

TREE FORCING AND DEFINABLE MAXIMAL INDEPENDENT SETS IN HYPERGRAPHS

JONATHAN SCHILHAN 

Abstract. We show that after forcing with a countable support iteration or a finite product of Sacks or splitting forcing over L , every analytic hypergraph on a Polish space admits a Δ_2^1 maximal independent set. This extends an earlier result by Schritterser (see [25]). As a main application we get the consistency of $\mathfrak{r} = \mathfrak{u} = \mathfrak{i} = \omega_2$ together with the existence of a Δ_2^1 ultrafilter, a Π_1^1 maximal independent family, and a Δ_2^1 Hamel basis. This solves open problems of Brendle, Fischer, and Khomskii [5] and the author [23]. We also show in ZFC that $\mathfrak{d} \leq \mathfrak{i}_{cl}$, addressing another question from [5].

§1. Introduction. Throughout mathematics, the existence of various kinds of maximal sets can typically only be obtained by an appeal to the *Axiom of Choice* or one of its popular forms, such as *Zorn's Lemma*. Under certain circumstances, it is possible though, to explicitly define such objects. The earliest result in this direction is probably due to Gödel who noted in [15, p. 67] that in the constructible universe L , there is a Δ_2^1 well-order of the reals (see [18, 25] for a modern treatment). Using similar ideas, many other special sets of reals, such as *Vitali sets*, *Hamel bases*, or *mad families*, just to name a few, can be constructed in L in a Δ_2^1 way. This has become by now a standard set theoretic technique. In many cases, these results also give an optimal bound for the complexity of such a set. For example, a Vitali set cannot be Lebesgue measurable and in particular cannot have a Σ_1^1 or Π_1^1 definition. In other cases, one can get stronger results by constructing Π_1^1 witnesses. This is typically done using a coding technique, originally developed by Erdős, Kunen, and Mauldin in [8], later streamlined by Miller (see [21]) and further generalized by Vidnyánszky (see [34]). For example, Miller showed that there are Π_1^1 Hamel bases and mad families in L . Other results of this type can be found, e.g., in [9], [14], or [11]. Since the assumption $V = L$ is quite restrictive, it is interesting to know in what forcing extensions of L , definable witnesses for the above mentioned kinds of sets still exist. Various such results exist in the literature, e.g., in [6], [10], [13], [26], or [12].

The starting observation for this paper is that almost all of these examples can be treated in the same framework, as *maximal independent sets in hypergraphs*.

DEFINITION 1.1. A *hypergraph* E on a set X is a collection of finite non-empty subsets of X , i.e., $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$. Whenever $Y \subseteq X$, we say that Y is *E -independent*

Received September 14, 2020.

2020 *Mathematics Subject Classification.* 03E15, 03E17, 03E35, 03E45, 05C65.

Key words and phrases. forcing, definability, constructible universe, cardinal characteristics, ultrafilters, descriptive set theory, splitting forcing.

© The Author(s), 2022. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

0022-4812/22/8704-0006

DOI:10.1017/jsl.2022.36

if $[Y]^{<\omega} \cap E = \emptyset$. Moreover, we say that Y is *maximal E -independent* if Y is maximal under inclusion in the collection of E -independent subsets of X .

Whenever X is a topological space, $[X]^{<\omega}$ is the disjoint sum of the spaces $[X]^n$ for $n \in \omega$. Here, as usual, $[X]^n$, the set of subsets of X of size n becomes a topological space by identification with the quotient of X^n under the equivalence relation $(x_0, \dots, x_{n-1}) \sim (y_0, \dots, y_{n-1})$ iff $\{x_0, \dots, x_{n-1}\} = \{y_0, \dots, y_{n-1}\}$. Whenever X is Polish, $[X]^{<\omega}$ is Polish as well and we can study its definable subsets. In particular, we can study definable hypergraphs on Polish spaces.

The main result of this paper is the following theorem.

THEOREM 1.2. *After forcing with the ω_2 -length countable support iteration (csi) of Sacks or splitting forcing over L , every analytic hypergraph on a Polish space has a Δ_2^1 maximal independent set.*

This extends a result by Schrittemser [25], who proved the above for Sacks forcing, which we denote by $\mathbb{S}$, and ordinary two-dimensional graphs (see also [26]). For equivalence relations this was already known by Budinas [7]. We will also prove the case of finite products but our main focus will be on the countable support iteration. *Splitting forcing* $\mathbb{SP}$ (Definition 4.1) is a less-known forcing notion that was originally introduced by Shelah in [27] and has been studied in more detail recently [17, 20, 29, 30]. Although it is very natural and gives a minimal way to add a *splitting real* (see more below), it has not been exploited a lot and to the best of our knowledge, there is no major set theoretic text treating it in more detail.

Our three guiding examples for Theorem 1.2 will be *ultrafilters*, *maximal independent families*, and *Hamel bases*.

Recall that an *ultrafilter* on ω is a maximal subset $\mathcal{U}$ of $\mathcal{P}(\omega)$ with the *strong finite intersection property*, i.e., the property that for any $\mathcal{A} \in [\mathcal{U}]^{<\omega}$, $|\bigcap \mathcal{A}| = \omega$.¹ Thus, letting $E_u := \{\mathcal{A} \in [\mathcal{P}(\omega)]^{<\omega} : |\bigcap \mathcal{A}| < \omega\}$, an ultrafilter is a maximal E_u -independent set. In [23], we studied the projective definability of ultrafilters and introduced the cardinal invariant u_B , which is the smallest size of a collection of Borel subsets of $\mathcal{P}(\omega)$ whose union is an ultrafilter. If there is a Σ_2^1 ultrafilter, then $u_B = \omega_1$, since every Σ_2^1 set is the union of ω_1 many Borel sets. Recall that the classical ultrafilter number u is the smallest size of an ultrafilter base. We showed in [23], that $u_B \leq u$ and asked whether it is consistent that $u_B < u$ or even whether a Δ_2^1 ultrafilter can exist while $\omega_1 < u$. The difficulty is that we have to preserve a definition for an ultrafilter, while its interpretation in L must be destroyed. This has been achieved before for mad families (see [6]).

An *independent family* is a subset $\mathcal{I}$ of $\mathcal{P}(\omega)$ so that for any disjoint $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{I}]^{<\omega}$, $|\bigcap_{x \in \mathcal{A}_0} x \cap \bigcap_{x \in \mathcal{A}_1} \omega \setminus x| = \omega$. It is called *maximal independent family* if it is additionally maximal under inclusion. Thus, letting $E_i = \{\mathcal{A}_0 \dot{\cup} \mathcal{A}_1 \in [\mathcal{P}(\omega)]^{<\omega} : |\bigcap_{x \in \mathcal{A}_0} x \cap \bigcap_{x \in \mathcal{A}_1} \omega \setminus x| < \omega\}$, a maximal independent family is a maximal E_i -independent set. The definability of maximal independent families was studied by Miller in [21], who showed that they cannot be analytic, and recently by Brendle, Fischer, and Khomskii in [5], where they introduced the invariant i_B , the least size of a collection of Borel sets whose union is a maximal independent family. The classical

¹In this article, all ultrafilters are considered non-principal.

independence number i is simply the smallest size of a maximal independent family. In [5], it was asked whether $i_B < i$ is consistent and whether there can be a Π_1^1 maximal independent family while $\omega_1 < i$. In the same article, it was shown that the existence of a Δ_2^1 maximal independent family is equivalent to that of a Π_1^1 such family. The difficulty in the problem is similar to that before.

A *Hamel basis* is a vector-space basis of $\mathbb{R}$ over the field of rationals $\mathbb{Q}$. Thus, letting $E_h := \{\mathcal{A} \in [\mathbb{R}]^{<\omega} : \mathcal{A} \text{ is linearly dependent over } \mathbb{Q}\}$, a Hamel basis is a maximal E_h -independent set. A Hamel basis must be as large as the continuum itself. This is reflected in the fact that, when adding a real, every ground-model Hamel basis is destroyed. But still it makes sense to ask how many Borel sets are needed to get one. Miller, also in [21], showed that a Hamel basis can never be analytic. As before, we may ask whether there can be a Δ_2^1 Hamel basis while CH fails. Again, destroying ground-model Hamel bases seems to pose a major obstruction.

The most natural way to increase u and i is by iteratively adding *splitting reals*. Recall that for $x, y \in \mathcal{P}(\omega)$, we say that x *splits* y iff $|x \cap y| = \omega$ and $|y \setminus x| = \omega$. A real x is called *splitting over* V iff for every $y \in \mathcal{P}(\omega) \cap V$, x splits y . The classical forcing notions adding splitting reals are *Cohen*, *Random*, and *Silver forcing* and all forcings that add so-called *dominating reals*. It was shown though, in [23], that after forcing with any of these, a Σ_2^1 definition with ground model parameters will not define an ultrafilter and the same argument can be applied to independent families. For this reason, we are going to use the forcing notion $\mathbb{SP}$ that we mentioned above. As an immediate corollary of Theorem 1.2, we get the following.

THEOREM 1.3. *It is consistent that $\tau = u = i = \omega_2$ while there is a Δ_2^1 ultrafilter, a Π_1^1 maximal independent family, and a Δ_2^1 Hamel basis. In particular, we get the consistency of $i_B, u_B < \tau, i, u$.*

Here, τ is the reaping number, the least size of a set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ so that there is no splitting real over $\mathcal{S}$. This solves the abovementioned questions from [5, 23]. Moreover, Theorem 1.2 gives a “black-box” way to get many results, saying that certain definable families exist in the Sacks model.

In [5], another cardinal invariant i_{cl} is introduced, which is the smallest size of a collection of closed sets, whose union is a maximal independent family. Similarly, one can define a closed version of the ultrafilter number, u_{cl} . Here, it is irrelevant whether we consider closed subsets of $[\omega]^\omega$ or $\mathcal{P}(\omega)$, since every closed subset of $[\omega]^\omega$ with the strong finite intersection property is σ -compact (see Lemma 5.6). In the model of Theorem 1.3, we have that $i_{cl} = i_B$ and $u_{cl} = u_B$, further answering the questions of Brendle, Fischer, and Khomskii. On the other hand we show that $\mathfrak{d} \leq i_{cl}$, mirroring Shelah’s result that $\mathfrak{d} \leq i$ (see [32]). Here, $\mathfrak{d}$ is the dominating number, the least size of a dominating family in $(\omega^\omega, <^*)$.

THEOREM 1.4. *(ZFC) $\mathfrak{d} \leq i_{cl}$.*

The paper is organized as follows. In Section 2, we will consider basic results concerning iterations of tree forcings. This section is interesting in its own right and can be read independently from the rest. More specifically, we prove a version of continuous reading of names for countable support iterations that is widely applicable (Lemma 2.2). In Section 3, we prove our main combinatorial lemma (Main Lemmas 3.4 and 3.14) which is at the heart of Theorem 1.2. As for Section 2,

Section 3 can be read independently of the rest, since our result is purely descriptive set theoretical. In Section 4, we introduce splitting and Sacks forcing and place it in bigger class of forcings to which we can apply the main lemma. This combines the results from Sections 2 and 3. In Section 4, we bring everything together and prove Theorems 1.2–1.4. We end with concluding remarks concerning the further outlook of our technique and pose some questions.

§2. Tree forcing. Let A be a fixed countable set, usually ω or 2 .

- (a) A *tree* T on A is a subset of $A^{<\omega}$ so that for every $t \in T$ and $n < |t|$, $t \restriction n \in T$, where $|t|$ denotes the length of t . For $s_0, s_1 \in A^{<\omega}$, we write $s_0 \perp s_1$ whenever $s_0 \not\subseteq s_1$ and $s_1 \not\subseteq s_0$.
- (b) T is *perfect* if for every $t \in T$ there are $s_0, s_1 \in T$ so that $s_0, s_1 \supseteq t$ and $s_0 \perp s_1$.
- (c) A node $t \in T$ is called a *splitting node*, if there are $i \neq j \in A$ so that $t \frown i, t \frown j \in T$. The set of splitting nodes in T is denoted $\text{split}(T)$. We define $\text{split}_n(T)$ to be the set of $t \in \text{split}(T)$ such that there are exactly n splitting nodes below t in T . The finite subtree of T generated by $\text{split}_n(T)$ is denoted $\text{split}_{\leq n}(T)$.
- (d) For any $t \in T$ we define the restriction of T to t as $T_t = \{s \in T : s \not\subseteq t\}$.
- (e) The set of branches through T is denoted by $[T] = \{x \in A^\omega : \forall n \in \omega (x \restriction n \in T)\}$.
- (f) A^ω carries a natural Polish topology generated by the clopen sets $[t] = \{x \in A^\omega : t \subseteq x\}$ for $t \in A^{<\omega}$. Then $[T]$ is closed in A^ω .
- (g) Whenever $X \subseteq A^\omega$ is closed, there is a continuous *retraction* $\varphi : A^\omega \rightarrow X$, i.e., $\varphi'' A^\omega = X$ and $\varphi \restriction X$ is the identity.
- (h) A *tree forcing* is a collection $\mathbb{P}$ of perfect trees ordered by inclusion.
- (i) By convention, all tree forcings are closed under restrictions, i.e., if $T \in \mathbb{P}$ and $t \in T$, then $T_t \in \mathbb{P}$, and the trivial condition is $A^{<\omega}$.
- (j) The set $\mathcal{T}$ of perfect subtrees of $A^{<\omega}$ is a G_δ subset of $\mathcal{P}(A^{<\omega}) \cong \mathcal{P}(\omega)$, where we identify $A^{<\omega}$ with ω , and thus carries a natural Polish topology. It is not hard to see that it is homeomorphic to ω^ω , when $|A| \geq 2$.
- (k) Often times, we will use a bar above a variable, as in “ $\bar{x}$,” to indicate that it denotes a sequence. In that case, we either write $x(\alpha)$ or x_α to denote the α -th element of that sequence, depending on the context.
- (l) Let $\langle T_i : i < \alpha \rangle$ be a sequence of trees where α is an arbitrary ordinal. Then we write $\bigotimes_{i < \alpha} T_i$ for the set of finite partial sequences $\bar{s}$ where $\text{dom } \bar{s} \in [\alpha]^{<\omega}$ and for every $i \in \text{dom } \bar{s}$, $s(i) \in T_i$.
- (m) $(A^\omega)^\alpha$ carries a topology generated by the sets $[\bar{s}] = \{\bar{x} \in (A^\omega)^\alpha : \forall i \in \text{dom } \bar{s} (x(i) \in [s(i)])\}$ for $\bar{s} \in \bigotimes_{i < \alpha} A^{<\omega}$.
- (n) Whenever $X \subseteq (A^\omega)^\alpha$ and $C \subseteq \alpha$, we define the *projection of X to C* as $X \restriction C = \{\bar{x} \restriction C : \bar{x} \in X\}$.

FACT. Let $\mathbb{P}$ be a tree forcing and G a $\mathbb{P}$ -generic filter over V . Then $\mathbb{P}$ adds a real $x_G := \bigcup \{s \in A^{<\omega} : \forall T \in G (s \in T)\} \in A^\omega$. Moreover, $V[G] = V[x_G]$.

DEFINITION 2.1. We say that $(\mathbb{P}, \leq)$ is *Axiom A* if there is a decreasing sequence of partial orders $\langle \leq_n : n \in \omega \rangle$ refining $\leq$ on $\mathbb{P}$ so that

- (1) for any $n \in \omega$ and $T, S \in \mathbb{P}$, if $S \leq_n T$, then $S \cap A^{<n} = T \cap A^{<n}$,
- (2) for any fusion sequence, i.e., a sequence $\langle p_n : n \in \omega \rangle$ where $p_{n+1} \leq_n p_n$ for every n , $p = \bigcap_{n \in \omega} p_n \in \mathbb{P}$ and $p \leq_n p_n$ for every n , and

- (3) for any maximal antichain $D \subseteq \mathbb{P}$, $p \in \mathbb{P}$, $n \in \omega$, there is $q \leq_n p$ so that $\{r \in D : r \not\leq q\}$ is countable.

Moreover we say that $(\mathbb{P}, \leq)$ is *Axiom A with continuous reading of names (crn)* if there is such a sequence of partial orders so that additionally,

- (4) for every $p \in \mathbb{P}$, $n \in \omega$, and $\dot{y}$ a $\mathbb{P}$ -name for an element of a Polish space² X , there is $q \leq_n p$ and a continuous function $f : [q] \rightarrow X$ so that

$$q \Vdash \dot{y}[G] = f(x_G).$$

Although (1) is typically not part of the definition of Axiom A, we include it for technical reasons. The only classical example that we are aware of, in which it is not clear whether (1)–(4) can be realized simultaneously, is Mathias forcing.

Let $\langle \mathbb{P}_\beta, \dot{Q}_\beta : \beta < \alpha \rangle$ be a countable support iteration of tree forcings that are Axiom A with crn, where for each $\beta < \alpha$,

$$\Vdash_{\mathbb{P}_\beta} \langle \dot{Q}_{\beta,n} : n \in \omega \rangle \text{ witnesses that } \dot{Q}_\beta \text{ is Axiom A with crn.}''$$

- (n) For each $n \in \omega$, $a \subseteq \alpha$, we define $\leq_{n,a}$ on $\mathbb{P}_\alpha$, where

$$\bar{q} \leq_{n,a} \bar{p} \leftrightarrow (\bar{q} \leq \bar{p} \wedge \forall \beta \in a (\bar{q} \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{q}(\beta) \dot{\leq}_{\beta,n} \dot{p}(\beta))).$$

- (o) The support of $\bar{p} \in \mathbb{P}_\alpha$ is the set $\text{supp}(\bar{p}) = \{\beta < \alpha : \bar{p} \restriction \beta \Vdash \dot{p}(\beta) \neq \mathbb{1}\}$.

Recall that a condition q is called a master condition over a model M if for any maximal antichain $D \in M$, $\{p \in D : q \not\leq p\} \subseteq M$. Equivalently, it means that for every generic filter G over V containing q , G is generic over M as well. Throughout this paper, when we say that M is *elementary*, we mean that it is elementary in a large enough model of the form $H(\theta)$. Sometimes, we will say that M is *a model of set theory* or just that M is *a model*. In most generality, this just means that $(M, \in)$ satisfies a strong enough fragment of ZFC. But this is a way too general notion for our purposes. For instance, such M may not even be correct about what ω is. Thus, let us clarify that in all our instances this will mean that M is either elementary or an extension of an elementary model by a countable (in M) forcing. In particular, some basic absoluteness (e.g., for Σ_1^1 or Π_1^1 formulas) holds true between M and V , M is transitive below ω_1 , and ω_1 is computed correctly.

