\mathbf{S}^{n-1}} (w\partial_r \Delta_0 v - v\Delta_0 w + \partial_r v \Delta_0 w - \partial_r w \Delta_0 v)(r\theta) r^{n-1} d\theta \\
&\quad = \int_{\mathbf{S}^{n-1}} \Big(-\widetilde{w}\partial_t^3 \widetilde{v} - \frac{n-4}{2}\widetilde{w}\partial_t^2 \widetilde{v} + \frac{n^2}{4}\widetilde{w}\partial_t \widetilde{v} + \frac{n^2(n-4)}{8}\widetilde{w}\widetilde{v} - \widetilde{w}\Delta_\theta \partial_t \widetilde{v} \\
&\qquad - \frac{n}{2}\widetilde{w}\Delta_\theta \widetilde{v} \\
&\qquad + \widetilde{v}\partial_t^3 \widetilde{w} + \frac{n-4}{2}\widetilde{v}\partial_t^2 \widetilde{w} - \frac{n^2}{4}\widetilde{v}\partial_t \widetilde{w} - \frac{n^2(n-4)}{8}\widetilde{v}\widetilde{w} + \widetilde{v}\Delta_\theta \partial_t \widetilde{w} + \frac{n}{2}\widetilde{v}\Delta_\theta \widetilde{w} \\
&\qquad - \partial_t \widetilde{v}\partial_t^2 \widetilde{w} + 2\partial_t \widetilde{v}\partial_t \widetilde{w} + \frac{n(n-4)}{4}\widetilde{w}\partial_t \widetilde{v} - \partial_t \widetilde{v}\Delta_\theta \widetilde{w} - \frac{n-4}{2}\widetilde{v}\partial_t^2 \widetilde{w} \\
&\qquad + (n-4)\widetilde{v}\partial_t \widetilde{w} + \frac{n(n-4)^2}{8}\widetilde{v}\widetilde{w} - \frac{n-4}{2}\widetilde{v}\Delta_\theta \widetilde{w} \\
&\qquad + \partial_t \widetilde{w}\partial_t^2 \widetilde{v} - 2\partial_t \widetilde{w}\partial_t \widetilde{v} - \frac{n(n-4)}{4}\widetilde{v}\partial_t \widetilde{w} + \partial_t \widetilde{w}\Delta_\theta \widetilde{v} + \frac{n-4}{2}\widetilde{w}\partial_t^2 \widetilde{v} \\
&\qquad -(n-4)\widetilde{w}\partial_t \widetilde{v} - \frac{n(n-4)^2}{8}\widetilde{w}\widetilde{v} + \frac{n-4}{2}\widetilde{w}\Delta_\theta \widetilde{v}\Big)\, d\theta \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\widetilde{v}\partial_t^3 \widetilde{w} - \widetilde{w}\partial_t^3 \widetilde{v} + \partial_t \widetilde{w}\partial_t^2 \widetilde{v} - \partial_t v \partial_t^2 \widetilde{w} + \frac{n(n-4)+8}{2}(\widetilde{w}\partial_t \widetilde{v} - \widetilde{v}\partial_t \widetilde{w}) \right) \\
&\qquad \times d\theta
\end{aligned}$$

Observe that each element of the deficiency space is asymptotically radial about each puncture point, so that the expansions

$$\mathfrak{F}(v(\cdot - p_i))(t,\theta) = \overline{v}(t) + \mathcal{O}(e^{-\delta t}), \qquad \mathfrak{F}(w(\cdot - p_i))(t,\theta) = \overline{w}(t) + \mathcal{O}(e^{-\delta t})$$

for each puncture p_i. Thus each term in the expansion above involving derivatives with respect to θ will vanish in the limit.

Next we use the fact that both v and w lie in the defnciency space $\mathcal{W}_g$ (see Definition 12). This means that near each puncture p_i the functions v and w have asymptotic expansions of the form

$$\widetilde{v} = \alpha_i^+ w_0^+(\varepsilon_i) + \alpha_i^- w_0^-(\varepsilon_i), \qquad \widetilde{w} = \beta_i^+ w_0^+(\varepsilon_i) + \beta_i^- w_0^-(\varepsilon_i), \tag{40}$$

for t sufficiently large.

