

ON THE SPECTRUM OF n -TUPLES OF p -HYPONORMAL OPERATORS

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(Received 27 June, 1996)

1. Introduction. Let $B(H)$ denote the algebra of operators (i.e., bounded linear transformations) on the Hilbert space H . $A \in B(H)$ is said to be p -hyponormal ($0 < p \leq 1$), if $(AA^*)^p \leq (A^*A)^p$. (Of course, a 1-hyponormal operator is hyponormal.) The p -hyponormal property is monotonic decreasing in p and a p -hyponormal operator is q -hyponormal operator for all $0 < q \leq p$. Let A have the polar decomposition $A = U|A|$, where U is a partial isometry and $|A|$ denotes the (unique) positive square root of A^*A . If A has equal defect and nullity, then the partial isometry U may be taken to be unitary. Let $\mathcal{HU}(p)$ denote the class of p -hyponormal operators for which U in $A = U|A|$ is unitary. $\mathcal{HU}(1/2)$ operators were introduced by Xia and $\mathcal{HU}(p)$ operators for a general $0 < p < 1$ were first considered by Aluthge (see [1, 14]); $\mathcal{HU}(p)$ operators have since been considered by a number of authors (see [3, 4, 5, 9, 10] and the references cited in these papers). Generally speaking, $\mathcal{HU}(p)$ operators have spectral properties similar to those of hyponormal operators. Indeed, let $A \in \mathcal{HU}(p)$, ($0 < p < 1/2$), have the polar decomposition $A = U|A|$, and define the $\mathcal{HU}(p+1/2)$ operator $\hat{A}$ by $\hat{A} = |A|^{1/2} U |A|^{1/2}$. Let $\tilde{A} = V|\hat{A}|$ with V unitary and $\tilde{A}$ be the hyponormal operator defined by $\tilde{A} = |\hat{A}|^{1/2} V |\hat{A}|^{1/2}$. Then we have the following result.

LEMMA 0. $\sigma_s(A) = \sigma_s(\tilde{A})$, where σ_s denotes either of the following: point spectrum, approximate point spectrum, eigenvalues of finite multiplicity, spectrum, Weyl spectrum, and essential spectrum.

Recall that an n -tuple $\mathcal{A} = (A_1, A_2, \dots, A_n)$ of operators is said to be doubly commuting if $A_i A_j - A_j A_i = 0$ and $A_i^* A_j - A_j^* A_i = 0$, for all $1 \leq i \neq j \leq n$. Doubly commuting n -tuples $\mathcal{A}$ of operators in $\mathcal{HU}(p)$ have been considered by Muneo Cho in [3], where it is shown that a weak Putnam theorem holds for $\mathcal{A}$ and that $\mathcal{A}$ is jointly normaloid. In this note we study the relationship between the spectral properties of $\mathcal{A}$ and $\tilde{\mathcal{A}} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$, and prove that $\sigma_s(\mathcal{A}) = \sigma_s(\tilde{\mathcal{A}})$, where σ_s is either the joint point spectrum or the joint approximate point spectrum or the joint (Taylor) spectrum. This then leads us to:

- (a) $\|\mathcal{A}\| = \|\tilde{\mathcal{A}}\|$;
- (b) if $\sigma(\mathcal{A}) \in \mathcal{R}'$, then A_i is self-adjoint, for all $1 \leq i \leq n$.

We show that the (Cho-Takaguchi) joint Weyl spectrum of $\mathcal{A}$ is contained in the (Taylor) spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ minus the set of isolated points of $\sigma(\mathcal{A})$ which are joint eigenvalues of finite multiplicity, and that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have the same (Harte) essential spectrum. We conclude this note with a result (in the spirit of Dash [8, Corollary 4.6]) on the joint eigenvalues of $\mathcal{A}$ in the Calkin algebra.

We assume henceforth, without loss of generality, that $0 < p < 1/2$. Most of the notation that we use in this note is standard (and usually explained at the first instance of

occurrence). The following theorem, the n -tuple version of the Berberian extension theorem, will play an important role in the sequel.