FACT (Fusion Lemma, see, e.g., [1, Lemmas 1.2 and 2.3]). *If $\langle a_n : n \in \omega \rangle$ is $\subseteq$ -increasing, $\langle \bar{p}_n : n \in \omega \rangle$ is such that $\forall n \in \omega (\bar{p}_{n+1} \leq_{n,a_n} \bar{p}_n)$, and $\bigcup_{n \in \omega} \text{supp}(\bar{p}_n) \subseteq \bigcup_{n \in \omega} a_n \subseteq \alpha$, then there is a condition $\bar{p} \in \mathbb{P}_\alpha$ so that for every $n \in \omega$, $\bar{p} \leq_{n,a_n} \bar{p}_n$; in fact, for every $\beta < \alpha$, $\bar{p} \restriction \beta \Vdash \dot{p}(\beta) = \bigcap_{n \in \omega} \dot{p}_n(\beta)$.*

Moreover, let M be a countable elementary model, $\bar{p} \in M \cap \mathbb{P}_\alpha$, $n \in \omega$, $a \subseteq M \cap \alpha$ finite, and $\langle \alpha_i : i \in \omega \rangle$ a cofinal increasing sequence in $M \cap \alpha$. Then there is $\bar{q} \leq_{n,a} \bar{p}$ a master condition over M so that for every name $\dot{y} \in M$ for an element of ω^ω and $j \in \omega$, there is $i \in \omega$ so that below $\bar{q}$, the value of $\dot{y} \restriction j$ only depends on the $\mathbb{P}_{\alpha_i}$ -generic.

- (p) For G a $\mathbb{P}_\alpha$ -generic, we write $\bar{x}_G$ for the generic element of $\prod_{\beta < \alpha} A^\omega$ added by $\mathbb{P}_\alpha$.

²In the generic extension $V[G]$ we reinterpret X as the completion of $(X)^V$. Similarly, we reinterpret spaces $(A^\omega)^\alpha$, continuous functions, and open and closed sets on these spaces. This should be standard.

Let us from now on assume that for each $\beta < \alpha$ and $n \in \omega$, $\mathbb{Q}_\beta$ and $\leq_{\beta,n}$ are fixed analytic subsets of $\mathcal{T}$ and $\mathcal{T}^2$ respectively, coded in V . Although the theory that we develop below can be extended to a large extent to non-definable iterands, we will only focus on this case, since we need stronger results later on.

LEMMA 2.2. *For any $\bar{p} \in \mathbb{P}_\alpha$, M a countable elementary model so that $\mathbb{P}_\alpha, \bar{p} \in M$, and $n \in \omega$, $a \subseteq M \cap \alpha$ finite, there is $\bar{q} \leq_{n,a} \bar{p}$ a master condition over M and a closed set $[\bar{q}] \subseteq (A^\omega)^\alpha$ so that*

$$(1) \bar{q} \Vdash \bar{x}_G \in [\bar{q}],$$

for every $\beta < \alpha$,

$$(2) \bar{q} \Vdash \dot{q}(\beta) = \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x}_G \restriction \beta \wedge s \subseteq z(\beta))\},$$

$$(3) \text{ the map sending } \bar{x} \in [\bar{q}] \restriction \beta \text{ to } \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\} \text{ is continuous and maps to } \mathbb{Q}_\beta,$$

$$(4) [\bar{q}] \restriction \beta \subseteq (A^\omega)^\beta \text{ is closed,}$$

and for every name $\dot{y} \in M$ for an element of a Polish space X ,

$$(5) \text{ there is a continuous function } f : [\bar{q}] \rightarrow X \text{ so that } \bar{q} \Vdash \dot{y} = f(\bar{x}_G).$$

(q) We call such $\bar{q}$ as in Lemma 2.2 a *good master condition over M* .

Before we prove Lemma 2.2, let us draw some consequences from the definition of a good master condition.

LEMMA 2.3. *Let $\bar{q} \in \mathbb{P}_\alpha$ be a good master condition over a model M and $\dot{y} \in M$ a name for an element of a Polish space X .*

(i) *Then $[\bar{q}]$ is unique, in fact it is the closure of $\{\bar{x}_G : G \ni \bar{q} \text{ is generic over } V\}$ in any forcing extension W of V where $(\sqsubset_\omega(|\mathbb{P}_\alpha|))^V$ is countable.*

(ii) *The continuous map $f : [\bar{q}] \rightarrow X$ given by (5) is unique, and*

(iii) *whenever $Y \in M$ is an analytic subset of X and $\bar{q} \Vdash \dot{y} \in Y$, then $f''[\bar{q}] \subseteq Y$.*

Moreover, there is a countable set $C \subseteq \alpha$, not depending on $\dot{y}$, so that

(iv) *$[\bar{q}] \restriction C$ is a closed subset of the Polish space $(A^\omega)^C$ and $[\bar{q}] = ([\bar{q}] \restriction C) \times (A^\omega)^{\alpha \setminus C}$,*

(v) *for every $\beta \in C$, there is a continuous function $g : [\bar{q}] \restriction (C \cap \beta) \rightarrow \mathbb{Q}_\beta$, so that for every $\bar{x} \in [\bar{q}]$,*

$$g(\bar{x} \restriction (C \cap \beta)) = \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \restriction \beta \wedge s \subseteq z(\beta))\},$$

(vi) *there is a continuous function $f : [\bar{q}] \restriction C \rightarrow X$, so that*

$$\bar{q} \Vdash \dot{y} = f(\bar{x}_G \restriction C).$$

PROOF. Let us write, for every $\beta < \alpha$ and $\bar{x} \in [\bar{q}] \restriction \beta$,

$$T_{\bar{x}} := \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}.$$

For (i), let W be an extension in which $(\sqsubset_\omega(|\mathbb{P}_\alpha|))^W$ is countable and let $\bar{s} \in \bigotimes_{i < \alpha} A^{<\omega}$ be arbitrary so that $[\bar{s}] \cap [\bar{q}]$ is non-empty. We claim that there is a generic G over V containing $\bar{q}$ so that $\bar{x}_G \in [\bar{s}]$. This is shown by induction on $\max(\text{dom}(\bar{s}))$. For $\bar{s} = \emptyset$ the claim is obvious. Now assume $\max(\text{dom}(\bar{s})) = \beta$, for $\beta < \alpha$. Then, by (3), $O := \{\bar{x} \in [\bar{q}] : s(\beta) \in T_{\bar{x} \restriction \beta}\}$ is open and it is non-empty since $[\bar{s}] \cap [\bar{q}] \neq \emptyset$. Applying the inductive hypothesis, there is a generic $G \ni \bar{q}$ so

that $\bar{x}_G \in O$. In $V[G \restriction \beta]$ we have, by (2), that $T_{\bar{x}_G \restriction \beta} = \dot{q}(\beta)[G]$. Moreover, since $\bar{x}_G \in O$, we have that $s(\beta) \in \dot{q}(\beta)[G]$. Then it is easy to force over $V[G \restriction \beta]$, to get a full $\mathbb{P}_\alpha$ generic $H \supseteq G \restriction \beta$ containing $\bar{q}$ so that $\bar{x}_H \restriction \beta = \bar{x}_G \restriction \beta$ and $s(\beta) \subseteq \bar{x}_H(\beta)$. By (1), for every generic G over V containing $\bar{q}$, $\bar{x}_G \in [\bar{q}]$. Thus we have shown that the set of such $\bar{x}_G$ is dense in $[\bar{q}]$. Uniqueness follows from $[\bar{q}]$ being closed and the fact that if two closed sets coded in V agree in W , then they agree in V . This follows easily from Π_1^1 absoluteness.

Now (ii) follows easily since any two continuous functions given by (5) have to agree on a dense set in an extension W and thus they agree in V . Again this is an easy consequence of Π_1^1 absoluteness.

For (iii), let us consider the analytic space $Z = \{0\} \times X \cup \{1\} \times Y$, which is the disjoint union of the spaces X and Y . Then there is a continuous surjection $F: \omega^\omega \rightarrow Z$ and by elementarity we can assume it is in M . Let us find in M a name $\dot{z}$ for an element of ω^ω so that in $V[G]$, if $\dot{y}[G] \in Y$, then $F(\dot{z}[G]) = (1, \dot{y}[G])$, and if $\dot{y}[G] \notin Y$, then $F(\dot{z}[G]) = (0, \dot{y}[G])$. By (5), there is a continuous function $g: [\bar{q}] \rightarrow \omega^\omega$ so that $\bar{q} \Vdash \dot{z} = g(\bar{x}_G)$. Since $\bar{q} \Vdash \dot{y} \in Y$, we have that for any generic G containing $\bar{q}$, $F(g(\bar{x}_G)) = (1, \dot{y}[G])$. By density, for every $\bar{x} \in [\bar{q}]$, $F(g(\bar{x})) = (1, f(\bar{x}))$ and in particular $f(\bar{x}) \in Y$.

Now let us say that the support of a function $g: [\bar{q}] \rightarrow X$ is the smallest set $C_g \subseteq \alpha$ so that the value of $g(\bar{x})$ only depends on $\bar{x} \restriction C_g$. The results of [3] imply that if g is continuous, then g has countable support. Note that for all $\beta \notin \text{supp}(\bar{q})$, the map in (3) is constant on the set of generics and by continuity it is constant everywhere. Thus it has empty support. Let C be the union of $\text{supp}(\bar{q})$ with all the countable supports given by instances of (3) and (5). Then C is a countable set. For (iv)–(vi), note that $[\bar{q}] \restriction C = \{\bar{y} \in (A^\omega)^C : \bar{y} \restriction (\bar{x} \restriction \alpha \setminus C) \in [\bar{q}]\}$ for $\bar{x} \in [\bar{q}]$ arbitrary, and recall that in a product, sections of closed sets are closed and continuous functions are coordinate-wise continuous. $\dashv$

PROOF OF LEMMA 2.2. Let us fix for each $\beta < \alpha$ a continuous surjection $F_\beta: \omega^\omega \rightarrow \mathbb{Q}_\beta$. The proof is by induction on α . If $\alpha = \beta + 1$, then $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{Q}_\beta$. Let $\bar{q}_0 \leq_{n,a} \bar{p} \restriction \beta$ be a master condition over M and $H \ni \bar{q}_0$ a $\mathbb{P}_\beta$ generic over V . Then, applying a standard fusion argument using Axiom A with continuous reading of names in $V[H]$ to $\mathbb{Q}_\beta$, we find $q(\beta) \leq_{\beta,n} p(\beta)$ a master condition over $M[H]$ (note that H is also M generic since $\bar{q}_0$ is a master condition over M) so that for each name $\dot{y} \in M[H]$ for an element of a Polish space X there is a continuous function $f: [q(\beta)] \rightarrow X$ so that $q(\beta) \Vdash \dot{y} = f(\dot{x}_G)$. Thus we find in V , a $\mathbb{P}_\beta$ -name $\dot{q}(\beta)$ so that $\bar{q}_0$ forces that it is such a condition. Let $M^+ \ni M$ be a countable elementary model containing $\dot{q}(\beta)$ and $\bar{q}_0$, and let $\bar{q}_{1/2} \leq_{n,a} \bar{q}_0$ be a master condition over M^+ . Again let $M^{++} \ni M^+$ be a countable elementary model containing $\bar{q}_{1/2}$. By the induction hypothesis we find $\bar{q}_1 \leq_{n,a} \bar{q}_{1/2}$ a good master condition over M^{++} . Finally, let $\bar{q} = \bar{q}_1 \dot{\cap} \dot{q}(\beta)$. Then $\bar{q} \leq_{n,a} \bar{p}$ and $\bar{q}$ is a master condition over M . Since $\dot{q}(\beta) \in M^+ \subseteq M^{++}$, there is a continuous function $f: [\bar{q}_1] \rightarrow \omega^\omega$, so that $\bar{q}_1 \Vdash_\beta F_\beta(f(\bar{x}_H)) = \dot{q}(\beta)$. Here note that F_β is in M by elementarity and we indeed find a name $\dot{z}$ in M^+ so that $\bar{q}_0 \Vdash F_\beta(\dot{z}) = \dot{q}(\beta)$. Let $[\bar{q}] = \{\bar{x} \in (A^\omega)^\alpha : \bar{x} \restriction \beta \in [\bar{q}_1] \wedge x(\beta) \in [F_\beta(f(\bar{x} \restriction \beta))]\}$. Then $[\bar{q}]$ is closed and (1)–(4) hold true. To see that $[\bar{q}]$ is closed, note that the graph of a continuous function is always closed, when the codomain is a Hausdorff space. For (5), let $\dot{y} \in M$ be a $\mathbb{P}_\alpha$ -name for an element of a Polish space X . If $H \ni \bar{q}_1$

is V -generic, then there is a continuous function $g: [q(\beta)] \rightarrow X$ in $V[H]$ so that $V[H] \models q(\beta) \Vdash g(\dot{x}_G) = \dot{y}$, where we view $\dot{y}$ as a $\mathbb{Q}_\beta$ -name in $M[H]$. Moreover there is a continuous retraction $\varphi: A^\omega \rightarrow [q(\beta)]$ in $V[H]$. Since M^+ was chosen elementary enough, we find names $\dot{g}$ and $\dot{\varphi}$ for g and φ in M^+ . The function $g \circ \varphi$ is an element of the space³ $C(A^\omega, X)$, but this is not a Polish space when A is infinite, i.e., when A^ω is not compact. It is though, always a coanalytic space (consult, e.g., [19, Proposition 2.6] to see how $C(A^\omega, X)$ is a coanalytic subspace of a suitable Polish space). Thus there is an increasing sequence $\langle Y_\xi : \xi < \omega_1 \rangle$ of analytic subspaces such that $\bigcup_{\xi < \omega_1} Y_\xi = C(A^\omega, X)$ and the same equality holds in any ω_1 -preserving extension. Since $\bar{q}_{1/2}$ is a master condition over M^+ , we have that $\bar{q}_{1/2} \Vdash \dot{g} \circ \dot{\varphi} \in Y_\xi$, where $\xi = M^+ \cap \omega_1$. Since $\bar{q}_1$ is a good master condition over M^{++} and $Y_\xi \in M^{++}$, by Lemma 2.3, there is a continuous function $g' \in V$, $g': [\bar{q}_1] \rightarrow Y_\xi$, so that $\bar{q}_1 \Vdash g'(\bar{x}_H) = \dot{g} \circ \dot{\varphi}$. Altogether we have that $\bar{q} \Vdash \dot{y} = g'(\bar{x}_G \upharpoonright \beta)(x_G(\beta))$.

For α limit, let $\langle \alpha_i : i \in \omega \rangle$ be a strictly increasing sequence cofinal in $M \cap \alpha$ and let $\bar{q}_0 \leq_{n,a} \bar{p}$ be a master condition over M so that for every name $\dot{y} \in M$ for an element of ω^ω , $j \in \omega$, the value of $\dot{y} \upharpoonright j$ only depends on the generic restricted to $\mathbb{P}_{\alpha_i}$ for some $i \in \omega$. Let us fix a “big” countable elementary model N , with $\bar{q}_0, M \in N$. Let $\langle a_i : i \in \omega \rangle$ be an increasing sequence of finite subsets of $N \cap \alpha$ so that $a_0 = a$ and $\bigcup_{i \in \omega} a_i = N \cap \alpha$. Now inductively define sequences $\langle M_i : i \in \omega \rangle$, $\langle \bar{r}_i : i \in \omega \rangle$, initial segments lying in N , so that for every $i \in \omega$,

- $M_0 = M$, $\bar{r}_0 = \bar{q}_0 \upharpoonright \alpha_0$,
- $M_{i+1} \ni \bar{q}_0$ is a countable model,
- $M_i, \bar{r}_i, a_i \in M_{i+1}$,
- $\bar{r}_i$ is a good $\mathbb{P}_{\alpha_i}$ master condition over M_i ,
- $r_{i+1} \leq_{n+i, a_i \cap \alpha_i} r_i \widehat{\cap} \bar{q}_0 \upharpoonright [\alpha_i, \alpha_{i+1})$.

Define for each $i \in \omega$, $\bar{q}_i = \bar{r}_i \widehat{\cap} \bar{q}_0 \upharpoonright [\alpha_i, \alpha)$. Then $\langle \bar{q}_i : i \in \omega \rangle$ is a fusion sequence in $\mathbb{P}_\alpha$ and we can find a condition $\bar{q} \leq_{n,a} \bar{q}_0 \leq_{n,a} \bar{p}$, where for each $\beta < \alpha$, $\bar{q} \upharpoonright \beta \Vdash \dot{q}(\beta) = \bigcap_{i \in \omega} \dot{q}_i(\beta)$. Finally let $[\bar{q}] := \bigcap_{i \in \omega} ([\bar{r}_i] \times (A^\omega)^{[\alpha_i, \alpha)})$. Then (1) is easy to check. For (5), we can assume without loss of generality that $\dot{y}$ is a name for an element of ω^ω since for any Polish space X , there is a continuous surjection from ω^ω to X . Now let $(i_j)_{j \in \omega}$ be increasing so that $\dot{y} \upharpoonright j$ is determined on $\mathbb{P}_{\alpha_{i_j}}$ for every $j \in \omega$. Since $\bar{r}_{i_j}$ is a good master condition over M , there is a continuous function $f_j: [\bar{r}_{i_j}] \rightarrow \omega^j$ so that $\bar{r}_{i_j} \Vdash \dot{y} \upharpoonright j = f_j(\bar{x}_{G_{\alpha_{i_j}}})$ for every $j \in \omega$. It is easy to put these functions together to a continuous function $f: [\bar{q}] \rightarrow 2^\omega$, so that $f(\bar{x}) \upharpoonright j = f_j(\bar{x} \upharpoonright \alpha_{i_j})$. Then we obviously have that $\bar{q} \Vdash \dot{y} = f(\bar{x}_G)$.