At each end, using bilinearity and skew-symmetry, we see that $\omega(v,w)$ is the limit as $t \to \infty$ of

$$(\alpha_i^+\beta_i^- - \alpha_i^-\beta_i^+) \int_{\mathbf{S}^{n-1}} \Big(w_0^+ \dddot{w}_0^- - w_0^- \dddot{w}_0^+ + \dot{w}_0^- \ddot{w}_0^+ - \dot{w}_0^+ \ddot{w}_0^- \tag{41}$$

$$+ \frac{n(n-4)+8}{2} (w_0^- \dot{w}_0^+ - w_0^+ \dot{w}_0^-) \Big) \, d\theta$$

If A_ε is the integrand in (41), then

$$\begin{aligned} \frac{d}{dt} A_\varepsilon &= w_0^+ \left(\ddddot{w}_0^- - \frac{n(n-4)+8}{2} \ddot{w}_0^- \right) - w_0^- \left(\ddddot{w}_0^+ - \frac{n(n-4)+8}{2} \ddot{w}_0^+ \right) \\ &= \left(\frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^{\frac{8}{n-4}} \right) (w_0^+ w_0^- - w_0^- w_0^+) = 0. \end{aligned}$$

Thus A_ε does not depend on t. Here we have used the ODE for $w_0^\pm$, namely

$$\ddddot{w}_0^\pm - \frac{n(n-4)+8}{2} \ddot{w}_0^\pm + \left(\frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^{\frac{8}{n-4}} \right) w_0^\pm = 0.$$

Let us find the value of the integrand in (41) at $t=0$ using the definitions of $w_0^\pm$ in (23). First observe that $w_0^+(\varepsilon) = \dot{v}_\varepsilon$, so $w_0^+(0) = \dot{v}_\varepsilon(0) = 0$, $\dot{w}_0^+(0) = \ddot{v}_\varepsilon(0) > 0$, $\ddot{w}_0^+(0) = \dddot{v}_\varepsilon(0) = 0$ and by (14) we get

$$\dddot{w}_0^+(0) = \ddddot{v}_\varepsilon(0) = \frac{n(n-4)+8}{2} \ddot{v}_\varepsilon(0) - \frac{n(n-4)}{16} \left(n(n-4)\varepsilon - (n^2-4)\varepsilon^{\frac{n+4}{n-4}} \right).$$

Furthermore, since $w_0^- = \dfrac{d}{d\varepsilon} v_\varepsilon$ and v_ε assumes its minimal value at $t=0$ we see $w_0^-(0) = 1$. Therefore, at $t=0$ it holds

$$\begin{aligned} A_\varepsilon(0) &= -\dddot{w}_0^+(0) - \ddot{v}_\varepsilon(0)\ddot{w}_0^- + \frac{n(n-4)+8}{2} \ddot{v}_0^+(0) \\ &= -\ddot{v}_\varepsilon(0)\ddot{w}_0^- + \frac{n(n-4)}{16} \left(n(n-4)\varepsilon - (n^2-4)\varepsilon^{\frac{n+4}{n-4}} \right) \end{aligned}$$

By (15) we have

$$\mathcal{H}_\varepsilon = \frac{1}{2} \ddot{v}_\varepsilon(0)^2 - \frac{n^2(n-4)^2}{32} \varepsilon^2 + \frac{(n-4)^2(n^2-4)}{32} \varepsilon^{\frac{2n}{n-4}}. \tag{42}$$

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As we remarked earlier, the energy is a decreasing function of ε, minimized by the cylinder, which has the largest possible necksize, so differentiating (42) we see

$$0 > \frac{d}{d\varepsilon}\mathcal{H}_\varepsilon = \ddot{v}_\varepsilon(0)\frac{d}{d\varepsilon}\ddot{v}_\varepsilon(0) - \frac{n^2(n-4)^2}{16}\varepsilon + \frac{n(n-4)(n^2-4)}{16}\varepsilon^{\frac{n+4}{n-4}} \tag{43}$$

$$= \ddot{v}_\varepsilon(0)\ddot{w}_0^-(0) - \frac{n(n-4)}{16}(n(n-4)\varepsilon - (n^2-4)\varepsilon^{\frac{n+4}{n-4}}).$$

This implies that

$$A_\varepsilon(0) = -\frac{d}{d\varepsilon}\mathcal{H}_\varepsilon > 0,$$

and so ω is nondegenerate. □

COROLLARY 17. *Let $g \in \mathcal{M}_\Lambda$ be nondegenerate. Then there exists an open neighborhood $\mathcal{U}$ of* g *in $\mathcal{M}_\Lambda$ that embeds into $\mathcal{W}_g$ as a Lagrangian submanifold, with respect to the symplectic form given by* (36)*.*

Proof. As in the proof of Theorem 3, we can identify the bounded null space $\mathcal{B}_g$ as the tangent space $T_g\mathcal{M}_\Lambda$. In particular, this identification shows $L_g(v) = 0$ for each $v \in \mathcal{B}_g$. On the other hand, the linear decomposition lemma allows us to identify $\mathcal{B}_g$ as a k-dimensional subspace of $\mathcal{W}_g$. The corollary now follows. □

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