THEOREM B. *If $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is an n -tuple of commuting operators on H , then there exists a Hilbert space $H^0 \supset H$ and an isometric $*$ -isomorphism $A_i \rightarrow A_i^0$, ($1 \leq i \leq n$), preserving order such that $\sigma_\pi(A_i) = \sigma_\pi(A_i^0) = \sigma_p(A_i^0)$ and $\sigma_\pi(\mathcal{A}) = \sigma_\pi(A_1, A_2, \dots, A_n) = \sigma_\pi(A_1^0, A_2^0, \dots, A_n^0) = \sigma_p(A_1^0, A_2^0, \dots, A_n^0) = \sigma_p(\mathcal{A}^0)$. (Here $\sigma_p(\mathcal{A})$ and $\sigma_\pi(\mathcal{A})$ denote, respectively, the joint spectrum and the joint approximate point spectrum (defined below) of $\mathcal{A}$.)*

It is my pleasure to thank Professor Muneo Chō for supplying me with off-prints and preprints of his papers.

2. Results. Throughout the following $\mathcal{A} = (A_1, A_2, \dots, A_n)$ will denote a doubly commuting (i.e., $A_i A_j - A_j A_i = 0$ and $A_i A_j^* - A_j^* A_i = 0$, for all $1 \leq i \neq j \leq n$) n -tuple of $\mathcal{H}U(p)$ operators A_i ($1 \leq i \leq n$). Given $A_i = U_i |A_i|$, define $\hat{A}_i$ by $\hat{A}_i = |A_i|^{1/2} U_i |A_i|^{1/2}$; also, letting $\hat{A}_i$ have the polar decomposition $\hat{A}_i = V_i |\hat{A}_i|$, define $\tilde{A}_i$ by

$$\tilde{A}_i = |\hat{A}_i|^{1/2} V_i |\hat{A}_i|^{1/2} \quad (1 \leq i \leq n).$$

The n -tuples $\hat{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ are then defined by $\hat{\mathcal{A}} = (\hat{A}_1, \hat{A}_2, \dots, \hat{A}_n)$ and $\tilde{\mathcal{A}}_1 = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$.

LEMMA 1. *$\mathcal{A}$ is doubly commuting $\Rightarrow \hat{\mathcal{A}}$ is doubly commuting $\Rightarrow \tilde{\mathcal{A}}$ is doubly commuting. Also, $\mathcal{A}$ is doubly commuting $\Rightarrow [A_i, |\hat{A}_j|] = 0 = [\tilde{A}_i, |\hat{A}_j|] = 0$, for $1 \leq i \neq j \leq n$, where $[A, B]$ denotes the commutator $AB - BA$ of A and B .*

Proof. Given $A_i = U_i |A_i|$ and $\hat{A}_i = V_i |\hat{A}_i|$, the doubly commuting hypothesis on $\mathcal{A}$ implies that

$$[U_i, U_j] = [|A_i|, |A_j|] = [|A_i|, U_j] = 0,$$

for all $1 \leq i \neq j \leq n$. (See [11, Theorems 2 and 4].) Consequently, $\hat{\mathcal{A}}$ is doubly commuting and so

$$[V_i, V_j] = [|\hat{A}_i|, |\hat{A}_j|] = [|\hat{A}_i|, V_j] = 0,$$

for all $1 \leq i \neq j \leq n$. This implies that $\tilde{\mathcal{A}}$ is doubly commuting. The argument above also implies that $[A_i, \hat{A}_j] = [A_i, \hat{A}_j^*] = [\hat{A}_i, \tilde{A}_j] = [\hat{A}_i, \tilde{A}_j^*] = 0$, for all $1 \leq i \neq j \leq n$. Hence, also, $[A_i, |\hat{A}_j|] = [\tilde{A}_i, |\hat{A}_j|] = 0$, for all $1 \leq i \neq j \leq n$.

In the following we shall denote the Taylor joint spectrum of $\mathcal{A}$ by $\sigma(\mathcal{A})$. (See [13] for the definition of Taylor spectrum of a commuting n -tuple of operators.) We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, ($\lambda_i \in \mathbb{C}$ for all $1 \leq i \leq n$), is in the joint approximate point spectrum $\sigma_\pi(\mathcal{A})$ of $\mathcal{A}$ if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(A_i - \lambda_i)x_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

for all $1 \leq i \leq n$; $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_i \in \mathbb{C}$ for all $1 \leq i \leq n$, is in the joint point spectrum $\sigma_p(\mathcal{A})$ of $\mathcal{A}$ if there exists a non-trivial vector $x \in H$ such that

$$(A_i - \lambda_i)x = 0, \quad \text{for all } 1 \leq i \leq n.$$

We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is in the *normal point spectrum* $\sigma_{np}(\mathcal{A})$ of $\mathcal{A}$ if there exists a non-trivial vector $x \in H$ such that $(A_i - \lambda_i)x = 0 \Leftrightarrow (A_i - \lambda_i)^*x = 0$, for all $1 \leq i \leq n$.