Now let us fix for each $i \in \omega$, $C_i \subseteq \alpha_i$ a countable set as given by Lemma 2.3 applied to $\bar{r}_i, M_i$, which by elementarity exists in N . Let $C = \bigcup_{i \in \omega} C_i$. Then $[\bar{q}] = [\bar{q}] \upharpoonright C \times (A^\omega)^{\alpha \setminus C}$ and $[\bar{q}] \upharpoonright C$ is closed. For every $\beta \in \alpha \setminus C$, the map given in (3) is constant and maps to $\mathbb{Q}_\beta$, as $A^{<\omega}$ is the trivial condition. Thus we may restrict our attention to $\beta \in C$. Let us write $X_i = ([\bar{r}_i] \times (A^\omega)^{[\alpha_i, \alpha)}) \upharpoonright C$ for every $i \in \omega$ and note that $\bigcap_{i \in \omega} X_i = [\bar{q}] \upharpoonright C$. For every $\beta \in C$, $\bar{x} \in [\bar{q}] \upharpoonright (C \cap \beta)$, and $i \in \omega$,

³The topology is such that for any continuous h mapping to $C(A^\omega, X)$, $(x, y) \mapsto h(x)(y)$ is continuous.

we write

$$T_{\bar{x}} := \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}] \upharpoonright C(\bar{z} \upharpoonright \beta = \bar{x} \wedge s \subseteq z(\beta))\}$$

and

$$T_{\bar{x}}^i = \{s \in A^{<\omega} : \exists \bar{z} \in X_i(\bar{z} \upharpoonright \beta = \bar{x} \wedge s \subseteq z(\beta))\}.$$

CLAIM 2.4. *For every $i \in \omega$, where $\beta \in a_i$, $T_{\bar{x}}^{i+1} \leq_{\beta,i} T_{\bar{x}}^i$. In particular, $\bigcap_{i \in \omega} T_{\bar{x}}^i \in \mathbb{Q}_\beta$.*

PROOF. If $\alpha_{i+1} \leq \beta$, then $T_{\bar{x}}^{i+1} = T_{\bar{x}}^i = A^{<\omega}$. Else consider a $\mathbb{P}_{\alpha_{i+2}}$ -name for $(T_{\bar{y}}^{i+1}, T_{\bar{y}}^i) \in \mathcal{T}^2$, where $\bar{y} = \bar{x}_G \upharpoonright (C \cap \beta)$. Such a name exists in M_{i+2} and $\beta \in a_i \subseteq M_{i+2}$. Thus $\leq_{\beta,i} \in M_{i+2}$ and by Lemma 2.3, we have that for every $\bar{y} \in [\bar{r}_{i+2}] \upharpoonright (C \cap \beta)$, $(T_{\bar{y}}^{i+1}, T_{\bar{y}}^i) \in \leq_{\beta,i}$, thus also for $\bar{y} = \bar{x}$. The rest follows from the fact that the statement, that for any fusion sequence in $\mathbb{Q}_\beta$, its intersection is in $\mathbb{Q}_\beta$, is Π_2^1 and thus absolute. $\dashv$

CLAIM 2.5. *For every γ , $\bigcap_{i \in \omega} (X_i \upharpoonright \gamma) = (\bigcap_{i \in \omega} X_i) \upharpoonright \gamma$.*

PROOF. That $\bigcap_{i \in \omega} (X_i \upharpoonright \gamma) \supseteq (\bigcap_{i \in \omega} X_i) \upharpoonright \gamma$ is obvious. Let us show by induction on $\delta \in C$, that for any $\delta' \in C \cap \delta$,

$$\bigcap_{i \in \omega} (X_i \upharpoonright \delta') \subseteq \left(\bigcap_{i \in \omega} (X_i \upharpoonright \delta) \right) \upharpoonright \delta'.$$

The base case $\delta = \min C$ is clear. For the limit case, let $\delta' \in C \cap \delta$ be given and let $(\delta_n)_{n \in \omega}$ be increasing cofinal in $(C \cap \delta) \setminus \delta'$. Whenever $\bar{y} \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta')$, by the inductive hypothesis, there is $\bar{y}_0 \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta_0)$ extending $\bar{y}$. In particular, there is $\bar{z}_0 \in X_0 \upharpoonright \delta$ extending $\bar{y}_0$. Next, there is $\bar{y}_1 \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta_1)$ extending $\bar{y}_0$ and $\bar{z}_1 \in X_1 \upharpoonright \delta$ extending $\bar{y}_1$. Continuing like this, we find a sequence $\langle \bar{z}_n : n \in \omega \rangle$ that converges to $\bar{z} \in (A^\omega)^{C \cap \delta}$. Since $\langle \bar{z}_n : n \geq m \rangle$ is contained within the closed set $X_m \upharpoonright \delta$ for each $m \in \omega$, $\bar{z} \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta)$. Since $\bar{z} \upharpoonright \delta' = \bar{y}$, this proves the limit case. Now assume $\delta = \xi + 1$. Let $\delta' < \delta$ be given and let $\bar{y} \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta')$. Then there is $\bar{z} \in \bigcap_{i \in \omega} (X_i \upharpoonright \xi)$ extending $\bar{y}$ by the inductive hypothesis. Since $\bigcap_{i \in \omega} T_{\bar{z}}^i \in \mathbb{Q}_\delta$, there is $u \in [\bigcap_{i \in \omega} T_{\bar{z}}^i]$ and $\bar{z} \cap u \in \bigcap_{i \in \omega} (X_i \upharpoonright \delta)$. To finish the proof apply the induction step one more time to $\delta = \sup\{\xi + 1 : \xi \in C\}$ and $\delta' = \gamma$. $\dashv$

Claim 2.5 shows that (4) holds as $[\bar{q}] \upharpoonright \beta = (\bigcap_{i \in \omega} X_i) \upharpoonright \beta \times (A^\omega)^{\beta \setminus C} = \bigcap_{i \in \omega} (X_i \upharpoonright \beta) \times (A^\omega)^{\beta \setminus C}$ and $\bigcap_{i \in \omega} (X_i \upharpoonright \beta)$ is closed, being an intersection of closed sets.

CLAIM 2.6. $T_{\bar{x}} = \bigcap_{i \in \omega} T_{\bar{x}}^i$.

PROOF. That $T_{\bar{x}} \subseteq \bigcap_{i \in \omega} T_{\bar{x}}^i$ is clear from the definitions. Thus let $s \in \bigcap_{i \in \omega} T_{\bar{x}}^i$. As $\bigcap_{i \in \omega} T_{\bar{x}}^i \in \mathbb{Q}_\beta$, there is $y \in [\bigcap_{i \in \omega} T_{\bar{x}}^i]$ with $s \subseteq y$. In particular, $\bar{x} \cap y \in \bigcap_{i \in \omega} (X_i \upharpoonright (\beta + 1)) = (\bigcap_{i \in \omega} X_i) \upharpoonright (\beta + 1)$. So there is $\bar{z} \in \bigcap_{i \in \omega} X_i$ with $\bar{z} \upharpoonright \beta = \bar{x}$ and $z(\beta) = y \supseteq s$. Thus $s \in T_{\bar{x}}$. $\dashv$

Now (2) follows easily. For the continuity of $\bar{x} \mapsto T_{\bar{x}}$, let $t \in A^{<\omega}$ be arbitrary and j large enough so that $|t| \leq j$ and $\beta \in a_j$. Then $\{\bar{x} \in [\bar{q}] \upharpoonright \beta : t \notin T_{\bar{x}}\} =$

$\{\bar{x} \in [\bar{q}] \upharpoonright \beta : t \notin T_{\bar{x}}^j\}$ and $\{\bar{x} \in [\bar{q}] \upharpoonright \beta : t \in T_{\bar{x}}\} = \{\bar{x} \in [\bar{q}] \upharpoonright \beta : t \in T_{\bar{x}}^j\}$ which are both open.⁴ Thus we have shown (3). $\dashv$

LEMMA 2.7. Let $C \subseteq \alpha$ be countable and $X \subseteq (A^\omega)^C$ be a closed set so that for every $\beta \in C$ and $\bar{x} \in X \upharpoonright \beta$,

$$\{s \in A^{<\omega} : \exists \bar{z} \in X(\bar{z} \upharpoonright \beta = \bar{x} \wedge s \subseteq z(\beta))\} \in \mathbb{Q}_\beta.$$

Let $M \ni X$ be countable elementary. Then there is a good master condition $\bar{r}$ over M so that $[\bar{r}] \upharpoonright C \subseteq X$.

PROOF. It is easy to construct $\bar{q} \in M$ recursively so that $\bar{q} \Vdash \bar{x}_G \upharpoonright C \in X$. By Lemma 2.2, we can extend $\bar{q}$ to a good master condition $\bar{r}$ over M . The unique continuous function $f : [\bar{r}] \rightarrow (A^\omega)^C$ so that for generic G , $f(\bar{x}_G) = \bar{x}_G \upharpoonright C$, is so that $f(\bar{x}) = \bar{x} \upharpoonright C$ for every $\bar{x} \in [\bar{r}]$. Since f maps to X , $[\bar{r}] \upharpoonright C \subseteq X$. $\dashv$

§3. The Main Lemma.

3.1. Mutual Cohen genericity. Let X be a Polish space and M a model of set theory with $X \in M$. Recall that $x \in X$ is *Cohen generic in X over M* if for any open dense $O \subseteq X$, such that $O \in M$, $x \in O$.

Let $x_0, \dots, x_{n-1} \in X$. Then we say that $x_0, \dots, x_{n-1}$ are *(X -)mutually Cohen generic (mCg) over M* if $(y_0, \dots, y_{K-1})$ is a Cohen generic real over M in the Polish space X^K , where $\langle y_i : i < K \rangle$ is some, equivalently any, enumeration of $\{x_0, \dots, x_{n-1}\}$. In particular, we allow for repetition in the definition of mutual genericity.

DEFINITION 3.1. Let $\langle X_l : l < k \rangle \in M$ be Polish spaces. Then we say that $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{l < k} X_l$ are *$\langle X_l : l < k \rangle$ -mutually Cohen generic (mCg) over M* , if their components are mutually added Cohen generics, i.e.,

$$(y_0^0, \dots, y_0^{K_0}, \dots, y_{k-1}^0, \dots, y_{k-1}^{K_{k-1}}) \text{ is Cohen generic in } \prod_{l < k} X_l^{K_l} \text{ over } M,$$

where $\langle y_l^i : i < K_l \rangle$ is some, equivalently any, enumeration of $\{x_i(l) : i < n\}$ for each $l < k$.

DEFINITION 3.2. Let X be a Polish space with a fixed countable basis $\mathcal{B}$. Then we define the forcing poset $\mathbb{C}(2^\omega, X)$ consisting of functions $h : 2^{\leq n} \rightarrow \mathcal{B} \setminus \{\emptyset\}$ for some $n \in \omega$ such that $\forall \sigma \subseteq \tau \in 2^{\leq n} (h(\sigma) \supseteq h(\tau))$. The poset is ordered by function extension.

The poset $\mathbb{C}(2^\omega, X)$ adds generically a continuous function $\chi : 2^\omega \rightarrow X$, given by $\chi(x) = y$ where $\bigcap_{n \in \omega} h(x \upharpoonright n) = \{y\}$ and $h = \bigcup G$ for G the generic filter. This forcing will be used in this section several times to obtain ZFC results. Note for instance that if G is generic over M , then for any $x \in 2^\omega$, $\chi(x)$ is Cohen generic in X over M , and moreover, for any $x_0, \dots, x_{n-1} \in 2^\omega$, $\chi(x_0), \dots, \chi(x_{n-1})$ are X -mutually Cohen generic over M . Sometimes we will use $\mathbb{C}(2^\omega, X)$ to force over a countable model a continuous function from a space homeomorphic to 2^ω , such as $(2^\omega)^\alpha$ for $\alpha < \omega_1$.

⁴Here we use clause (1) in the definition of Axiom A.

LEMMA 3.3. Let M be a model of set theory, $K, n \in \omega$, $X_j \in M$ a Polish space for every $j < n$, and G a $\prod_{j < n} \mathbb{C}(2^\omega, X_j)$ -generic over M yielding $\chi_j: 2^\omega \rightarrow X_j$ for every $j < n$. Then, whenever $\bar{x}$ is Cohen generic in $(2^\omega)^K$ over $M[G]$ and $u_0, \dots, u_{n-1} \in 2^\omega \cap M[\bar{x}]$ are pairwise distinct,

$$\bar{x} \restriction \langle \chi_j(u_i) : i < n, j < n \rangle$$

is Cohen generic in

$$(2^\omega)^K \times \prod_{i < n} X_i$$

over M .

PROOF. Since $\bar{x}$ is generic over M it suffices to show that $\langle \chi_j(u_i) : i < n, j \rangle$ is generic over $M[\bar{x}]$. Let $\dot{O} \in M$ be a $(2^{<\omega})^K$ -name for a dense open subset of $\prod_{j < n} (X_j)^n$ and $\dot{u}_i$ a $(2^{<\omega})^K$ -name for u_i , $i < n$, such that the trivial condition forces that the $\dot{u}_i$ are pairwise distinct. Then consider the set

$$D := \{(\bar{h}, \bar{s}) \in \prod_{i < n} \mathbb{C}(2^\omega, X_i) \times (2^{<\omega})^K : \exists t_0, \dots, t_{n-1} \in 2^{<\omega} \\ (\forall i < n (\bar{s} \Vdash t_i \subseteq \dot{u}_i) \wedge \bar{s} \Vdash \prod_{i, j < n} h_j(t_i) \subseteq \dot{O})\}.$$

We claim that this set is dense in $\prod_{i < n} \mathbb{C}(2^\omega, X_i) \times (2^{<\omega})^K$ which finishes the proof. Namely let $(\bar{h}, \bar{s})$ be arbitrary, wlog $\text{dom } h_j = 2^{\leq n_0}$ for every $j < n$. Then we can extend $\bar{s}$ to $\bar{s}'$ so that there are incompatible t_i , with $|t_i| \geq n_0$, so that $\bar{s}' \Vdash t_i \subseteq \dot{u}_i$ and there are $U_{i,j} \subseteq h_j(t_i \restriction n_0)$ basic open subsets of X_j in M for every $i < n$ and $j < n$, so that $\bar{s}' \Vdash \prod_{i, j < n} U_{i,j} \subseteq \dot{O}$. Then we can extend $\bar{h}$ to $\bar{h}'$ so that $h'_j(t_i) = U_{i,j}$ for every $i, j < n$. We see that $(\bar{h}', \bar{s}') \in D$. $\dashv$

3.2. Finite products. This subsection can be skipped entirely if one is only interested in the results for the countable support iteration. The lemma that we will prove below is relevant to finite products instead (see Theorem 5.5). It is very similar to Main Lemma 3.14 in the next subsection, but the proofs are completely different. Main Lemma 3.4 is based on a forcing-theoretic proof of the Halpern–Läuchli theorem that is commonly attributed to L. Harrington (see, e.g., [16, Lemma 4.2.4] as a reference). On the other hand, Main Lemma 3.14 uses an inductive argument.

MAIN LEMMA 3.4. Let $k \in \omega$ and $E \subseteq [(2^\omega)^k]^{<\omega}$ an analytic hypergraph on $(2^\omega)^k$. Then there is a countable model M so that either

(1) for any $n \in \omega$ and $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^k$ that are $\langle 2^\omega : l < k \rangle$ -mCg over M ,

$$\{\bar{x}_0, \dots, \bar{x}_{n-1}\} \text{ is } E\text{-independent}$$

or for some $N \in \omega$,

(2) there are $\phi_0, \dots, \phi_{N-1}: (2^\omega)^k \rightarrow (2^\omega)^k$ continuous, $\bar{s} \in \bigotimes_{l < k} 2^{<\omega}$ so that for any $n \in \omega$ and $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^k \cap [\bar{s}]$, that are $\langle 2^\omega : l < k \rangle$ -mCg over M ,

$$\{\phi_j(\bar{x}_i) : j < N, i < n\} \text{ is } E\text{-independent but } \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

REMARK 3.5. Note that $N = 0$ is possible in the second option. For example whenever $[(2^\omega)^k]^1 \subseteq E$, then $\emptyset$ is the only E -independent set. In this case the last line simplifies to “ $\{\bar{x}_0\} \in E$.”

PROOF. Let $\kappa = \beth_{2k-1}(\aleph_0)^+$. Recall that by Erdős–Rado (see [18, Theorem 9.6]), for any $c: [\kappa]^{2k} \rightarrow H(\omega)$, there is $B \in [\kappa]^{\aleph_1}$ which is monochromatic for c , i.e., $c \upharpoonright [B]^{2k}$ is constant. Let $\mathbb{Q}$ be the forcing adding κ many Cohen reals

$$\langle z_{(l,\alpha)} : \alpha < \kappa \rangle \text{ in } 2^\omega \text{ for each } l < k$$

with finite conditions, i.e., $\mathbb{Q} = \prod_{\kappa}^{<\omega} (2^{<\omega})^k$. We will use the notational convention that elements of $[\kappa]^d$, for $d \in \omega$, are sequences $\bar{\alpha} = (\alpha_0, \dots, \alpha_{d-1})$ ordered increasingly. For any $\bar{\alpha} \in [\kappa]^k$ we define $\bar{z}_{\bar{\alpha}} := (z_{(0,\alpha_0)}, \dots, z_{(k-1,\alpha_{k-1})}) \in (2^\omega)^k$.

Let $\dot{A}$ be a $\mathbb{Q}$ -name for a maximal E -independent subset of $\{\bar{z}_{\bar{\alpha}} : \bar{\alpha} \in [\kappa]^k\}$, reinterpreting E in the extension by $\mathbb{Q}$. For any $\bar{\alpha} \in [\kappa]^k$, we fix $p_{\bar{\alpha}} \in \mathbb{Q}$ so that either

$$p_{\bar{\alpha}} = 1 \wedge p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \in \dot{A} \quad (1)$$

or

$$p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \notin \dot{A}. \quad (2)$$

In case (2) we additionally fix $N_{\bar{\alpha}} < \omega$ and $(\bar{\beta}^i)_{i < N_{\bar{\alpha}}} = (\bar{\beta}^i(\bar{\alpha}))_{i < N_{\bar{\alpha}}}$, and we assume that

$$p_{\bar{\alpha}} \Vdash \{\bar{z}_{\bar{\beta}^i} : i < N_{\bar{\alpha}}\} \subseteq \dot{A} \wedge \{\bar{z}_{\bar{\alpha}}\} \cup \{\bar{z}_{\bar{\beta}^i} : i < N_{\bar{\alpha}}\} \in E.$$

We also define $H_l(\bar{\alpha}) = \{\beta_l^i : i < N_{\bar{\alpha}}\} \cup \{\alpha_l\} \in [\kappa]^{<\omega}$ for each $l < k$.