LEMMA 2. $\sigma_p(\mathcal{A}) = \sigma_{np}(\mathcal{A}) = \sigma_{np}(\tilde{\mathcal{A}}) = \sigma_p(\tilde{\mathcal{A}})$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma_p(\mathcal{A})$ and let $x \in H$ be such that $x \neq 0$ and $(A_i - \lambda_i)x = 0$, for all $1 \leq i \leq n$. It is easily seen that $\tilde{A}_i |\hat{A}_i|^{1/2} |A_i|^{1/2} = |\hat{A}_i|^{1/2} |A_i|^{1/2} A_i$; hence

$$\tilde{A}_i |\hat{A}_i|^{1/2} |A_i|^{1/2} x = \lambda_i |\hat{A}_i|^{1/2} |A_i|^{1/2} x,$$

for all $1 \leq i \leq n$. Let

$$y = \prod_{i=1}^{n'} |\hat{A}_i|^{1/2} |A_i|^{1/2} x,$$

where “ $\prod$ ” on the product “ $\prod_{i=1}^n$ ” denotes that only those $|A_i|s$, (and so also $|\hat{A}_i|s$), appear in the product for which λ_i in $A_i x = \lambda_i x$ does not equal 0. Then y is non-trivial, and

$$\tilde{A}_i y = \lambda_i y, \quad \text{for all } i = 1, 2, \dots, n \quad \text{for which } \lambda_i \neq 0.$$

If $\lambda_i = 0$, i.e. $A_i x = 0$, then $|A_i|^{1/2} x = 0$. This implies that $\hat{A}_i x = 0$. Since this in turn implies that $|\hat{A}_i|^{1/2} x = 0$, we conclude that $\tilde{A}_i x = 0$. Since $[\tilde{A}_i, \tilde{A}_j] = 0$ for all $1 \leq i \neq j \leq n$, we have that $\tilde{A}_i y = 0$. Consequently, $\lambda \in \sigma_p(\tilde{\mathcal{A}})$ and $\sigma_p(\mathcal{A}) \subseteq \sigma_p(\tilde{\mathcal{A}})$.

If, on the other hand, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma_p(\tilde{\mathcal{A}})$, then there is a non-trivial $x \in H$ such that $(\tilde{A}_i - \lambda_i)x = 0$ and $(\tilde{A}_i^* - \bar{\lambda}_i)x = 0$ for all $1 \leq i \leq n$. Since $A_i^* |\hat{A}_i|^{1/2} |\hat{A}_i|^{1/2} = |A_i|^{1/2} |\hat{A}_i|^{1/2} \tilde{A}_i^*$,

$$A_i^* |A_i|^{1/2} |\hat{A}_i|^{1/2} x = \bar{\lambda}_i |A_i|^{1/2} |\hat{A}_i|^{1/2} x$$

for all $1 \leq i \leq n$. Defining $(0 \neq) y$ by

$$y = \prod_{i=1}^{n'} |A_i|^{1/2} |\hat{A}_i|^{1/2} x,$$

where $\prod_{i=1}^n$ has meaning similar to that above, we have $A_i^* y = \bar{\lambda}_i y$, for all $i = 1, 2, \dots, n$ such that $\lambda_i \neq 0$. Since $\lambda_i \in \sigma_p(\tilde{A}_i)$ implies $\lambda_i \in \sigma_p(A_i) = \sigma_{np}(A_i)$ (see Lemma 0), $A_i y = \lambda_i y$ for all $i = 1, 2, \dots, n$ such that $\lambda_i \neq 0$. Now if $\tilde{A}_i x = 0$, then $0 \in \sigma_p(\hat{A}_i) = \sigma_p(A_i)$ and

$$\begin{aligned} \tilde{A}_i^* x = 0 &\Rightarrow |\hat{A}_i|^{1/2} V_i^* |\hat{A}_i|^{1/2} x = 0 \\ &\Rightarrow \hat{A}_i^* |\hat{A}_i|^{1/2} x = 0 \Leftrightarrow \hat{A}_i |\hat{A}_i|^{1/2} x = 0 \\ &\Rightarrow |\hat{A}_i|^{1/2} x = 0 \Rightarrow \hat{A}_i x = 0 \Leftrightarrow \hat{A}_i^* x = 0 \\ &\Rightarrow A_i^* |A_i|^{1/2} x = 0 \Leftrightarrow A_i |A_i|^{1/2} x = 0 \\ &\Rightarrow |A_i|^{1/2} x = 0 \Rightarrow A_i x = 0 \Leftrightarrow A_i^* x = 0. \end{aligned}$$

(Line 2 follows since $0 \in \sigma_p(A_i)$. Line 4 follows because $0 \in \sigma_p(A_i) = \sigma_{np}(A_i)$.) Consequently, $A_i y = 0$ for such an i . Hence $\sigma_p(\tilde{\mathcal{A}}) \subseteq \sigma_p(\mathcal{A})$. Since $\sigma_p(A_i) = \sigma_{np}(A_i)$ and $\sigma_p(\tilde{A}_i) = \sigma_{np}(\tilde{A}_i)$, for all $1 \leq i \leq n$, this completes the proof.

LEMMA 3. $\sigma_\pi(\mathcal{A}) = \sigma_{n\pi}(\mathcal{A}) = \sigma_{n\pi}(\tilde{\mathcal{A}}) = \sigma_\pi(\tilde{\mathcal{A}})$.

Proof. Letting $A^0 = (A_1^0, A_2^0, \dots, A_n^0)$ denote the Berberian extension of $\mathcal{A}$ (see Theorem B), it follows from Lemma 2 that

$$\sigma_\pi(\mathcal{A}) = \sigma_0(\mathcal{A}^0) = \sigma_{np}(\mathcal{A}^0) = \sigma_{np}(\tilde{\mathcal{A}}^0) = \sigma_p(\tilde{\mathcal{A}}^0) = \sigma_\pi(\tilde{\mathcal{A}}).$$

We are now in a position to prove the equality of the (Taylor) spectra of $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

THEOREM 1. $\sigma(\mathcal{A}) = \sigma(\tilde{\mathcal{A}})$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma(\mathcal{A})$. Then there exists a partition

$$\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_s\} \quad \text{of} \quad \{1, 2, \dots, n\}$$

and a sequence $\{x_k\}$ of unit vectors in H such that

$$(A_{i_r} - \lambda_{i_r})x_k \rightarrow 0 \quad \text{and} \quad (A_{j_t}^* - \bar{\lambda}_{j_t})x_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

for all $1 \leq r \leq m$ and $1 \leq t \leq s$. (See [7, Corollary 3.3].) Let $\mathcal{A}^0$ denote the Berberian extension $(A_{i_1}^0, \dots, A_{i_m}^0, A_{j_1}^0, \dots, A_{j_s}^0)$ of $\mathcal{A}$, and let $\mathcal{B} = (A_{i_1}^0, \dots, A_{i_m}^0, A_{j_1}^{0*}, \dots, A_{j_s}^{0*})$. Then

$$(\lambda_{i_1}, \dots, \lambda_{i_m}, \bar{\lambda}_{j_1}, \dots, \bar{\lambda}_{j_s}) \in \sigma_p(\mathcal{B}).$$

Since $\sigma_p(A_{i_r}^0) = \sigma_p(\tilde{A}_{i_r}^0) = \sigma_{np}(\tilde{A}_{i_r}^0)$, for all $1 \leq r \leq m$, and since

$$\tilde{A}_{j_t}^* |\hat{A}_{j_t}|^{1/2} V_{j_t}^* |A_{j_t}|^{1/2} U_{j_t}^* = |\hat{A}_{j_t}|^{1/2} V_{j_t}^* |A_{j_t}|^{1/2} U_{j_t}^* A_{j_t}^*,$$

it follows (from an argument similar to that used in the proof of Lemma 2) that $\sigma_p(\mathcal{B}) \subseteq \sigma_p(\tilde{\mathcal{B}})$ and

$$\bar{\lambda} \in \sigma_p(\tilde{\mathcal{A}}^{0*}) = \sigma_\pi(\tilde{\mathcal{A}}^*) \subseteq \sigma(\tilde{\mathcal{A}}^*).$$

Hence $\lambda \in \sigma(\tilde{\mathcal{A}})$, and $\sigma(\mathcal{A}) \subseteq \sigma(\tilde{\mathcal{A}})$.

Conversely, if $\lambda \in \sigma(\tilde{\mathcal{A}})$, then (from an argument similar to that above) $\bar{\lambda} \in \sigma_p(\tilde{\mathcal{A}}^{0*})$. This implies that $\bar{\lambda} \in \sigma_\pi(\mathcal{A}^*) \subseteq \sigma(\mathcal{A}^*)$, $\lambda \in \sigma(\mathcal{A})$ and $\sigma(\tilde{\mathcal{A}}) \subseteq \sigma(\mathcal{A})$. Hence $\sigma(\mathcal{A}) = \sigma(\tilde{\mathcal{A}})$, and the proof is complete.