Now for $\bar{\alpha} \in [\kappa]^{2k}$ we collect the following information:

- (i) whether $p_{\bar{\alpha} \upharpoonright k} = p_{\alpha_0, \dots, \alpha_{k-1}} \Vdash \bar{z}_{\bar{\alpha} \upharpoonright k} \in \dot{A}$ or not,
- (ii) $\bar{s} = (p_{\bar{\alpha} \upharpoonright k}(0, \alpha_0), \dots, p_{\bar{\alpha} \upharpoonright k}(k-1, \alpha_{k-1})) \in (2^{<\omega})^k$,
- (iii) the relative position of the $p_{\bar{\gamma}}$ for $\bar{\gamma} \in \Gamma := \prod_{l < k} \{\alpha_{2l}, \alpha_{2l+1}\}$ to each other. More precisely, consider $\bigcup_{\bar{\gamma} \in \Gamma} \text{dom } p_{\bar{\gamma}} = \{0\} \times d_0 \cup \dots \cup \{k-1\} \times d_{k-1}$ where $d_0, \dots, d_{k-1} \subseteq \kappa$. Let $M_l = |d_l|$ for $l < k$ and for each $\bar{j} \in \prod_{l < k} \{2l, 2l+1\}$, collect a function $r_{\bar{j}}$ with $\text{dom } r_{\bar{j}} \subseteq \{0\} \times M_0 \cup \dots \cup \{k-1\} \times M_{k-1}$ that is a copy of $p_{\bar{\gamma}}$, where $\bar{\gamma} = (\alpha_{j_0}, \dots, \alpha_{j_{k-1}})$, $\bar{j} = (j_0, \dots, j_{k-1})$. Namely, $r_{\bar{j}}(l, m) = p_{\bar{\gamma}}(l, \beta)$, whenever β is the m -th element of d_l .

In case $p_{\bar{\alpha} \upharpoonright k} \Vdash \bar{z}_{\bar{\alpha} \upharpoonright k} \notin \dot{A}$ we additionally remember

- (iv) $N = N_{\bar{\alpha} \upharpoonright k}$,
- (v) $N_l = |H_l(\bar{\alpha} \upharpoonright k)|$, for each $l < k$,
- (vi) $\bar{b}^i \in \prod_{l < k} N_l$ so that β_l^i is the b_l^i -th element of $H_l(\bar{\alpha} \upharpoonright k)$, for each $i < N$,
- (vii) $\bar{a} \in \prod_{l < k} N_l$ so that α_l is the a_l -th member of $H_l(\bar{\alpha} \upharpoonright k)$,
- (viii) the partial function r with domain a subset of $\bigcup_{l < k} \{l\} \times N_l$, so that $r(l, m) = t \in 2^{<\omega}$ iff $p_{\bar{\alpha} \upharpoonright k}(l, \beta) = t$ where β is the m -th element of $H_l(\bar{\alpha} \upharpoonright k)$.

And finally we also remember

- (ix) for each pair $\bar{\gamma}, \bar{\delta} \in \prod_{l < k} \{\alpha_{2l}, \alpha_{2l+1}\}$, where $\bar{\gamma} = (\alpha_{j_l})_{l < k}$ and $\bar{\delta} = (\alpha_{j'_l})_{l < k}$, finite partial injections $e_{l, \bar{j}, \bar{j}'} : N_l \rightarrow N_l$ so that $e_{l, \bar{j}, \bar{j}'}(m) = m'$ iff the m -th element of $H_l(\bar{\gamma})$ equals the m' -th element of $H_l(\bar{\delta})$.

This information is finite and defines a coloring $c: [\kappa]^{2k} \rightarrow H(\omega)$. Let $B \in [\kappa]^{\omega_1}$ be monochromatic for c . Let $M \preccurlyeq H(\theta)$ be countable for θ large enough so that $\kappa, c, B, \langle p_{\bar{\alpha}} : \bar{\alpha} \in [B]^k \rangle, E, \dot{A} \in M$.

CLAIM 3.6. *If for every $\bar{\alpha} \in [B]^k$, $p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \in \dot{A}$, then (1) of the main lemma holds true.*

PROOF. Let $\bar{x}_0, \dots, \bar{x}_{n-1}$ be arbitrary mCg over M . Say $\{x_i(l) : i < n\}$ is enumerated by $\langle y_l^i : i < K_l \rangle$ for every $l < k$. Now find

$$\alpha_0^0 < \dots < \alpha_0^{K_0-1} < \dots < \alpha_{k-1}^0 < \dots < \alpha_{k-1}^{K_{k-1}-1}$$

in $M \cap B$. Then there is a $\mathbb{Q}$ -generic G over M so that for any $\bar{j} \in \prod_{l < k} K_l$,

$$\bar{z}_{\bar{\beta}}[G] = (y_0^{j_0}, \dots, y_{k-1}^{j_{k-1}}),$$

where $\bar{\beta} = (\alpha_0^{j_0}, \dots, \alpha_{k-1}^{j_{k-1}})$. In particular, for each $i < n$, there is $\bar{\beta}_i \in [B \cap M]^k$ so that $\bar{z}_{\bar{\beta}_i}[G] = \bar{x}_i$. Since $p_{\bar{\beta}_i} = \mathbb{1} \in G$ for every $\bar{\beta}_i$ we have that

$$M[G] \models \bar{x}_i \in \dot{A}[G]$$

for every $i < n$ and in particular

$$M[G] \models \{\bar{x}_i : i < n\} \text{ is } E\text{-independent.}$$

By absoluteness $\{\bar{x}_i : i < n\}$ is indeed E -independent. $\dashv$

Assume from now on that $p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \notin \dot{A}$ for every $\bar{\alpha} \in [B]^k$. Then we may fix $\bar{s}$, N , $(N_l)_{l < k}$, $\bar{b}^i$ for $i < N$, $\bar{a}$, r and $e_{l, \bar{j}, \bar{j}'}$ for all $l < k$ and $\bar{j}, \bar{j}' \in \prod_{l' < k} \{2l', 2l' + 1\}$ corresponding to the coloring on $[B]^{2k}$.

CLAIM 3.7. *For any $\bar{\alpha} \in [B]^{2k}$ and $\bar{\gamma}, \bar{\delta} \in \prod_{l < k} \{\alpha_{2l}, \alpha_{2l+1}\}$,*

$$p_{\bar{\gamma}} \upharpoonright (\text{dom } p_{\bar{\gamma}} \cap \text{dom } p_{\bar{\delta}}) = p_{\bar{\delta}} \upharpoonright (\text{dom } p_{\bar{\gamma}} \cap \text{dom } p_{\bar{\delta}}).$$

PROOF. Suppose not. By homogeneity we find a counterexample $\bar{\alpha}$, $\bar{\gamma}$, $\bar{\delta}$ where $B \cap (\alpha_{2l'}, \alpha_{2l'+1})$ is non-empty for every $l' < k$. So let $(l, \beta) \in \text{dom } p_{\bar{\gamma}} \cap \text{dom } p_{\bar{\delta}}$ such that $p_{\bar{\gamma}}(l, \beta) = u \neq v = p_{\bar{\delta}}(l, \beta)$. Let $\bar{\rho} \in [B]^k$ be such that for every $l' < k$,

$$\begin{cases} \rho_{l'} \in (\gamma_{l'}, \delta_{l'}), & \text{if } \gamma_{l'} < \delta_{l'}, \\ \rho_{l'} \in (\delta_{l'}, \gamma_{l'}), & \text{if } \delta_{l'} < \gamma_{l'}, \\ \rho_{l'} = \gamma_{l'}, & \text{if } \gamma_{l'} = \delta_{l'}. \end{cases}$$

Now note that $\bar{\rho}$'s relative position to $\bar{\gamma}$ is the same as that of $\bar{\delta}$ to $\bar{\gamma}$. More precisely, let $\bar{j}, \bar{j}' \in \prod_{l' < k} \{2l', 2l' + 1\}$ so that $\bar{\gamma} = (\alpha_{j_0}, \dots, \alpha_{j_{k-1}})$ and $\bar{\delta} = (\alpha_{j'_0}, \dots, \alpha_{j'_{k-1}})$. Then there is $\bar{\beta} \in [B]^{2k}$ so that $\bar{\gamma} = (\beta_{j_0}, \dots, \beta_{j_{k-1}})$ and $\bar{\rho} = (\beta_{j'_0}, \dots, \beta_{j'_{k-1}})$. Thus by homogeneity of $[B]^{2k}$ via c , $p_{\bar{\rho}}(l, \beta) = v$. Similarly $\bar{\delta}$ is in the same position relative to $\bar{\rho}$ as to $\bar{\gamma}$. Thus also $p_{\bar{\rho}}(l, \beta) = u$ and we find that $v = u$ —we get a contradiction. $\dashv$

CLAIM 3.8. *For any $l < k$ and $\bar{j}, \bar{j}' \in \prod_{l' < k} \{2l', 2l' + 1\}$, $e_{l, \bar{j}, \bar{j}'}(m) = m$ for every $m \in \text{dom } e_{l, \bar{j}, \bar{j}'}$.*

PROOF. Let $\alpha_0 < \dots < \alpha_{2k} \in B$ so that $(\alpha_{2l'}, \alpha_{2l'+1}) \cap B \neq \emptyset$ for every $l' < k$. Consider $\bar{\gamma} = (\alpha_{j_{l'}})_{l' < k}$, $\bar{\delta} = (\alpha_{j'_{l'}})_{l' < k}$ and again we find $\bar{\rho} \in [B]^k$ so that $\rho_{l'}$ is between (possibly equal to) $\alpha_{j_{l'}}$ and $\alpha_{j'_{l'}}$. If $e_{l, \bar{j}, \bar{j}'}(m) = m'$, then if β is the m -th element of $H_l(\bar{\gamma})$, then β is m' -th element of $H_l(\bar{\delta})$ as well as of $H_l(\bar{\rho})$. But also β is the m -th element of $H_l(\bar{\rho})$, thus $m = m'$. $\dashv$

Note that by the above claim $e_{l, \bar{j}, \bar{j}'} = (e_{l, \bar{j}', \bar{j}})^{-1} = e_{l, \bar{j}', \bar{j}}$ and the essential information given by $e_{l, \bar{j}, \bar{j}'}$ is its domain.

Next let us introduce some notation. Whenever $x, y \in 2^\omega$, we write $x < y$ to say that x is lexicographically below y , i.e., $x(n) < y(n)$, where $n = \min\{m \in \omega : x(m) \neq y(m)\}$. For any $g \in \{-1, 0, 1\}^k$ we naturally define a relation $\tilde{R}_g$ between k -length sequences $\bar{v}$ and $\bar{\mu}$, either of elements of 2^ω , or of ordinals $< \kappa$, as follows:

$$\bar{v} \tilde{R}_g \bar{\mu} \leftrightarrow \forall l < k \begin{cases} v_l < \mu_l, & \text{if } g(l) = -1, \\ v_l = \mu_l, & \text{if } g(l) = 0, \\ v_l > \mu_l, & \text{if } g(l) = 1. \end{cases}$$

Further we write $\bar{v} R_g \bar{\mu}$ iff $\bar{v} \tilde{R}_g \bar{\mu}$ or $\bar{\mu} \tilde{R}_g \bar{v}$. Enumerate $\{R_g : g \in \{-1, 0, 1\}^k\}$ without repetition as $\langle R_i : i < K \rangle$ (it is easy to see that $K = \frac{3^k+1}{2}$). Note that for any $\bar{v}, \bar{\mu}$ there is a unique $i < K$ so that $\bar{v} R_i \bar{\mu}$. Now for each $l < k$ and $i < K$, we let

$$I_{l,i} := \text{dom } e_{l, \bar{j}, \bar{j}'} \subseteq N_l,$$

where $\bar{j} R_i \bar{j}'$. By homogeneity of $[B]^{2k}$ and the observation that $e_{l, \bar{j}, \bar{j}'} = e_{l, \bar{j}', \bar{j}}$, we see that $I_{l,i}$ does not depend on the particular choice of $\bar{j}, \bar{j}'$, such that $\bar{j} R_i \bar{j}'$.

For each $l < k$ and $m < N_l$, we define a relation $E_{l,m}$ on $(2^\omega)^k$ as follows:

$$\bar{x} E_{l,m} \bar{y} \leftrightarrow m \in I_{l,i} \text{ where } i \text{ is such that } \bar{x} R_i \bar{y}.$$

CLAIM 3.9. $E_{l,m}$ is an equivalence relation.

PROOF. The reflexivity and symmetry of $E_{l,m}$ is obvious. Assume that $\bar{x}_0 E_{l,m} \bar{x}_1$ and $\bar{x}_1 E_{l,m} \bar{x}_2$, and say $\bar{x}_0 R_{i_0} \bar{x}_1$, $\bar{x}_1 R_{i_1} \bar{x}_2$, and $\bar{x}_0 R_{i_2} \bar{x}_2$. Find $\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^2 \in [B]^k$ so that

$$\{\gamma_0^i : i < 3\} < \dots < \{\gamma_{k-1}^i : i < 3\}$$

and

$$\bar{\gamma}^0 R_{i_0} \bar{\gamma}^1, \bar{\gamma}^1 R_{i_1} \bar{\gamma}^2, \bar{\gamma}^0 R_{i_2} \bar{\gamma}^2.$$

If β is the m -th element of $H_l(\bar{\gamma}^0)$, then β is also the m -th element of $H_l(\bar{\gamma}^1)$, since we can find an appropriate $\bar{\alpha} \in [B]^{2k}$ and $\bar{j}, \bar{j}'$ so that $\bar{\gamma}^0 = (\alpha_{j_l})_{l < k}$ and $\bar{\gamma}^1 = (\alpha_{j'_l})_{l < k}$, $\bar{j} R_{i_0} \bar{j}'$ and we have that $m \in I_{l,i_0}$. Similarly β is the m -th element of $H_l(\bar{\gamma}^2)$.

But now we find again $\bar{\alpha} \in [B]^{2k}$ and $\bar{j}, \bar{j}'$ so that $\bar{\gamma}^0 = (\alpha_{j_l})_{l < k}$ and $\bar{\gamma}^2 = (\alpha_{j'_l})_{l < k}$. Thus $m \in I_{l,i_2}$, as $e_{l, \bar{j}, \bar{j}'}(m) = m$ and $\bar{x}_0 E_{l,m} \bar{x}_2$. $\dashv$

CLAIM 3.10. $E_{l,m}$ is smooth as witnessed by a continuous function, i.e., there is a continuous map $\varphi_{l,m} : (2^\omega)^k \rightarrow 2^\omega$ so that $\bar{x} E_{l,m} \bar{y}$ iff $\varphi_{l,m}(\bar{x}) = \varphi_{l,m}(\bar{y})$.

PROOF. We will check the following:

- (a) For every open $O \subseteq (2^\omega)^k$, the $E_{l,m}$ saturation of O is Borel,
- (b) every $E_{l,m}$ equivalence class is G_δ .

By a theorem of Srivastava [31, Theorem 4.1.], (a) and (b) imply that $E_{l,m}$ is smooth, i.e., we can find $\varphi_{l,m}$ Borel.

- (a) The $E_{l,m}$ saturation of O is the set $\{\bar{x} : \exists \bar{y} \in O(\bar{x}E_{l,m}\bar{y})\}$. It suffices to check for each $g \in \{-1, 0, 1\}^k$ that the set $X = \{\bar{x} : \exists \bar{y} \in O(\bar{x}\tilde{R}_g\bar{y})\}$ is Borel. Let $S = \{\bar{\sigma} \in (2^{<\omega})^k : [\sigma_0] \times \cdots \times [\sigma_{k-1}] \subseteq O\}$. Consider

$$\varphi(\bar{x}) : \leftrightarrow \exists \bar{\sigma} \in S \forall l' < k \begin{cases} x_{l'} <_{\text{lex}} \sigma_{l'} 0^\omega, & \text{if } g(l') = -1, \\ x_{l'} \in [\sigma_{l'}], & \text{if } g(l') = 0, \\ \sigma_{l'} \cap 1^\omega <_{\text{lex}} x_{l'}, & \text{if } g(l') = 1. \end{cases}$$

If $\varphi(\bar{x})$ holds true then let $\bar{\sigma}$ witness this. We then see that there is $\bar{y} \in [\sigma_0] \times \cdots \times [\sigma_{k-1}]$ with $\bar{x}\tilde{R}_g\bar{y}$. On the other hand, if $\bar{y} \in O$ is such that $\bar{x}\tilde{R}_g\bar{y}$, then we find $\bar{\sigma} \in S$ defining a neighborhood of $\bar{y}$ witnessing $\varphi(\bar{x})$. Thus X is defined by φ and is thus Borel.

- (b) Since finite unions of G_δ 's are G_δ it suffices to check that $\{\bar{x} : \bar{x}\tilde{R}_g\bar{y}\}$ is G_δ for every $\bar{y}$ and $g \in \{-1, 0, 1\}^k$. But this is obvious from the definition.

Now note that given $\varphi_{l,m}$ Borel, we can find perfect $X_0, \dots, X_{k-1} \subseteq 2^\omega$ so that $\varphi_{l,m}$ is continuous on $X_0 \times \cdots \times X_{k-1}$ ($\varphi_{l,m}$ is continuous on a dense G_δ). But there is a $<_{\text{lex}}$ preserving homeomorphism from X_l to 2^ω for each $l < k$ so we may simply assume $X_l = 2^\omega$. $\dashv$

Fix such $\varphi_{l,m}$ for every $l < k, m < N_l$, so that $\varphi_{l,a_l}(\bar{x}) = x_l$ (note that $\bar{x}E_{l,a_l}\bar{y}$ iff $x_l = y_l$). Now let M_0 be countable elementary, containing all relevant information and such that $\varphi_{l,m} \in M_0$ for every $l < k, m < N_l$. Let $\chi_{l,m} : 2^\omega \rightarrow [r(l, m)]$ for $l < k$ and $m \neq a_l$ be generic continuous functions over M_0 , i.e., the sequence $(\chi_{l,m})_{l < k, m \in N_l \setminus \{a_l\}}$ is $\prod_{l < k, m \in N_l \setminus \{a_l\}} \mathbb{C}(2^\omega, [r(l, m)])$ generic over M_0 . Let us denote with M the generic extension of M_0 . Also let $\chi_{l,m}$ for $m = a_l$ be the identity. Finally we set

$$\phi_i(\bar{x}) = ((\chi_{l,b_l^i} \circ \varphi_{l,b_l^i})(\bar{x}))_{l < k}$$

for each $i < N$.

CLAIM 3.11. (2) of the main lemma holds true with $M, \bar{s}$, and $\phi_i, i < N$, that we just defined.