The joint spectral radius $r(\mathcal{T})$ and the joint operator norm $\|\mathcal{T}\|$ of an n -tuple $\mathcal{T} = (T_1, T_2, \dots, T_n)$ are defined by

$$r(\mathcal{T}) = \sup \left\{ |\lambda| = \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} : \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma(\mathcal{T}) \right\}$$

and

$$\|\mathcal{T}\| = \sup \left\{ \left(\sum_{i=1}^n \|T_i x\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}.$$

See [6]. The operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ being jointly normaloid (see [3, Theorem 9] and [6, Theorem 3.4]), $r(\mathcal{A}) = \|\mathcal{A}\|$ and $r(\tilde{\mathcal{A}}) = \|\tilde{\mathcal{A}}\|$. Theorem 1 thus implies the following result.

COROLLARY 1. $\|\mathcal{A}\| = \|\tilde{\mathcal{A}}\| = \|\tilde{\mathcal{A}}\|$.

That $\|A\| = \|\tilde{A}\|$ for a single operator $A \in \mathcal{HU}(p)$ has been proved by M. Fujii *et al.* in [10].

Given a semi-normal (i.e., hyponormal or co-hyponormal) operator $T = X + iY$, a well known result of Putnam [12] states that if a real number $r \in \sigma(X)$ (or $r + is \in \sigma(T)$, for some real numbers r and s), then there exists a real number s such that $r + is \in \sigma(T)$ (resp., $r \in \sigma(X)$ and $s \in \sigma(Y)$). This result extends to doubly commuting n -tuples of hyponormal operators [4]. Does a similar result hold (for $A \in \mathcal{HU}(p)$ and) doubly commuting n -tuples in $\mathcal{HU}(p)$? The technique of this paper (seemingly) does not lend to a proof of this. We do however have the following analogue for $\mathcal{HU}(p)$ operators of a result on n -tuples of doubly commuting hyponormal operators with spectrum in $\mathbb{R}^n$. (See [4, Corollary].)

COROLLARY 2. If $\sigma(\mathcal{A}) \subseteq \mathbb{R}^n$, then A_i is self-adjoint, for all $1 \leq i \leq n$.

Proof. Since $\sigma(\tilde{\mathcal{A}}) = \sigma(\mathcal{A}) \subseteq \mathbb{R}^n$, $\tilde{A}_i$ is self-adjoint, for all $1 \leq i \leq n$, by [4]. Recall that A_i is normal if and only if $\tilde{A}_i$ is normal [9, Corollary 2]; hence A_i is self-adjoint, for all $1 \leq i \leq n$.

Following Chō [2], we define the joint Weyl spectrum $\sigma_\omega(\mathcal{T})$ of a commuting n -tuple $\mathcal{T}$ by

$$\sigma_\omega(\mathcal{T}) = \cap \{ \sigma(\mathcal{T} + \mathcal{K}); \mathcal{K} \text{ is an } n\text{-tuple of compact operators and } (\mathcal{T} + \mathcal{K}) \text{ is a commuting } n\text{-tuple} \}.$$

Let $\sigma_{00}(\mathcal{T})$ denote the set of isolated points of $\sigma(\mathcal{T})$ which are joint eigen-values of finite multiplicity of $\mathcal{T}$. It is clear from Theorem 1 that, if λ is an isolated point of $\sigma(\mathcal{A})$, then λ is an isolated point of $\sigma(\tilde{\mathcal{A}})$. The operator $\tilde{\mathcal{A}}$ being a doubly commutative n -tuple of hyponormal operators, an isolated point λ of $\sigma(\tilde{\mathcal{A}})$ is a point of $\sigma_p(\tilde{\mathcal{A}})$. Hence by Lemma 2 we have the following result.

COROLLARY 3. If λ is an isolated point of $\sigma(\mathcal{A})$, then $\lambda \in \sigma_p(\mathcal{A})$.

Recall that if A is p -hyponormal, then $\sigma_\omega(A) = \sigma(A) - \sigma_{00}(A)$ by [9] and if $\mathcal{T}$ is a doubly commuting n -tuple of hyponormal operators, then $\sigma_\omega(\mathcal{T}) \subseteq \sigma(\mathcal{T}) - \sigma_{00}(\mathcal{T})$ by [2].

THEOREM 2. $\sigma_\omega(\mathcal{A}) \subseteq \sigma(\mathcal{A}) - \sigma_{00}(\mathcal{A})$.

Proof. Suppose $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma_{00}(\mathcal{A})$, and let $N = \ker \left\{ \sum_{i=1}^n (A_i - \lambda_i)^*(A_i - \lambda_i) \right\}$. Since $\lambda \in \sigma_p(\mathcal{A})$ if and only if $0 \in \sigma_p \left(\sum_{i=1}^n (A_i - \lambda)^*(A_i - \lambda) \right)$, N is finite dimensional. By Lemma 2, $\sigma_p(\mathcal{A}) = \sigma_{np}(\mathcal{A})$; hence N reduces $\mathcal{A}$, $\mathcal{A}_0 = \mathcal{A} \upharpoonright N = (A_1 \upharpoonright N, A_2 \upharpoonright N, \dots, A_n \upharpoonright N)$ is normal and $\mathcal{A}_1 = \mathcal{A} \upharpoonright N^\perp = (A_1 \upharpoonright N^\perp, A_2 \upharpoonright N^\perp, \dots, A_n \upharpoonright N^\perp)$ is a doubly commuting n -tuple of $\mathcal{HU}(p)$ operators. Let P be the orthogonal projection of H onto N . P is then a compact operator which satisfies $[A_i, P] = [A_i^*, P] = 0$, for all $i = 1, 2, \dots, n$. The operator

$$\mathcal{A} + \mathcal{P} = \left(A_1 + \frac{1}{\sqrt{n}}P, A_2 + \frac{1}{\sqrt{n}}P, \dots, A_n + \frac{1}{\sqrt{n}}P \right)$$

is a doubly commuting n -tuple. Let

$$\mathcal{R} = (\mathcal{A} + \mathcal{P}) \mid N = \left(\left(A_1 + \frac{1}{\sqrt{n}} P \right) \mid N, \left(A_2 + \frac{1}{\sqrt{n}} P \right) \mid N, \dots, \left(A_n + \frac{1}{\sqrt{n}} P \right) P \mid N \right),$$

$$\mathcal{S} = (\mathcal{A} + \mathcal{P}) \mid N^\perp.$$

$\mathcal{R}$ and $\mathcal{S}$ are then doubly commuting n -tuples such that $\sigma(\mathcal{A} + \mathcal{P}) = \sigma(\mathcal{R})U\sigma(\mathcal{S})$.

Suppose that $\lambda \in \sigma(\mathcal{A} + \mathcal{P})$. Then $\lambda \notin \sigma(\mathcal{R})$ and so λ must be an isolated point of $\sigma(\mathcal{S})$. There exists a partition $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_s\}$ of $\{1, 2, \dots, n\}$ and a sequence $\{x_k\}$ of unit vectors in $N^\perp$ such that

$$\left(A_{i_r} - \lambda_{i_r} + \frac{1}{\sqrt{n}} P \right) x_k \rightarrow 0 \quad \text{and} \quad \left(A_{j_s}^* - \bar{\lambda}_{j_s} + \frac{1}{\sqrt{n}} P \right) x_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

But then $\lambda \in \sigma(\mathcal{A}_1)$ and hence (by Corollary 3) $\lambda \in \sigma_p(\mathcal{A}_1)$. Thus there exists an $x \in N^\perp$ such that $(A_i - \lambda_i)x = 0$, for all $i = 1, 2, \dots, n$. Since this is a contradiction, we must have $\lambda \notin \sigma_\omega(\mathcal{A})$.

REMARKS. (i) the Taylor–Weyl spectrum of $\mathcal{T}$, $\sigma_{Tw}(\mathcal{T})$, is defined to be the set of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $(\mathcal{T} - \lambda)$ is not Taylor–Weyl (where $\mathcal{T} - \lambda$ is said to be Taylor–Weyl if $\mathcal{T} - \lambda$ is Fredholm and $\text{index}(\mathcal{T} - \lambda) = 0$). Theorem 2 implies that $\sigma(\mathcal{A}) \setminus \sigma_{Tw}(\mathcal{A}) \supseteq \sigma_{00}(\mathcal{A})$. The inclusion $\sigma(\mathcal{A}) \setminus \sigma_{Tw}(\mathcal{A}) \subseteq \sigma_{00}(\mathcal{A})$ does not hold (even for hyponormal $\mathcal{A}$).

(ii) Given a p -hyponormal operator A , $\sigma_\omega(A) = \sigma_\omega(\tilde{A})$ by [9]. Does $\sigma_\omega(\mathcal{A}) = \sigma_\omega(\tilde{\mathcal{A}})$?