PROOF. Let $\bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{s}]$ be $\langle 2^\omega : l < k \rangle$ -mCg over M . Let us write $\{\bar{x}_i(l) : i < n\} = \{y_l^i : i < K_l\}$ for every $l < k$, where $y_l^0 <_{\text{lex}} \cdots <_{\text{lex}} y_l^{K_l-1}$. Now find

$$\alpha_0^0 < \cdots < \alpha_0^{K_0-1} < \cdots < \alpha_{k-1}^0 < \cdots < \alpha_{k-1}^{K_{k-1}-1}$$

in $B \cap M$. For every $\bar{j} \in \prod_{l < k} K_l$, define $\bar{y}_{\bar{j}} := (y_0^{j(0)}, \dots, y_{k-1}^{j(k-1)})$ and $\bar{\alpha}_{\bar{j}} := (\alpha_0^{j(0)}, \dots, \alpha_{k-1}^{j(k-1)})$. Then, for each $i < n$, we have $\bar{j}_i \in \prod_{l < k} K_l$ so that $\bar{x}_i = \bar{y}_{\bar{j}_i}$. For each $i < n$ define the function $g_i : \bigcup_{l < k} \{l\} \times H_l(\bar{\alpha}_{\bar{j}_i}) \rightarrow 2^\omega$, setting

$$g_i(l, \beta) = \chi_{l,m}(\varphi_{l,m}(\bar{x}_i)),$$

whenever β is the m -th element of $H_l(\bar{\alpha}_{\bar{j}_i})$.

Now we have that the g_i agree on their common domain. Namely let $i_0, i_1 < n$ and $(l, \beta) \in \text{dom } g_{i_0} \cap \text{dom } g_{i_1}$. Then if we set i to be so that $\bar{x}_{i_0} R_i \bar{x}_{i_1}$, we have that $m \in I_{l,i}$, where β is the m -th element of $H_l(\bar{\alpha}_{\bar{j}_{i_0}})$ and of $H_l(\bar{\alpha}_{\bar{j}_{i_1}})$. In particular $\bar{x}_{i_0} E_{l,m} \bar{x}_{i_1}$ and $\varphi_{l,m}(\bar{x}_{i_0}) = \varphi_{l,m}(\bar{x}_{i_1})$ and thus

$$g_{i_0}(l, \beta) = \chi_{l,m}(\varphi_{l,m}(\bar{x}_{i_0})) = \chi_{l,m}(\varphi_{l,m}(\bar{x}_{i_1})) = g_{i_1}(l, \beta).$$

Let $g := \bigcup_{i < n} g_i$. Then we see by Lemma 3.3 that g is Cohen generic in $\prod_{(l,\beta) \in \text{dom } g} 2^\omega$ over M_0 . Namely consider $K = \sum_{l < k} K_l$ and $(y_0^0, \dots, y_{k-1}^{K_{k-1}-1})$ as a $(2^{<\omega})^K$ -generic over M . Then, if $\langle u_i : i < n' \rangle$ enumerates $\{\varphi_{l,m}(\bar{x}_i) : i < n, l < k, m < N_l\}$, we have that every value of g is contained in $\{\chi_{l,m}(u_i) : i < n', l < k, m < N_l\}$. Also note that by construction for every $i < n$, $p_{\bar{\alpha}_{\bar{j}_i}} \upharpoonright \text{dom } g$ is in the generic filter defined by g . Since $\{p_{\bar{\alpha}_{\bar{j}_i}} : i < n\}$ is centered we can extend the generic filter of g to a $\mathbb{Q}$ -generic G over M_0 so that $p_{\bar{\alpha}_{\bar{j}_i}} \in G$ for every $i < n$.

Now we have that

$$\bar{z}_{\bar{\alpha}_{\bar{j}_i}}[G] = \bar{x}_i \text{ and } \bar{z}_{\bar{\beta}^j(\bar{\alpha}_{\bar{j}_i})}[G] = \phi_j(\bar{x}_i)$$

for every $i < n$ and $j < N$. Thus we get that

$$M_0[G] \models \bigcup_{i < n} \{\phi_j(\bar{x}_i) : j < N\} \subseteq \dot{A}[G] \wedge \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

Again, by absoluteness, we get the required result. $\dashv$

3.3. Infinite products.

DEFINITION 3.12. Let $\langle X_i : i < \alpha \rangle \in M$ be Polish spaces indexed by a countable ordinal α . Then we say that $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i < \alpha} X_i$ are $\langle X_i : i < \alpha \rangle$ -mutually Cohen generic (mCg) over M if there are $\xi_0 = 0 < \dots < \xi_k = \alpha$ for some $k \in \omega$ so that

$$\bar{x}_0, \dots, \bar{x}_{n-1} \text{ are } \langle Y_l : l < k \rangle\text{-mutually Cohen generic over } M,$$

where $Y_l = \prod_{i \in [\xi_l, \xi_{l+1})} X_i$ for every $l < k$ and we identify $\bar{x}_i$ with

$$(\bar{x}_i \upharpoonright [\xi_0, \xi_1), \dots, \bar{x}_i \upharpoonright [\xi_{k-1}, \xi_k)) \in \prod_{l < k} Y_l,$$

for every $i < n$.

The identification of a sequence $\bar{x}$ with $(\bar{x} \upharpoonright [\xi_0, \xi_1), \dots, \bar{x} \upharpoonright [\xi_{k-1}, \xi_k))$ for a given $\langle \xi_l : l < k \rangle$ will be implicitly made throughout the rest of the paper in order to reduce the notational load.

Note that whenever $\bar{x}_0, \dots, \bar{x}_{n-1}$ are $\langle X_i : i < \alpha \rangle$ -mCg over M and $\beta \leq \alpha$, then $\bar{x}_0 \upharpoonright \beta, \dots, \bar{x}_{n-1} \upharpoonright \beta$ are $\langle X_i : i < \beta \rangle$ -mCg over M . Also note that Definition 3.12 agrees with the notion of mCg for finite α .

DEFINITION 3.13. We say that $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i < \alpha} X_i$ are *strongly* $\langle X_i : i < \alpha \rangle$ -mCg over M if they are $\langle X_i : i < \alpha \rangle$ -mCg over M and for any $i, j < n$, if $\xi = \min\{\beta < \alpha : x_i(\beta) \neq x_j(\beta)\}$, then $x_i(\beta) \neq x_j(\beta)$ for all $\beta \geq \xi$.

MAIN LEMMA 3.14. Let $\alpha < \omega_1$ and $E \subseteq [(2^\omega)^\alpha]^{<\omega}$ be an analytic hypergraph. Then there is a countable model M , $\alpha + 1 \subseteq M$, so that either

- (1) for any $n \in \omega$ and $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ that are strongly $\langle 2^\omega : i < \alpha \rangle$ -mCg over M ,

$$\{\bar{x}_0, \dots, \bar{x}_{n-1}\} \text{ is } E\text{-independent}$$

or for some $N \in \omega$,

- (2) there are $\phi_0, \dots, \phi_{N-1} : (2^\omega)^\alpha \rightarrow (2^\omega)^\alpha$ continuous, $\bar{s} \in \bigotimes_{i < \alpha} 2^{<\omega}$ so that for any $n \in \omega$ and $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg over M ,

$$\{\phi_j(\bar{x}_i) : j < N, i < n\} \text{ is } E\text{-independent but } \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

PROOF. We are going to show something slightly stronger. Let R be an analytic hypergraph on $(2^\omega)^\alpha \times \omega$, M a countable model with $R \in M, \alpha + 1 \subseteq M$, and $k \in \omega$. Then consider the following two statements.

- (1) $_{R, M, k}$: For any pairwise distinct $\bar{x}_0, \dots, \bar{x}_{n-1}$ that are strongly $\langle 2^\omega : i < \alpha \rangle$ -mCg over M , and any $k_0, \dots, k_{n-1} < k$,

$$\{(\bar{x}_0, k_0), \dots, (\bar{x}_{n-1}, k_{n-1})\} \text{ is } R\text{-independent.}$$

- (2) $_{R, M, k}$: There is $N \in \omega$, there are $\phi_0, \dots, \phi_{N-1} : (2^\omega)^\alpha \rightarrow (2^\omega)^\alpha$ continuous, such that for every $\bar{x} \in (2^\omega)^\alpha$ and $j_0 < j_1 < N$, $\phi_{j_0}(\bar{x}) \neq \phi_{j_1}(\bar{x})$ and $\phi_{j_0}(\bar{x}) \neq \bar{x}$, there are $k_0, \dots, k_{N-1} \leq k$ and $\bar{s} \in \bigotimes_{i < \alpha} 2^{<\omega}$, so that for any pairwise distinct $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly $\langle 2^\omega : i < \alpha \rangle$ -mCg over M ,

$$\{(\phi_j(\bar{x}_i), k_j) : j < N, i < n\} \text{ is } R\text{-independent, but}$$

$$\{(\bar{x}_0, k)\} \cup \{(\phi_j(\bar{x}_0), k_j) : j < N\} \in R.$$

In fact, if $k > 0$,

$$\{(\bar{x}_i, k-1) : i < n\} \cup \{(\phi_j(\bar{x}_i), k_j) : j < N, i < n\} \text{ is } R\text{-independent.}$$

We are going to show by induction on α that for any R, M, k , (1) $_{R, M, k}$ implies that either (1) $_{R, M, k+1}$ or there is a countable model $M^+ \supseteq M$ so that (2) $_{R, M^+, k}$. From this we easily follow the statement of the main lemma. Namely, whenever E is a hypergraph on $(2^\omega)^\alpha$, consider the hypergraph R on $(2^\omega)^\alpha \times \omega$ where $\{(\bar{x}_0, k_0), \dots, (\bar{x}_{n-1}, k_{n-1})\} \in R$ iff $\{\bar{x}_0, \dots, \bar{x}_{n-1}\} \in E$. Then, if M is an arbitrary countable elementary model with $R, \alpha \in M$ and if $k = 0$, (1) $_{R, M, k}$ holds vacuously true. Applying the claim we find M^+ so that either (1) $_{R, M, 1}$ or (2) $_{R, M^+, 0}$. The two options easily translate to the conclusion of the main lemma.

Let us first consider the successor step. Assume that $\alpha = \beta + 1$, R is an analytic hypergraph on $(2^\omega)^\alpha \times \omega$, and M a countable model with $R \in M, \alpha + 1 \subseteq M$ so that (1) $_{R, M, k}$ holds true for some given $k \in \omega$. Let $\mathbb{Q}$ be the forcing adding mutual Cohen reals $\langle z_{0,i,j}, z_{1,i,j} : i, j \in \omega \rangle$ in 2^ω . Then we define the hypergraph $\tilde{R}$ on $(2^\omega)^\beta \times \omega$ where $\{(\bar{y}_0, m_0), \dots, (\bar{y}_{n-1}, m_{n-1})\} \in \tilde{R} \cap [(2^\omega)^\beta \times \omega]^n$ iff there is $p \in \mathbb{Q}$ and there are $K_i \in \omega, k_{i,0}, \dots, k_{i,K_i-1} < k$ for every $i < n$, so that

$$p \Vdash_{\mathbb{Q}} \bigcup_{i < n} \{(\bar{y}_i \restriction z_{0,i,j}, k_{i,j}) : j < K_i\} \cup \{(\bar{y}_i \restriction z_{1,i,j}, k) : j < m_i\} \in R.$$

Then $\tilde{R}$ is analytic (see, e.g., [19, Theorem 29.22]).

CLAIM 3.15. $(1)_{\tilde{R}, M, 1}$ is satisfied.

PROOF. Suppose $\bar{y}_0, \dots, \bar{y}_{n-1}$ are pairwise distinct and strongly mCg over M , but $\{(\bar{y}_0, 0), \dots, (\bar{y}_{n-1}, 0)\} \in \tilde{R}$ as witnessed by $p \in \mathbb{Q}$, $\langle K_i : i < n \rangle$ and $\langle k_{i,j} : i < n, j < K_i \rangle$, each $k_{i,j} < k$. More precisely,

$$p \Vdash_{\mathbb{Q}} \bigcup_{i < n} \{(\bar{y}_i \hat{\smallfrown} z_{0,i,j}, k_{i,j}) : j < K_i\} \in R. \quad (*_0)$$

By absoluteness, $(*_0)$ is satisfied in $M[\bar{y}_0, \dots, \bar{y}_{n-1}]$. Thus, let $\langle z_{0,i,j}, z_{1,i,j} : i, j \in \omega \rangle$ be generic over $M[\bar{y}_0, \dots, \bar{y}_{n-1}]$ with p in the associated generic filter. Then the $\bar{y}_i \hat{\smallfrown} z_{(0,i,j)}$ for $i < n, j < K_i$ are pairwise distinct and strongly mCg over M , but

$$\bigcup_{i < n} \{(\bar{y}_i \hat{\smallfrown} z_{0,i,j}, k_{i,j}) : j < K_i\} \in R.$$

This poses a contradiction to $(1)_{R, M, k}$. $\dashv$

CLAIM 3.16. If $(1)_{\tilde{R}, M, m}$ is satisfied for every $m \in \omega$, then also $(1)_{R, M, k+1}$.

PROOF. Let $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ be pairwise distinct, strongly mCg over M and let $k_0, \dots, k_{n-1} \leq k$. Then we may write $\{(\bar{x}_0, k_0), \dots, (\bar{x}_{n-1}, k_{n-1})\}$ as

$$\bigcup_{i < n'} \{(\bar{y}_i \hat{\smallfrown} z_{0,i,j}, k_{i,j}) : j < K_i\} \cup \{(\bar{y}_i \hat{\smallfrown} z_{1,i,j}, k) : j < m_i\}, \quad (*_1)$$

for some pairwise distinct $\bar{y}_0, \dots, \bar{y}_{n'-1}$, $\langle K_i : i < n' \rangle$, $\langle k_{i,j} : i < n', j < K_i \rangle$, $\langle m_i : i < n' \rangle$ and $\langle z_{0,i,j} : i, j \in \omega \rangle$, $\langle z_{1,i,j} : i, j \in \omega \rangle$ 2^ω -mutually Cohen generic over $M[\bar{y}_0, \dots, \bar{y}_{n'-1}]$. Letting $m = \max_{i < n'} m_i + 1$, we follow the R -independence of the set in $(*_1)$ from $(1)_{\tilde{R}, M, m}$. $\dashv$

CLAIM 3.17. If there is $m \in \omega$ so that $(1)_{\tilde{R}, M, m}$ fails, then there is a countable model $M^+ \supseteq M$ so that $(2)_{R, M^+, k}$.

PROOF. Let $m \geq 1$ be least so that $(2)_{\tilde{R}, M_0, m}$ for some countable model $M_0 \supseteq M$. We know that such m exists, since from $(1)_{\tilde{R}, M, 1}$ we follow that either $(1)_{\tilde{R}, M, 2}$ or $(2)_{\tilde{R}, M_0, 1}$ for some M_0 , then, if $(1)_{\tilde{R}, M, 2}$, either $(1)_{\tilde{R}, M, 3}$ or $(2)_{\tilde{R}, M_0, 2}$ for some M_0 , and so on. Let $\phi_0, \dots, \phi_{N-1}, m_0, \dots, m_{N-1} \leq m$ and $\bar{s} \in \bigotimes_{i < \beta} (2^{<\omega})$ witness $(2)_{\tilde{R}, M_0, m}$. Let M_1 be a countable elementary model such that $\phi_0, \dots, \phi_{N-1}, M_0 \in M_1$. Then we have that for any $\bar{y}$ that is Cohen generic in $(2^\omega)^\beta \cap [\bar{s}]$ over M_1 , in particular over M_0 , that

$$\{(\bar{y}, m)\} \cup \{(\phi_j(\bar{y}), m_j) : j < N\} \in \tilde{R},$$

i.e., there is $p \in \mathbb{Q}$, there are $K_i \in \omega$, $k_{i,0}, \dots, k_{i,K_i-1} < k$ for every $i \leq N$, so that

By extending $\bar{s}$, we can assume wlog that $p, \langle K_i : i \leq N \rangle, \langle k_{i,j} : i \leq N, j < K_i \rangle$ are the same for each $\bar{y} \in [\bar{s}]$ generic over M_1 , since $(*_2)$ can be forced over M_1 . Also, from the fact that ϕ_j is continuous for every $j < N$, that $\phi_j(\bar{y}) \neq \bar{y}$ for every $j < N$, and that $\phi_{j_0}(\bar{y}) \neq \phi_{j_1}(\bar{y})$ for every $j_0 < j_1 < N$, we can assume wlog that for any $\bar{y}_0, \bar{y}_1 \in [\bar{s}]$ and $j_0 < j_1 < N$,

$$\phi_{j_0}(\bar{y}_0) \neq \bar{y}_1 \text{ and } \phi_{j_0}(\bar{y}_0) \neq \phi_{j_1}(\bar{y}_1). \quad (*_3)$$

Let us force in a finite support product over M_1 continuous functions $\chi_{0,i,j}: (2^\omega)^\beta \rightarrow [p(0, i, j)]$ and $\chi_{1,i,j}: (2^\omega)^\beta \rightarrow [p(1, i, j)]$ for $i, j \in \omega$ and write $M^+ = M_1[\langle \chi_{0,i,j}, \chi_{1,i,j} : i, j \in \omega \rangle]$. For every $i < N$ and $j < K_i$ and $\bar{x} \in (2^\omega)^\alpha$, define

$$\phi_{0,i,j}(\bar{x}) := \phi_i(\bar{x} \restriction \beta) \wedge \chi_{0,i,j}(\phi_i(\bar{x} \restriction \beta)) \text{ and } k_{0,i,j} = k_{i,j}.$$

For every $i < N$ and $j < m_i$ and $\bar{x} \in (2^\omega)^\alpha$, define

$$\phi_{1,i,j}(\bar{x}) := \phi_i(\bar{x} \restriction \beta) \wedge \chi_{1,i,j}(\phi_i(\bar{x} \restriction \beta)) \text{ and } k_{1,i,j} = k.$$

For every $j < K_N$ and $\bar{x} \in (2^\omega)^\alpha$, define

$$\phi_{0,N,j}(\bar{x}) := \bar{x} \restriction \beta \wedge \chi_{0,N,j}(\bar{x} \restriction \beta) \text{ and } k_{0,N,j} = k_{N,j}.$$

At last, define for every $j < m - 1$ and $\bar{x} \in (2^\omega)^\alpha$,

$$\phi_{1,N,j}(\bar{x}) := \bar{x} \restriction \beta \wedge \chi_{1,N,j}(\bar{x} \restriction \beta) \text{ and } k_{1,N,j} = k.$$

Let $\bar{t} \in \bigotimes_{i < \alpha} 2^{<\alpha}$ be $\bar{s}$ with $p(1, N, m - 1)$ added in coordinate β . Now we have that for any $\bar{x} \in [\bar{t}]$ that is Cohen generic in $(2^\omega)^\alpha$ over M^+ ,

$$\{(\bar{x}, k)\} \cup \{(\phi_{0,i,j}(\bar{x}), k_{0,i,j}) : i \leq N, j < K_N\} \cup \{(\phi_{1,i,j}(\bar{x}), k_{1,i,j}) : i < N, j < m_i\} \\ \cup \{(\phi_{1,N,j}(\bar{x}), k_{1,N,j}) : j < m - 1\} \in R.$$

This follows from $(*)_2$ and applying Lemma 3.3 to see that the $\chi_{0,i,j}(\phi_i(\bar{x} \restriction \beta))$, $\chi_{1,i,j}(\phi_i(\bar{x} \restriction \beta))$, $\chi_{0,N,j}(\bar{x} \restriction \beta)$, $\chi_{1,N,j}(\bar{x} \restriction \beta)$, and $x(\beta)$ are mutually Cohen generic over $M_1[\bar{x} \restriction \beta]$. Moreover they correspond to the reals $z_{0,i,j}$, $z_{1,i,j}$ added by a $\mathbb{Q}$ -generic over $M_1[\bar{x} \restriction \beta]$, containing p in its generic filter. Also, remember that $(*)_2$ is absolute between models containing the relevant parameters, which $M_1[\bar{y}]$ is, with $\bar{y} = \bar{x} \restriction \beta$.

On the other hand, whenever $\bar{x}_0, \dots, \bar{x}_{n'-1} \in (2^\omega)^\alpha \cap [\bar{t}]$ are pairwise distinct and strongly mCg over M^+ , letting $\bar{y}_0, \dots, \bar{y}_{n'-1}$ enumerate $\{\bar{x}_i \restriction \beta : i < n\}$, we have that

$$\{(\bar{y}_i, m - 1) : i < n'\} \cup \{(\phi_j(\bar{y}_i), m_j) : i < n', j < N\} \text{ is } \tilde{R}\text{-independent.} \quad (*_4)$$

According to the definition of $\tilde{R}$, $(*_4)$ is saying, e.g., that whenever $A \cup B \subseteq (2^\omega)^\alpha$ is an arbitrary set of strongly mCg reals over M_1 , where $A \restriction \beta, B \restriction \beta \subseteq \{\bar{y}_i, \phi_j(\bar{y}_i) : i < n', j < N\}$ and in B , $\bar{y}_i$ is extended at most $m - 1$ many times and $\phi_j(\bar{y}_i)$ at most m_j many times for every $i < n'$, $j < N$, and, assuming for now that $k > 0$, if $f: A \rightarrow k$, then

$$\{(\bar{x}, f(\bar{x})) : \bar{x} \in A\} \cup (B \times \{k\}) \text{ is } R\text{-independent.}$$

As an example for such sets A and B , we have

$$A = \{\phi_{0,i,j}(\bar{x}_l) : l < n, i \leq N, j < K_i\} \cup \{\bar{x}_l : l < n'\} \text{ and}$$

$$B = \{\phi_{1,i,j}(\bar{x}_l) : l < n, i < N, j < m_i\} \cup \{\phi_{1,N,j}(\bar{x}_l) : l < n', j < m - 1\}.$$

Again, to see this we apply Lemma 3.3 to show that the relevant reals are mutually generic over the model $M_1[\bar{y}_0, \dots, \bar{y}_{n'-1}]$. Also, remember from the definition of $\phi_{1,i,j}$ for $i < N$ and $j < m_i$ that, if $\phi_i(\bar{x}_{l_0} \restriction \beta) = \phi_i(\bar{x}_{l_1} \restriction \beta)$, then also $\phi_{1,i,j}(\bar{x}_{l_0}) = \phi_{1,i,j}(\bar{x}_{l_1})$, for all $l_0, l_1 < n$. Equally, if $\bar{x}_{l_0} \restriction \beta = \bar{x}_{l_1} \restriction \beta$, then $\phi_{1,N,j}(\bar{x}_{l_0}) = \phi_{1,N,j}(\bar{x}_{l_1})$

for every $j < m - 1$. Use $(*_3)$ to note that $\{\bar{y}_i : i < n'\}$, $\{\phi_0(\bar{y}_i) : i < n'\}$, ..., $\{\phi_{N-1}(\bar{y}_i) : i < n'\}$ are pairwise disjoint. From this we can follow that indeed, each $\bar{y}_i$ is extended at most $m - 1$ many times in B and $\phi_j(\bar{y}_i)$ at most m_i many times. In total, we get that

$$\begin{aligned} & \{(\phi_{0,i,j}(\bar{x}_l), k_{0,i,j}) : l < n, i \leq N, j < K_i\} \cup \{(\bar{x}_l, k - 1) : l < n'\} \cup \\ & \{(\phi_{1,i,j}(\bar{x}_l), k) : l < n, i < N, j < m_i\} \cup \{(\phi_{1,N,j}(\bar{x}_l), k) : l < n', j < m - 1\} \\ & \text{is } R\text{-independent.} \end{aligned}$$

It is now easy to check that we have the witnesses required in the statement of $(2)_{R,M^+,k}$. For example, $\phi_{0,i,j}(\bar{x}) \neq \bar{x}$ when $i < N$ follows from $\phi_i(\bar{x}) \neq \bar{x}$. For the values $\phi_{0,N,j}(\bar{x})$ we simply have that $\chi_{0,N,j}(\bar{x} \upharpoonright \beta) \neq x(\beta)$, as the two values are mutually generic. Everything else is similar and consists only of a few case distinctions. Also, the continuity of the functions is clear.

If $k = 0$, then we can simply forget the set A above, since K_i must be 0 for every $i \leq N$. In this case we just get that

$$\begin{aligned} & \{(\phi_{1,i,j}(\bar{x}_l), k) : l < n, i < N, j < m_i\} \cup \{(\phi_{1,N,j}(\bar{x}_l), k) : l < n', j < m - 1\} \\ & \text{is } R\text{-independent,} \end{aligned}$$

which then yields $(2)_{R,M^+,k}$. $\dashv$

This finishes the successor step. Now assume that α is a limit ordinal. We fix some arbitrary tree $T \subseteq \omega^{<\omega}$ such that for every $t \in T$, $|\{n \in \omega : t \restriction n \in T\}| = \omega$ and for any branches $x \neq y \in [T]$, if $d = \min\{i \in \omega : x(i) \neq y(i)\}$ then $x(j) \neq y(j)$ for every $j \geq d$. We will use T only for notational purposes. For every sequence $\xi_0 < \dots < \xi_{k'} = \alpha$, we let $\mathbb{Q}_{\xi_0, \dots, \xi_{k'}} = \left(\prod_{l < k'} (\bigotimes_{i \in [\xi_l, \xi_{l+1}]} 2^{<\omega})^{<\omega}\right) \times (\bigotimes_{i \in [\xi_0, \alpha]} 2^{<\omega})^{<\omega}$. $\mathbb{Q}_{\xi_0, \dots, \xi_{k'}}$ adds, in the natural way, reals $\langle \bar{z}_{l,i}^0 : l < k', i \in \omega \rangle$ and $\langle \bar{z}_i^0 : i \in \omega \rangle$, where $\bar{z}_{l,i}^0 \in (2^\omega)^{[\xi_l, \xi_{l+1}]}$ and $\bar{z}_i^0 \in (2^\omega)^{[\xi_0, \alpha]}$ for every $l < k', i \in \omega$. Whenever $t \in T \cap \omega^{k'}$, we write $\bar{z}_t^0 = \bar{z}_{0,t(0)}^0 \cap \dots \cap \bar{z}_{k'-1,t(k'-1)}^0$. Note that for generic $\langle \bar{z}_{l,i}^0 : i \in \omega, l < k' \rangle$, the reals $\langle \bar{z}_t^0 : t \in T \cap \omega^{k'} \rangle$ are strongly $\langle 2^\omega : i \in [\xi_0, \alpha] \rangle$ -mCg.

Now, let us define for each $\xi < \alpha$ an analytic hypergraph R_ξ on $(2^\omega)^\xi \times 2$ so that $\{(\bar{y}_i^0, 0) : i < n_0\} \cup \{(\bar{y}_i^1, 1) : i < n_1\} \in R_\xi \cap [(2^\omega)^\xi \times 2]^{n_0+n_1}$, where $|\{(\bar{y}_i^0, 0) : i < n_0\}| = n_0$ and $|\{(\bar{y}_i^1, 1) : i < n_1\}| = n_1$, iff there are $\xi_0 = \xi < \dots < \xi_{k'} = \alpha$, $(p, q) \in \mathbb{Q}_{\xi_0, \dots, \xi_{k'}}$, $K_i \in \omega$, $k_{i,0}, \dots, k_{i,K_i-1} < k$ and distinct $t_{i,0}, \dots, t_{i,K_i-1} \in T \cap \omega^{k'}$ for every $i < n_0$, so that $t_{i_0,j_0}(0) \neq t_{i_1,j_1}(0)$ for all $i_0 < i_1 < n_0$ and $j_0 < K_{i_0}$, $j_1 < K_{i_1}$, and

$$(p, q) \Vdash_{\mathbb{Q}_\xi} \bigcup_{i < n_0} \{(\bar{y}_i^0 \cap \bar{z}_{k_{i,j}}^0, k_{i,j}) : j < K_i\} \cup \{(\bar{y}_i^1 \cap \bar{z}_i^1, k) : i < n_1\} \in R.$$

Note that each R_ξ can be defined within M . It should be clear, similar to the proof of Claim 3.15, that from $(1)_{R,M,k}$, we can show the following.

CLAIM 3.18. *For every $\xi < \alpha$, $(1)_{R_\xi, M, 1}$.*

CLAIM 3.19. *Assume that for every $\xi < \alpha$, $(1)_{R_\xi, M, 2}$. Then also $(1)_{R, M, k+1}$.*

PROOF. Let $\bar{x}_0^0, \dots, \bar{x}_{n_0-1}^0, \bar{x}_0^1, \dots, \bar{x}_{n_1-1}^1$ be pairwise distinct and strongly mCg over M and $k_0, \dots, k_{n_0-1} < k$. Then there is $\xi < \alpha$ large enough so that $\bar{x}_0^0 \restriction \xi, \dots, \bar{x}_{n_0-1}^0 \restriction \xi, \bar{x}_0^1 \restriction \xi, \dots, \bar{x}_{n_1-1}^1 \restriction \xi$ are pairwise distinct and in particular, $\bar{x}_0^0 \restriction [\xi, \alpha), \dots, \bar{x}_{n_0-1}^0 \restriction [\xi, \alpha), \bar{x}_0^1 \restriction [\xi, \alpha), \dots, \bar{x}_{n_1-1}^1 \restriction [\xi, \alpha)$ are pairwise different in every coordinate. Let $\xi_0 = \xi, \xi_1 = \alpha, K_i = 1$ for every $i < n_0$ and $t_{0,0}, \dots, t_{n_0-1,0} \in T \cap \omega^1$ pairwise distinct. Also, write $k_{0,0} = k_0, \dots, k_{n_0-1,0} = k_{n_0-1}$. Then, from $(1)_{R_\xi, M, 2}$, we have that

$$\mathbb{1} \Vdash_{\xi_0, \xi_1} \{((\bar{x}_i^0 \restriction \xi) \frown \bar{z}_{t_{i,0}}^0, k_{i,0}) : i < n_0\} \cup \{((\bar{x}_i^1 \restriction \xi) \frown \bar{z}_i^1, k) : i < n_1\}$$

is R -independent.

By absoluteness, this holds true in $M[\langle \bar{x}_i^0 \restriction \xi, \bar{x}_j^1 \restriction \xi : i < n_0, j < n_1 \rangle]$ and we find that

$$\{(\bar{x}_i^0, k_i) : i < n_0\} \cup \{(\bar{x}_i^1, k)\} \text{ is } R\text{-independent,}$$

as required. ⊥

CLAIM 3.20. *If there is $\xi < \alpha$ so that $(1)_{R_\xi, M, 2}$ fails, then there is a countable model $M^+ \supseteq M$ so that $(2)_{R, M^+, k}$.*

PROOF. If $(1)_{R_\xi, M, 2}$ fails, then there is a countable model $M_0 \supseteq M$ so that $(2)_{R_\xi, M, 1}$ holds true as witnessed by $\bar{s} \in \bigotimes_{i < \xi} 2^{<\omega}, \phi_0^0, \dots, \phi_{N_0-1}^0, \phi_0^1, \dots, \phi_{N_1-1}^1 : (2^\omega)^\xi \rightarrow (2^\omega)^\xi$ such that for any pairwise distinct $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\xi \cap [\bar{s}]$ that are strongly mCg over M_0 ,

$$\{(\bar{y}_i, 0) : i < n\} \cup \{(\phi_j^0(\bar{y}_i), 0) : i < n, j < N_0\} \cup \{(\phi_j^1(\bar{y}_i), 1) : i < n, j < N_1\} \quad (*5)$$

is R_ξ -independent, but

$$\{(\bar{y}_0, 1)\} \cup \{(\phi_j^0(\bar{y}_0), 0) : j < N_0\} \cup \{(\phi_j^1(\bar{y}_0), 1) : j < N_1\} \in R_\xi. \quad (*6)$$

As before, we may pick $M_1 \ni M_0$ elementary containing all relevant information, assume that $(*6)$ is witnessed by fixed $\xi_0 = \xi < \dots < \xi_{k'} = \alpha, (p, q) \in \mathbb{Q}_{\xi_0, \dots, \xi_{k'}}, K_0, \dots, K_{N_0-1}, k_{i,0}, \dots, k_{i, K_i-1}$, and $t_{i,0}, \dots, t_{i, K_i-1} \in T \cap \omega^{k'}$ for every $i < N_0$, so that for every generic $\bar{y}_0 \in (2^\omega)^\xi \cap [\bar{s}]$ over M_1 ,

$$(p, q) \Vdash_{\mathbb{Q}_\xi} \{(\bar{y}_0 \frown \bar{z}_{N_1}^1, k)\} \cup \bigcup_{i < N_0} \{(\phi_i^0(\bar{y}_0) \frown \bar{z}_{t_{i,j}}^0, k_{i,j}) : j < K_i\} \cup \{(\phi_j^1(\bar{y}_0) \frown \bar{z}_j^1, k) : j < N_1\} \in R. \quad (*7)$$

As before, we may also assume that $\bar{y}_0 \neq \phi_{i_0}^{j_0}(\bar{y}_0) \neq \phi_{i_1}^{j_1}(\bar{y}_1)$ for every $\bar{y}_0, \bar{y}_1 \in [\bar{s}]$ and $(j_0, i_1) \neq (j_1, i_1)$. We let $\bar{s}' = \bar{s} \frown q(N_1)$. Now we force continuous functions $\chi_{l,i}^0 : (2^\omega)^\xi \rightarrow (2^\omega)^{[\xi, \xi_{l+1})} \cap [p(l, i)]$ and $\chi_i^1 : (2^\omega)^\xi \rightarrow (2^\omega)^{[\xi, \alpha)} \cap [q(i)]$ over M_1 for every $i \in \omega, l < k'$ and we let $M^+ = M_1[\langle \chi_{l,i}^0, \chi_i^1 : i \in \omega, l < k' \rangle]$. Finally we let

$$\phi_{0,i,j}(\bar{x}) := \phi_i^0(\bar{x} \restriction \xi) \frown \chi_{0,t_{i,j}(0)}(\phi_i^0(\bar{x} \restriction \xi)) \frown \dots \frown \chi_{k'-1,t_{i,j}(k'-1)}(\phi_i^0(\bar{x} \restriction \xi))$$

for every $i < N_0$ and $j < K_i$, $\bar{x} \in (2^\omega)^\alpha$, and

$$\phi_{1,i}(\bar{x}) := \phi_i^1(\bar{x} \upharpoonright \xi) \frown \chi_{1,i}(\phi_i^1(\bar{x} \upharpoonright \xi))$$

for every $i < N_1$, $\bar{x} \in (2^\omega)^\alpha$.

We get from $(*_7)$, and, as usual, applying Lemma 3.3, that for any $\bar{x} \in (2^\omega)^\alpha \cap [\bar{s}']$ which is generic over M^+ ,

$$\{(\bar{x}, k)\} \cup \bigcup_{i < N_0} \{(\phi_{0,i,j}(\bar{x}), k_{i,j}) : j < K_i\} \cup \{(\phi_{1,i}(\bar{x}), k) : i < N_1\} \in R.$$

On the other hand, whenever $\bar{x}_0, \dots, \bar{x}_{n'-1} \in (2^\omega)^\alpha \cap [\bar{s}']$ are strongly mCg over M^+ , and letting $\bar{y}_0, \dots, \bar{y}_{n-1}$ enumerate $\{\bar{x}_i \upharpoonright \xi : i < n'\}$, knowing that the set in $(*_5)$ is R_ξ -independent, we get that

$$\{(\bar{x}_l, k-1) : l < n'\} \cup \bigcup_{i < N_0} \{(\phi_{0,i,j}(\bar{x}_l), k_{i,j}) : j < K_i, l < n'\} \cup \{(\phi_{1,i}(\bar{x}_l), k) : i < N_1, l < n'\} \text{ is } R\text{-independent,}$$

in case $k > 0$. To see this, we let $\eta_0 < \dots < \eta_{k''}$ be a partition refining $\xi_0 < \dots < \xi_{k'}$ witnessing the mCg of $\bar{x}_0 \upharpoonright [\xi, \alpha), \dots, \bar{x}_{n'-1} \upharpoonright [\xi, \alpha)$ and we find appropriate $u_{0,0}, \dots, u_{0,L_0-1}, \dots, u_{n-1,0}, \dots, u_{n-1,L_{n-1}-1} \in T \cap \omega^{k''}$ and $v_{i,j} \in T \cap \omega^{k''}$ for $i < N_0$, $j < K_i$ to interpret the above set in the form

$$\{(\bar{y}_l \frown \bar{z}_{u_{l,i}}^0, k-1) : l < n, i < L_i\} \cup \bigcup_{i < N_0} \{(\phi_i^0(\bar{y}_l) \frown \bar{z}_{v_{i,j}}^0, k_{i,j}) : i < N_0, j < K_i, l < n\} \\ \cup \{(\phi_i^1(\bar{y}_l) \frown \bar{z}_i^1, k) : i < N_1, l < n\},$$

for $\mathbb{Q}_{\eta_0, \dots, \eta_{k''-1}}$ -generic $\langle \bar{z}_{l,i}^0, \bar{z}_i^1 : l < k'', i \in \omega \rangle$ over $M_1[\bar{y}_0, \dots, \bar{y}_{n-1}]$. We leave the details to the reader. In case $k = 0$, all K_i are 0 and we get that

$$\{(\phi_{1,i}(\bar{x}_l), k) : i < N_1, l < n'\} \text{ is } R\text{-independent.}$$

Everything that remains, namely showing, e.g., that $\bar{x} \neq \phi_{1,i}(\bar{x})$ is clear. $\dashv$

As a final note, let us observe that the case $\alpha = 0$ is trivial, since $(2^\omega)^\alpha$ has only one element. $\dashv$

REMARK 3.21. If we replace “strong mCg” with “mCg” in the above lemma, then it already becomes false for $\alpha = \omega$. Namely consider the equivalence relation E on $(2^\omega)^\omega$, where $\bar{x} E \bar{y}$ if they eventually agree, i.e., if $\exists n \in \omega \forall m \geq n (x(n) = y(n))$.⁵ Then we can never be in case (1) since we can always find two distinct $\bar{x}$ and $\bar{y}$ that are mCg and $\bar{x} E \bar{y}$. On the other hand, in case (2) we get a continuous selector ϕ_0 for E (note that $N = 0$ is not possible). More precisely we have that for any $\bar{x}, \bar{y}$ that are mCg, $\bar{x} E \phi_0(\bar{x})$ and $\phi_0(\bar{x}) = \phi_0(\bar{y})$ iff $\bar{x} E \bar{y}$. But for arbitrary mCg $\bar{x}$ and $\bar{y}$ so that $\bar{x} \neg E \bar{y}$, we easily find a sequence $\langle \bar{x}_n : n \in \omega \rangle$ so that $\bar{x}$ and $\bar{x}_n$ are mCg and $\bar{x} E \bar{x}_n$, but $\bar{x}_n \upharpoonright n = \bar{y} \upharpoonright n$ for all n . In particular $\lim_{n \in \omega} \bar{x}_n = \bar{y}$. Then $\phi_0(\bar{y}) = \lim_{n \in \omega} \phi_0(\bar{x}_n) = \lim_{n \in \omega} \phi_0(\bar{x}) = \phi_0(\bar{x})$.