The Harte spectrum $\sigma_H(\mathcal{T})$ of the commutative n -tuple $\mathcal{T}$ is defined to be $\sigma_H(\mathcal{T}) = \sigma^l(\mathcal{T}) \cup \sigma^r(\mathcal{T})$, where $\sigma^l(\mathcal{T})$ (respectively, $\sigma^r(\mathcal{T})$) is the set of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\{T_i - \lambda_i\}_{1 \leq i \leq n}$ generates a proper left (resp., right) ideal in $B(H)$. The (Harte) essential spectrum $\sigma_e(\mathcal{T})$ is defined by $\sigma_e(\mathcal{T}) = \sigma_e(a)$, where $a = (a_1, a_2, \dots, a_n) = \pi(\mathcal{T})$ and π is the canonical homomorphism of $B(H)$ onto the Calkin algebra $B(H)/K(H)$; $K(H)$ is the algebra of compact operators on H . For a single linear operator, the (Harte) essential spectrum coincides with the essential spectrum; the following extends the conclusion $\sigma_e(A) = \sigma_e(\tilde{A})$ of Lemma 0 to $\sigma_e(\mathcal{A})$.

THEOREM 3. $\sigma_e(\mathcal{A}) = \sigma_e(\tilde{\mathcal{A}})$.

Proof. Suppose $\lambda \in \sigma_e(\tilde{\mathcal{A}})$. Then, $\tilde{\mathcal{A}}$ being a hyponormal n -tuple, there exists a sequence $\{x_k\}$ of unit vectors converging weakly to 0 in H such that

$$\|(\tilde{A}_i - \lambda_i)^* x_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad \text{for all } 1 \leq i \leq n,$$

by [8, Theorem 2.6]. Let $\{y_k\}$ be the sequence defined by

$$y_k = \left(\prod_{i=1}^n |A_i|^{1/2} |\tilde{A}_i|^{1/2} x_k \right) / \left\| \prod_{i=1}^n |A_i|^{1/2} |\tilde{A}_i|^{1/2} x_k \right\|,$$

where “ $\prod$ ” on the product $\prod_{i=1}^n$ denotes that only those $|A_i|$ s and $|\tilde{A}_i|$ s appear in the product

for which $\lambda_i \neq 0$. (Notice that if $\| |A_i|^{1/2} x_k \|$ or $\| |A_i|^{1/2} |\hat{A}_i|^{1/2} x_k \| \rightarrow 0$ as $k \rightarrow \infty$, for some i with $1 \leq i \leq n$, then $\| |\hat{A}_i|^{1/2} x_k \|$ and $\| \tilde{A}_i x_k \| \rightarrow 0$ as $k \rightarrow \infty$.) Since $(x_k, h) \rightarrow 0$ as $k \rightarrow \infty$ for all $h \in H$, $(y_k, h) \rightarrow 0$ as $k \rightarrow \infty$ and

$$\| (A_j - \lambda_j)^* y_k \| = \left\| \frac{\prod_{i=1}^n |A_i|^{1/2} |\hat{A}_i|^{1/2}}{\left\| \prod_{i=1}^n |A_i|^{1/2} |\hat{A}_i|^{1/2} x_k \right\|} (\tilde{A}_j - \lambda_j)^* x_k \right\| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

for all $1 \leq j \leq n$. Thus $\lambda \in \sigma_e(\mathcal{A})$ and $\sigma_e(\tilde{\mathcal{A}}) \subseteq \sigma_e(\mathcal{A})$.

Consider now $\lambda \in \sigma_e(\mathcal{A}) = \sigma_e'(\mathcal{A}) \cup \sigma_e''(\mathcal{A})$. Suppose that $\lambda \in \sigma_e'(\mathcal{A})$; then there exists a sequence $\{x_k\}$ of unit vectors converging weakly to 0 in H such that $\|(A_i - \lambda_i)x_k\| \rightarrow 0$ as $k \rightarrow \infty$, for all $1 \leq i \leq n$. Defining the sequence $\{y_k\}$ by

$$y_k = \frac{\left(\prod_{i=1}^n |\hat{A}_i|^{1/2} |A_i|^{1/2} x_k \right)}{\left\| \prod_{i=1}^n |\hat{A}_i|^{1/2} |A_i|^{1/2} x_k \right\|},$$

(where $\prod_{i=1}^n$ has a meaning similar to that above), an argument similar to that above shows that $\{y_k\}$ is a sequence of unit vectors converging weakly to 0 in H such that

$$\| (\tilde{A}_j - \lambda_j) y_k \| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad \text{for all } 1 \leq j \leq n.$$

Hence $\lambda \in \sigma_e'(\tilde{\mathcal{A}})$. A similar argument shows that if $\lambda \in \sigma_e''(\mathcal{A})$ then $\lambda \in \sigma_e'(\tilde{\mathcal{A}})$. Thus $\sigma_e(\mathcal{A}) \subseteq \sigma_e(\tilde{\mathcal{A}})$, and the proof is complete.