⁵This equivalence relation is usually called E_1 .

The proofs of Main Lemmas 3.4 and 3.14 can be generalized to E that is ω -universally Baire, in particular they also hold for coanalytic hypergraphs. The only assumptions on analytic sets that we used in the proofs are summarized below.

PROPOSITION 3.22. *Let Γ be a pointclass closed under countable unions, countable intersections, and continuous preimages and assume that for every $A \in \Gamma \cap \mathcal{P}(\omega^\omega)$, there are formulas φ, ψ (with parameters) in the language of set theory, such that for every countable elementary model M (with the relevant parameters) and G a generic over M for a finite support product of Cohen forcing,*

- (1) *for $x \in M[G] \cap \omega^\omega$, $M[G] \models \varphi(x)$ iff $x \in A$ and $M[G] \models \psi(x)$ iff $x \notin A$,*
- (2) *for $\dot{x} \in M[G]$ a $\mathbb{C}$ -name for a real, $p \in \mathbb{C}$, $M[G] \models "p \Vdash \varphi(\dot{x})"$ iff $p \Vdash \varphi(\dot{x})$,*
- (3) *for $\dot{y}$ a $\mathbb{C}$ -name for a real, $p \in \mathbb{C}$ and a continuous function $f: \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$, $\{x \in \omega^\omega: p \Vdash \varphi(f(x, \dot{y}))\} \in \Gamma$.*

Then Main Lemmas 3.4 and 3.14 hold, where “analytic” is replaced by Γ .

DEFINITION 3.23. For $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i < \alpha} X_i$, we define

$$\Delta(\bar{x}_0, \dots, \bar{x}_{n-1}) := \{\Delta_{\bar{x}_i, \bar{x}_j}: i \neq j < n\} \cup \{0, \alpha\},$$

where $\Delta_{\bar{x}_i, \bar{x}_j} := \min\{\xi < \alpha: x_i(\xi) \neq x_j(\xi)\}$ if this exists and $\Delta_{\bar{x}_i, \bar{x}_j} = \alpha$ if $\bar{x}_i = \bar{x}_j$.

REMARK 3.24. Whenever $\bar{x}_0, \dots, \bar{x}_{n-1}$ are strongly mCg, then they are mCg as witnessed by the partition $\xi_0 < \dots < \xi_k$, where $\{\xi_0, \dots, \xi_k\} = \Delta(\bar{x}_0, \dots, \bar{x}_{n-1})$.

§4. Sacks and splitting forcing.

4.1. Splitting forcing.

DEFINITION 4.1. We say that $S \subseteq 2^{<\omega}$ is *fat* if there is $m \in \omega$ so that for all $n \geq m$, there are $s, t \in S$ so that $s(n) = 0$ and $t(n) = 1$. A tree T on 2 is called *splitting tree* if for every $s \in T$, T_s is fat. We call *splitting forcing* the tree forcing $\mathbb{SP}$ consisting of splitting trees.

Note that for $T \in \mathbb{SP}$ and $s \in T$, T_s is again a splitting tree. Recall that $x \in 2^\omega$ is called *splitting over V* , if for every $y \in 2^\omega \cap V$, $\{n \in \omega: y(n) = x(n) = 1\}$ and $\{n \in \omega: x(n) = 1 \wedge y(n) = 0\}$ are infinite. The following is easy to see.

FACT. *Let G be $\mathbb{SP}$ -generic over V . Then x_G , the generic real added by $\mathbb{SP}$, is splitting over V .*

Whenever S is fat let us write $m(S)$ for the minimal $m \in \omega$ witnessing this.

DEFINITION 4.2. Let S, T be splitting trees and $n \in \omega$. Then we write $S \leq_n T$ iff $S \leq T$, $\text{split}_{\leq n}(S) = \text{split}_{\leq n}(T)$ and $\forall s \in \text{split}_{\leq n}(S) (m(S_s) = m(T_s))$.

PROPOSITION 4.3. *The sequence $\langle \leq_n: n \in \omega \rangle$ witnesses that $\mathbb{SP}$ has Axiom A with continuous reading of names.*

PROOF. It is clear that $\leq_n$ is a partial order refining $\leq$ and that $\leq_{n+1} \subseteq \leq_n$ for every $n \in \omega$. Let $\langle T_n: n \in \omega \rangle$ be a fusion sequence in $\mathbb{SP}$, i.e., for every n , $T_{n+1} \leq_n T_n$. Then we claim that $T := \bigcap_{n \in \omega} T_n$ is a splitting tree. More precisely, for $s \in T$, we claim that $m := m((T_{|s|})_s)$ witnesses that T_s is fat. To see this, let $n \geq m$ be arbitrary and note that $n \geq m \geq |s|$ must be the case. Then, since $\text{split}_{\leq n+1}(T_{n+1}) \subseteq T$ we

have that $s \in \text{split}_{\leq n+1}(T_{n+1})$ and $m((T_{n+1})_s) = m$. So find $t_0, t_1 \in T_{n+1}$ so that $t_0(n) = 0$, $t_1(n) = 1$ and $|t_0| = |t_1| = n + 1$. But then $t_0, t_1 \in T$, because $t_0, t_1 \in \text{split}_{\leq n+1}(T_{n+1}) \subseteq T$.

Now let $D \subseteq \mathbb{SP}$ be open dense, $T \in \mathbb{SP}$ and $n \in \omega$. We will show that there is $S \leq_n T$ so that for every $x \in [S]$, there is $t \subseteq x$, with $S_t \in D$. This implies condition (3) in Definition 2.1.

CLAIM 4.4. *Let S be a splitting tree. Then there is $A \subseteq S$ an antichain (seen as a subset of $2^{<\omega}$) so that for every $k \in \omega$, $j \in 2$, if $\exists s \in S(s(k) = j)$, then $\exists t \in A(t(k) = j)$.*

PROOF. Start with $\{s_i : i \in \omega\} \subseteq S$ an arbitrary infinite antichain and let $m_i := m(S_{s_i})$ for every $i \in \omega$. Then find for each $i \in \omega$, a finite set $H_i \subseteq S_{s_i}$ so that for all $k \in [m_i, m_{i+1})$, there are $t_0, t_1 \in H_i$, so that $t_0(k) = 0$ and $t_1(k) = 1$. Moreover let $H \subseteq S$ be finite so that for all $k \in [0, m_0)$ and $j \in 2$, if $\exists s \in S(s(k) = j)$, then $\exists t \in H(t(k) = j)$. Then define $F_i = H_i \cup (H \cap S_{s_i})$ for each $i \in \omega$ and let $F_{-1} := H \setminus \bigcup_{i \in \omega} F_i$. Since F_i is finite for every $i \in \omega$, it is easy to extend each of its elements to get a set F'_i that is additionally an antichain in S_{s_i} . Also extend the elements of F_{-1} to get an antichain F'_{-1} in S . It is easy to see that $A := \bigcup_{i \in [-1, \omega)} F'_i$ works. $\dashv$

Now enumerate $\text{split}_n(T)$ as $\langle \sigma_i : i < N \rangle$, $N := 2^n$. For each $i < N$, let $A_i \subseteq T_{\sigma_i}$ be an antichain as in the claim applied to $S = T_{\sigma_i}$. For every $i < N$ and $t \in A_i$, let $S^t \in D$ be so that $S^t \leq T_t$. For every $i < N$ pick $t_i \in A_i$ arbitrarily and $F_i \subseteq A_i$ a finite set so that for every $k \in [0, m(S^{t_i}))$ and $j \in 2$, if $\exists s \in A_i(s(k) = j)$, then $\exists t \in F_i(t(k) = j)$. Then we see that $S := \bigcup_{i < N} (\bigcup_{t \in F_i} S^t \cup S^{t_i})$ works. We constructed S so that $S \leq_n T$. Moreover, whenever $x \in [S]$, then there is $i < N$ be so that $\sigma_i \subseteq x$. Then $x \in [\bigcup_{t \in F_i} S_t \cup S_{t_i}]$ and since F_i is finite, there is $t \in F_i \cup \{t_i\}$ so that $t \subseteq x$. But then $S_t \leq S^t \in D$.

Finally, in order to show the continuous reading of names, let $\dot{y}$ be a name for an element of ω^ω , $n \in \omega$, and $T \in \mathbb{SP}$. It suffices to consider such names, since for every Polish space X , there is a continuous surjection $F : \omega^\omega \rightarrow X$. Then we have that for each $i \in \omega$, $D_i := \{S \in \mathbb{SP} : \exists s \in \omega^i (S \Vdash \dot{y} \restriction i = s)\}$ is dense open. Let $\langle T_i : i \in \omega \rangle$ be so that $T_0 \leq_n T$, $T_{i+1} \leq_{n+i} T_i$ and for every $x \in [T_i]$, there is $t \subseteq x$ so that $(T_i)_t \in D_i$. Then $S = \bigcap_{i \in \omega} T_i \leq_n T$. For every $x \in [S]$, define $f(x) = \bigcup \{s \in \omega^{<\omega} : \exists t \subseteq x (S_t \Vdash s \subseteq \dot{y})\}$. Then $f : [S] \rightarrow \omega^\omega$ is continuous and $S \Vdash \dot{y} = f(x_G)$. $\dashv$

COROLLARY 4.5. $\mathbb{SP}$ is proper and ω^ω -bounding.

4.2. Weighted tree forcing. In this subsection we define a class of forcings that we call *weighted tree forcing*. The definition is slightly ad-hoc, but simple enough to formulate and to check for various forcing notions. In earlier versions of this paper we proved many of the results only for splitting forcing, but we noted that similar combinatorial arguments apply more generally. The notion of a *weight* resulted directly from analysing the proof of Proposition 4.16 for splitting forcing. It also turned out to be particularly helpful in the next subsection.

DEFINITION 4.6. Let T be a perfect tree. A *weight* on T is a map $\rho : T \times T \rightarrow [T]^{<\omega}$ so that $\rho(s, t) \subseteq T_s \setminus T_t$ for all $s, t \in T$. Whenever ρ_0, ρ_1 are weights on T we write $\rho_0 \subseteq \rho_1$ to say that for all $s, t \in T$, $\rho_0(s, t) \subseteq \rho_1(s, t)$.

Note that if $t \subseteq s$ then $\rho(s, t) = \emptyset$ must be the case.

DEFINITION 4.7. Let T be a perfect tree, ρ a weight on T , and S a tree. Then we write $S \leq_\rho T$ if $S \subseteq T$ and there is a dense set of $s_0 \in S$ with an injective sequence $(s_n)_{n \in \omega}$ in S_{s_0} such that $\forall n \in \omega (\rho(s_n, s_{n+1}) \subseteq S)$.

REMARK 4.8. Whenever $\rho_0 \subseteq \rho_1$, we have that $S \leq_{\rho_1} T$ implies $S \leq_{\rho_0} T$.

DEFINITION 4.9. Let $\mathbb{P}$ be a tree forcing. Then we say that $\mathbb{P}$ is *weighted* if for any $T \in \mathbb{P}$ there is a weight ρ on T so that for any tree S , if $S \leq_\rho T$ then $S \in \mathbb{P}$.

LEMMA 4.10. $\mathbb{SP}$ is weighted.

PROOF. Let $T \in \mathbb{SP}$. For any $s, t \in T$ let $\rho(s, t) \subseteq T_s \setminus T_t$ be finite so that for any $k \in \omega$ and $i \in 2$, if there is $r \in T_s$ so that $r(k) = i$ and there is no such $r \in T_t$, then there is such r in $\rho(s, t)$. This is possible since T_t is fat. Let us show that ρ works. Assume that $S \leq_\rho T$ and let $s \in S$ be arbitrary. Then there is $s_0 \supseteq s$ in S with a sequence $(s_n)_{n \in \omega}$ as in the definition of $\leq_\rho$. Let $k \geq m(T_{s_0})$ and $i \in 2$ and suppose there is no $r \in S_{s_0}$ with $r(k) = i$. In particular this means that no such r is in $\rho(s_n, s_{n+1})$ for any $n \in \omega$, since $\rho(s_n, s_{n+1}) \subseteq S_{s_0}$. But then, using the definition of ρ and $m(T_{s_0})$, we see inductively that for each $n \in \omega$ such r must be found in T_{s_n} . Letting n large enough so that $k < |s_n|$, $s_n(k) = i$ must be the case. But $s_n \in S_{s_0}$, which is a contradiction. $\dashv$

DEFINITION 4.11. *Sacks forcing* is the tree forcing $\mathbb{S}$ consisting of all perfect subtrees of $2^{<\omega}$. It is well-known that it is Axiom A with continuous reading of names.

LEMMA 4.12. $\mathbb{S}$ is weighted.

PROOF. Let $T \in \mathbb{S}$. For $s, t \in T$, we let $\rho(s, t)$ contain all $r \frown i \in T_s \setminus T_t$ such that $r \frown (1 - i) \in T$ and where $|r|$ is minimal with this property. $\dashv$

Recall that for finite trees T_0, T_1 we say that T_1 is an end-extension of T_0 , written as $T_0 \sqsubset T_1$, if $T_0 \subsetneq T_1$ and for every $t \in T_1 \setminus T_0$ there is a terminal node $\sigma \in \text{term}(T_0)$ so that $\sigma \subseteq t$. A node $\sigma \in T_0$ is called terminal if it has no proper extension in T_0 .

DEFINITION 4.13. Let T be a perfect tree, ρ a weight on T , and T_0, T_1 finite subtrees of T . Then we write $T_0 \triangleleft_\rho T_1$ iff $T_0 \sqsubset T_1$ and

$$\forall \sigma \in \text{term}(T_0) \exists N \geq 2 \exists \langle s_i \rangle_{i < N} \in ((T_1)_\sigma)^N \text{ injective} \\ (s_0 = \sigma \wedge s_{N-1} \in \text{term}(T_1) \wedge \forall i < N (\rho(s_i, s_{i+1}) \subseteq T_1)). \quad (*)$$

LEMMA 4.14. Let T be a perfect tree, ρ be a weight on T , and $\langle T_n : n \in \omega \rangle$ be a sequence of finite subtrees of T so that $T_n \triangleleft_\rho T_{n+1}$ for every $n \in \omega$. Then $\bigcup_{n \in \omega} T_n \leq_\rho T$.

PROOF. Let $S := \bigcup_{n \in \omega} T_n$. To see that $S \leq_\rho T$ note that $\bigcup_{n \in \omega} \text{term}(T_n)$ is dense in S , in a very strong sense. Let $\sigma \in \text{term}(T_n)$ for some $n \in \omega$, then let $s_0, \dots, s_{N_0-1}$ be as in $(*)$ for T_n, T_{n+1} . Since $s_{N_0-1} \in \text{term}(T_{n+1})$ we again find $s_{N_0}, \dots, s_{N_1-1}$ so that $s_{N_0-1}, \dots, s_{N_1-1}$ is as in $(*)$ for T_{n+1}, T_{n+2} . Continuing like this, we find a sequence $\langle s_i : i \in \omega \rangle$ in S starting with $s_0 = \sigma$ so that $\rho(s_i, s_{i+1}) \subseteq S$ for all $i \in \omega$, as required. $\dashv$

LEMMA 4.15. *Let T be a perfect tree, p a weight on T , and T_0 a finite subtree of T . Moreover, let $k \in \omega$ and $D \subseteq (T)^k$ be dense open. Then there is $T_1 \triangleright_p T_0$ so that*

$$\begin{aligned} \forall \{\sigma_0, \dots, \sigma_{k-1}\} \in [\text{term}(T_0)]^k \forall \sigma'_0, \dots, \sigma'_{k-1} \in \text{term}(T_1) \\ (\forall l < k (\sigma_l \subseteq \sigma'_l) \rightarrow (\sigma'_0, \dots, \sigma'_{k-1}) \in D). \end{aligned} \quad (*_1)$$

PROOF. First let us enumerate $\text{term}(T_0)$ by $\sigma_0, \dots, \sigma_{K-1}$. We put $s_0^l = \sigma_l$ for each $l < K$. Next find for each $l < K$, $s_1^l \in T$, $s_0^l \subsetneq s_1^l$ above a splitting node in $T_{s_0^l}$. Moreover we find $s_2^l \in T_{s_0^l}$ so that $s_2^l \perp s_1^l$ and s_2^l is longer than any node appearing in $\rho(s_0^l, s_1^l)$. This is possible since we chose s_1^l to be above a splitting node in $T_{s_0^l}$. For each $l < K$ we let $\tilde{T}_2^l$ be the tree generated by (i.e., the downwards closure of) $\{s_1^l, s_2^l\} \cup \rho(s_0^l, s_1^l) \cup \rho(s_1^l, s_2^l)$. Note that $s_2^l \in \text{term}(\tilde{T}_2^l)$ as $\rho(s_1^l, s_2^l) \perp s_2^l$.