COROLLARY 4. $\sigma_e(\mathcal{A}) = \sigma_e'(\mathcal{A})$.

Proof. The argument of the proof of Theorem 3 implies that

$$\sigma_e'(\mathcal{A}) \subseteq \sigma_e(\mathcal{A}) = \sigma_e(\tilde{\mathcal{A}}) = \sigma_e'(\tilde{\mathcal{A}}) \subseteq \sigma_e'(\mathcal{A}).$$

COROLLARY 5. $\sigma_H(\mathcal{A}) = \sigma_e(\mathcal{A}) \cup \sigma_p(\mathcal{A}^*)^*$.

Proof. Let $\sigma_\delta(\mathcal{A}) = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) : \text{there exists a sequence } \{x_k\} \text{ of unit vectors in } H \text{ such that } \|(A_i - \lambda_i)^* x_k\| \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for all } i = 1, 2, \dots, n\}$ denote the joint approximate defect spectrum of $\mathcal{A}$. Then

$$\sigma_H(\mathcal{A}) = \sigma'(\mathcal{A}) \cup \sigma''(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \cup \sigma_\delta(\mathcal{A}).$$

By Lemma 3, $\sigma_\pi(\mathcal{A}) = \sigma_\pi(\tilde{\mathcal{A}})$; applying an argument similar to that used in the proof of Lemma 2 to A_i^{0*} it is seen that $\sigma_\delta(\mathcal{A}) = \sigma_\delta(\tilde{\mathcal{A}})$. We have

$$\sigma_H(\mathcal{A}) = \sigma_\pi(\mathcal{A}) \cup \sigma_\delta(\mathcal{A}) = \sigma_\pi(\tilde{\mathcal{A}}) \cup \sigma_\delta(\tilde{\mathcal{A}}) = \sigma_\delta(\tilde{\mathcal{A}}) = \sigma_H(\tilde{\mathcal{A}}),$$

since $\tilde{\mathcal{A}}$ is a hyponormal n -tuple. Also, since

$$\sigma_H(\tilde{\mathcal{A}}) = \sigma_\delta(\tilde{\mathcal{A}}) = \sigma_e(\tilde{\mathcal{A}}) \cup \sigma_p(\tilde{\mathcal{A}}^*)^* = \sigma_e(\mathcal{A}) \cup \sigma_p(\mathcal{A}^*)^*,$$

the proof is complete.

The n -tuple $(A_1, A_2, \dots, A_n)$ is said to be *essentially doubly commuting* (resp., *essentially $\mathcal{HU}(p)$*) if the n -tuple $(a_1, a_2, \dots, a_n)$, where $a_i = \pi(A_i)$ for all $1 \leq i \leq n$, and $\pi: B(H) \rightarrow B(H) \setminus K(H)$, is doubly commuting (resp., $\mathcal{HU}(p)$). We close this note with the following result.

THEOREM 4. Suppose $(A_1, A_2, \dots, A_n)$ is an n -tuple of essentially doubly commuting essentially $\mathcal{HU}(p)$ operators. Then $A_1, A_2, \dots, A_n$ have a common reducing subspace “modulo the compact operators”.

Proof. The hypotheses imply that $a_i \in \mathcal{HU}(p)$ for all $1 \leq i \leq n$ and that the a_i s are doubly commuting. Since $\sigma'_e(\mathcal{A}) \cap \sigma'_e(\mathcal{A})$ is not empty (this is consequence of the definition of essential spectrum—see [8, Lemma 4.2]), there exists $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma'_e(\mathcal{A}) \cap \sigma'_e(\mathcal{A})$ and a non-zero projection q in (the Calkin algebra) $B(H)/K(H)$ such that

$$a_i q = \lambda_i q \quad (1 \leq i \leq n).$$

Since $\sigma_p(a_i) = \sigma_{np}(a_i)$, this implies that $a_i^* q = \bar{\lambda}_i q$ ($1 \leq i \leq n$). Consequently $a_i q = (\bar{\lambda}_i q)^* = (a_i^* q)^* = q a_i$ ($1 \leq i \leq n$), or, letting $\pi(Q) = q$, $(A_i Q - Q A_i)$ is a compact operator, for all $1 \leq i \leq n$. This completes the proof.

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