Let us enumerate by $(f_j)_{2 \leq j < N}$ all functions $f: K \rightarrow \{1, 2\}$ starting with f_2 the constant function mapping to 1. We are going to construct recursively a sequence $\langle \tilde{T}_j^l : 2 \leq j \leq N \rangle$ where $\tilde{T}_j^l \subseteq \tilde{T}_{j+1}^l$, and $\langle s_j^l : 2 \leq j \leq N \rangle$ without repetitions, for each $l < K$ such that at any step $j < N$:

- (1) For every $l < K$, $s_j^l \in \text{term}(\tilde{T}_j^l)$ and $\begin{cases} s_2^l \subseteq s_j^l, & \text{if } f_j(l) = 1, \\ s_1^l \subseteq s_j^l, & \text{if } f_j(l) = 2. \end{cases}$
- (2) For any $\{l_i : i < k\} \in [K]^k$ and $(t_i)_{i < k}$ where $t_i \in \text{term}(\tilde{T}_{j+1}^{l_i})$ and $\begin{cases} s_1^{l_i} \subseteq t_i, & \text{if } f_j(l_i) = 1, \\ s_1^{l_i} \perp t_i, & \text{if } f_j(l_i) = 2, \end{cases}$ for every $i < k$, $(t_0, \dots, t_{k-1}) \in D$.
- (3) For every $l < K$, $\rho(s_j^l, s_{j+1}^l) \subseteq \tilde{T}_{j+1}^l$.

Note that (1) holds true at the initial step $j = 2$ since $f_2(l) = 1$, $s_2^l \subseteq s_2^l$, and $s_2^l \in \text{term}(\tilde{T}_2^l)$ for each $l < K$. Now suppose that for some $j < N$ we have constructed $\tilde{T}_j^l$ and s_j^l for each l with (1) holding true. Then we proceed as follows. Let $\{t_i^l : i < N_l\}$ enumerate $\{t : t \in \text{term}(\tilde{T}_j^l) \wedge s_1^l \subseteq t \text{ if } f_j(l) = 1 \wedge s_1^l \perp t \text{ if } f_j(l) = 2\}$ for each $l < K$. Now it is simple to find $r_i^l \in T$, $t_i^l \subseteq r_i^l$ for each $i < N_l$, $l < K$ so that $\{r_i^l : i < N_l, l < K\}^k \subseteq D$.

Let R_l be the tree generated by $\tilde{T}_j^l$ and $\{r_i^l : i < N_l\}$ for each $l < K$. It is easy to see that $\tilde{T}_j^l \subseteq R_l$ since we only extended elements from $\text{term}(\tilde{T}_j^l)$ (namely the t_i^l 's). Note that it is still the case that $s_j^l \in \text{term}(R_l)$ since $s_j^l \perp t_i^l$ for all $i < N_l$. Next we choose s_{j+1}^l extending an element of $\text{term}(R_l)$, distinct from all previous choices and so that $s_2^l \subseteq s_{j+1}^l$ if $f_{j+1}(l) = 1$ and $s_1^l \subseteq s_{j+1}^l$ if $f_{j+1}(l) = 2$.

Taking $\tilde{T}_{j+1}^l$ to be the tree generated by $R_l \cup \{s_{j+1}^l\} \cup \rho(s_j^l, s_{j+1}^l)$ gives the next step of the construction. Again $R_l \subseteq \tilde{T}_{j+1}^l$, as we only extended terminal nodes of R_l . Then (3) obviously holds true and $s_{j+1}^l \in \text{term}(\tilde{T}_{j+1}^l)$ since $\rho(s_j^l, s_{j+1}^l) \perp s_{j+1}^l$. It follows from the construction that (2) holds true for each $\tilde{T}_{j+1}^l$ replaced by R_l . Since $R_l \subseteq \tilde{T}_{j+1}^l$ we easily see that (2) is satisfied.

Finally we put $T_1 = \bigcup_{l < K} \tilde{T}_N^l$. It is clear that $(*_0)$ is true, in particular that $T_0 \triangleleft_p T_1$. For $(*_1)$ let $\{l_i : i < k\} \in [K]^k$ be arbitrary and assume that $t_i \in \text{term}(\tilde{T}_N^{l_i})$

for each $i < k$. Let $f : K \rightarrow \{1, 2\}$ be so that for each $i < k$ if $s_1^{l_i} \subseteq t_i$ then $f(l_i) = 1$, and if $s_1^{l_i} \perp t_i$ then $f(l_i) = 2$. Then there is $j \in [2, N)$ so that $f_j = f$. Clause (2) ensured that for initial segments $t'_i \subseteq t_i$ where $t'_i \in \text{term}(\tilde{T}_{j+1}^{l_i})$, $(t'_0, \dots, t'_{k-1}) \in D$. In particular $(t_0, \dots, t_{k-1}) \in D$ which proves $(*)_1$. $\dashv$

PROPOSITION 4.16. *Let M be a countable model of set theory, $R_l \in M$ a perfect tree, and ρ_l a weight on R_l for every $l < k \in \omega$. Then there is $S_l \leq_{\rho_l} R_l$ for every $l < k$ so that any $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{l < k} [S_l]$ are $\langle [R_l] : l < k \rangle$ -mutually Cohen generic over M .*

PROOF. Let $T := \{\emptyset\} \cup \{\langle l \rangle \frown s : s \in R_l, l < k\}$ be the disjoint sum of the trees R_l for $l < k$. Also let ρ be a weight on T extending arbitrarily the weights ρ_l defined on the copy of R_l in T . As M is countable, let $(D_n, k_n)_{n \in \omega}$ enumerate all pairs $(D, m) \in M$, such that D is a dense open subset of T^m and $m \in \omega \setminus \{0\}$, infinitely often. Let us find a sequence $(T_n)_{n \in \omega}$ of finite subtrees of T , such that for each $n \in \omega$, $T_n \triangleleft_\rho T_{n+1}$ and

$$\begin{aligned} \forall \{\sigma_0, \dots, \sigma_{k_n-1}\} \in [\text{term}(T_n)]^{k_n} \forall \sigma'_0, \dots, \sigma'_{k_n-1} \in \text{term}(T_{n+1}) \\ (\forall l < k (\sigma_l \subseteq \sigma'_l) \rightarrow (\sigma'_0, \dots, \sigma'_{k_n-1}) \in D_n). \end{aligned} \quad (*_1)$$

We start with $T_0 = k^{<2} = \{\emptyset\} \cup \{\langle l \rangle : l < k\}$ and then apply Lemma 4.15 recursively. Let $S := \bigcup_{n \in \omega} T_n$. Then we have that $S \leq_\rho T$.

CLAIM 4.17. *For any $m \in \omega$ and distinct $x_0, \dots, x_{m-1} \in [S]$, $(x_0, \dots, x_{m-1})$ is T^m -generic over M .*

PROOF. Let $D \subseteq T^m$ be open dense with $D \in M$. Then there is a large enough $n \in \omega$ with $(D_n, k_n) = (D, m)$ and $\sigma_0, \dots, \sigma_{m-1} \in \text{term}(T_n)$ distinct such that $\sigma_0 \subseteq x_0, \dots, \sigma_{m-1} \subseteq x_{m-1}$. Then there are unique $\sigma'_0, \dots, \sigma'_{m-1} \in \text{term}(T_{n+1})$ such that $\sigma'_0 \subseteq x_0, \dots, \sigma'_{m-1} \subseteq x_{m-1}$. By $(*)_1$, $(\sigma'_0, \dots, \sigma'_{m-1}) \in D$. $\dashv$

Finally let $S_l = \{s : \langle l \rangle \frown s \in S\}$ and note that $S_l \leq_{\rho_l} R_l$ for every $l < k$. The above claim clearly implies the statement of the proposition.

REMARK 4.18. Proposition 4.16 implies directly the main result of [30]. A modification of the above construction for splitting forcing can be used to show that for $T \in M$, we can in fact find a master condition $S \leq T$ so that for any distinct $x_0, \dots, x_{n-1} \in [S]$, $(x_0, \dots, x_{n-1})$ is $\mathbb{SP}^n$ -generic over M . In that case $(S, \dots, S) \in \mathbb{SP}^n$ is an $\mathbb{SP}^n$ -master condition over M . We won't provide a proof of this since our only application is Corollary 4.20, which seems to be implicit in [30]. The analogous statement for Sacks forcing is a standard fusion argument.

COROLLARY 4.19. *Let $\mathbb{P}$ be a weighted tree forcing with continuous reading of names. Then $\mathbb{P}$ adds a minimal real. In fact for any $\mathbb{P}$ -generic G , if $y \in 2^\omega \cap V[G] \setminus V$, then there is a continuous map $f : 2^\omega \rightarrow A^\omega$ in V so that $x_G = f(y)$.*

PROOF. Using the continuous reading of names let $T \in G$ be so that there is a continuous map $g : [T] \rightarrow 2^\omega$ with $T \Vdash \dot{y} = g(x_G)$. It is easy to see from the definition, that in any weighted tree forcing, the set of finitely branching trees is dense.⁶ Thus, let us assume that $[T]$ is compact. Moreover let M be countable

⁶In particular, weighted tree forcing with the crn is ω^ω -bounding.

elementary with $g, T \in M$. Now let $S \leq T$ be so that any $x_0, x_1 \in [S]$ are $[T]$ -mCg over M .

Suppose that there are $x_0 \neq x_1 \in [S]$, with $g(x_0) = g(x_1)$. Then there must be $s \subseteq x_0$ and $t \subseteq x_1$, so that $M \models (s, t) \Vdash_{T^2} g(\dot{c}_0) = g(\dot{c}_1)$, where $\dot{c}_0, \dot{c}_1$ are names for the generic branches added by T^2 . But then note that for any $x \in S_t$, since x and x_0 are mCg and $s \subseteq x_0, t \subseteq x$, we have that $g(x) = g(x_0)$. In particular g is constant on S_t and $S_t \Vdash g(x_G) = g(\dot{x}_0) \in V$.

On the other hand, if g is injective on $[S]$, then g^{-1} is continuous as $[S]$ is compact and it is easy to extend g^{-1} to a continuous function $f: A^\omega \rightarrow 2^\omega$. $\dashv$

COROLLARY 4.20. *$V^{\mathbb{SP}}$ is a minimal extension of V , i.e., whenever W is a model of ZFC so that $V \subseteq W \subseteq V^{\mathbb{SP}}$, then $W = V$ or $W = V^{\mathbb{SP}}$.*

PROOF. Let G be an $\mathbb{SP}$ -generic filter over V . It suffices to show that if $\langle \alpha_\xi : \xi < \delta \rangle \in W \setminus V$ is an increasing sequence of ordinals, then $x_G \in W$ (see also [18, Theorem 13.28]). So let $\langle \dot{\alpha}_\xi : \xi < \delta \rangle$ be a name for such a sequence of ordinals and $T \in \mathbb{SP}$ be such that $T \Vdash \langle \dot{\alpha}_\xi : \xi < \delta \rangle \notin V$. Note that this is in fact equivalent to saying that $(T, T) \Vdash_{\mathbb{SP}^2} \langle \dot{\alpha}_\xi[\dot{x}_0] : \xi < \delta \rangle \neq \langle \dot{\alpha}_\xi[\dot{x}_1] : \xi < \delta \rangle$, where $\dot{x}_0, \dot{x}_1$ are names for the generic reals added by $\mathbb{SP}^2$. Let M be a countable elementary model so that $T, \langle \dot{\alpha}_\xi : \xi < \delta \rangle \in M$ and let $T' \leq T$ be a master condition over M as in Remark 4.18. Then also $T' \Vdash \langle \dot{\alpha}_\xi : \xi \in \delta \cap M \rangle \notin V$. Namely, suppose towards a contradiction that there are $x_0, x_1 \in [T']$ generic over V so that $\langle \dot{\alpha}_\xi[x_0] : \xi \in \delta \cap M \rangle = \langle \dot{\alpha}_\xi[x_1] : \xi \in \delta \cap M \rangle$, then (x_0, x_1) is $\mathbb{SP}^2$ -generic over M and $M[x_0][x_1] \models \langle \dot{\alpha}_\xi[x_0] : \xi < \delta \rangle = \langle \dot{\alpha}_\xi[x_1] : \xi < \delta \rangle$ which yields a contradiction to the sufficient elementarity of M . Since $T' \Vdash \langle \dot{\alpha}_\xi : \xi \in \delta \cap M \rangle \subseteq M$ we can view $\langle \dot{\alpha}_\xi : \xi \in \delta \cap M \rangle$ as a name for a real, for M is countable. Back in W , we can define $\langle \alpha_\xi : \xi \in \delta \cap M \rangle$ since $M \in V \subseteq W$. But then, applying Corollary 4.19, we find that $x_G \in W$. $\dashv$

4.3. The countable support iteration. Recall that for any perfect subtree T of $2^{<\omega}$, $\text{split}(T)$ is order-isomorphic to $2^{<\omega}$ in a canonical way, via a map $\eta_T: \text{split}(T) \rightarrow 2^{<\omega}$. This map induces a homeomorphism $\tilde{\eta}_T: [T] \rightarrow 2^\omega$ and note that the value of $\tilde{\eta}_T(x)$ depends continuously on T and x . Whenever ρ is a weight on T , η_T also induces a weight $\tilde{\rho}$ on $2^{<\omega}$, so that whenever $S \leq_{\tilde{\rho}} 2^{<\omega}$, then $\eta_T^{-1}(S)$ generates a tree S' with $S' \leq_{\rho} T$.

Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \lambda \rangle$ be a countable support iteration where for each $\beta < \lambda$, $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta \in \{\mathbb{SP}, \mathbb{S}\}$. We fix in this section a $\mathbb{P}_\lambda$ name $\dot{y}$ for an element of a Polish space X , a good master condition $\bar{p} \in \mathbb{P}_\lambda$ over a countable model M_0 , where $\dot{y}, X \in M_0$, and let $C \subseteq \lambda$ be a countable set as in Lemma 2.3. For every $\beta \in C$ and $\bar{y} \in [\bar{p}] \upharpoonright (C \cap \beta)$, let us write

$$T_{\bar{y}} = \{s \in 2^{<\omega} : \exists \bar{x} \in [\bar{p}] [\bar{x} \upharpoonright (C \cap \beta) = \bar{y} \wedge s \subseteq x(\beta)]\}.$$

According to Lemma 2.3, the map $\bar{y} \mapsto T_{\bar{y}}$ is a continuous function from $[\bar{p}] \upharpoonright (C \cap \beta)$ to $\mathcal{T}$. Let $\alpha := \text{otp}(C) < \omega_1$ as witnessed by an order-isomorphism $\iota: \alpha \rightarrow C$. Then we define the homeomorphism $\Phi: [\bar{p}] \upharpoonright C \rightarrow (2^\omega)^\alpha$ so that for every $\bar{y} \in [\bar{p}] \upharpoonright C$ and every $\delta < \alpha$,

$$\Phi(\bar{y}) \upharpoonright (\delta + 1) = \Phi(\bar{y}) \upharpoonright \delta \smallfrown \tilde{\eta}_{T_{\bar{y} \upharpoonright \iota(\delta)}}(y(\iota(\delta))).$$

Note that for $\mathbb{P} \in \{\mathbb{SP}, \mathbb{S}\}$, the map sending $T \in \mathbb{P}$ to the weight ρ_T defined in Lemma 4.10 or Lemma 4.12 is a Borel function from $\mathbb{P}$ to the Polish space of partial functions from $(2^{<\omega})^2$ to $[2^{<\omega}]^{<\omega}$. Thus for $\beta \in C$ and $\bar{x} \in [\bar{p}] \upharpoonright (C \cap \beta)$, letting $\rho_{\bar{x}} := \rho_{T_{\bar{x}}}$, we get that $\bar{x} \mapsto \rho_{\bar{x}}$ is a Borel function on $[\bar{p}] \upharpoonright (C \cap \beta)$. For each $\delta < \alpha$ and $\bar{y} \in (2^\omega)^\delta$, we may then define $\tilde{\rho}_{\bar{y}}$ a weight on $2^{<\omega}$, induced by $\rho_{\bar{x}}$ and $\eta_{T_{\bar{x}}}$, where $\bar{x} = \Phi^{-1}(\bar{y} \smallfrown \bar{z}) \upharpoonright \beta$ for arbitrary, equivalently for every, $\bar{z} \in (2^\omega)^{\alpha \setminus \delta}$. The map sending $\bar{y} \in (2^\omega)^\delta$ to $\tilde{\rho}_{\bar{y}}$ is then Borel as well.

LEMMA 4.21. *Let M_1 be a countable elementary model with $M_0, \bar{p}, \mathbb{P}_\lambda \in M_1$ and let $\bar{s} \in \bigotimes_{i < \alpha} 2^{<\omega}$. Then there is $\bar{q} \leq \bar{p}$, a good master condition over M_0 , so that*

$$\forall \bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{q}](\Phi(\bar{x}_0 \upharpoonright C), \dots, \Phi(\bar{x}_{n-1} \upharpoonright C) \in (2^\omega)^\alpha \cap [\bar{s}])$$

are strongly $\langle 2^\omega : i < \alpha \rangle$ -mCg over M_1).

Moreover $[\bar{q}] \upharpoonright C$ is a closed subset of $[\bar{p}] \upharpoonright C$ and $[\bar{q}] = ([\bar{q}] \upharpoonright C) \times (2^\omega)^{\lambda \setminus C}$ (cf. Lemma 2.3).

PROOF. We can assume without loss of generality that $\bar{s} = \emptyset$, i.e., $[\bar{s}] = (2^\omega)^\alpha$. It will be obvious that this assumption is inessential. Next, let us introduce some notation. For any $\delta \leq \alpha$ and $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\delta$, recall from Definition 3.23 that

$$\Delta(\bar{y}_0, \dots, \bar{y}_{n-1}) := \{\Delta_{\bar{y}_i, \bar{y}_j} : i \neq j < n\} \cup \{0, \delta\}.$$

Let us write

$$\text{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) := (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle),$$

where $\{\xi_0 < \dots < \xi_k\} = \Delta(\bar{y}_0, \dots, \bar{y}_{n-1})$, $K_l = |\{\bar{y}_i \upharpoonright [\xi_l, \xi_{l+1}) : i < n\}|$ for every $l < k$ and $\langle U_i : i < n \rangle$ are the clopen subsets of $(2^\omega)^\delta$ of the form $U_i = [\bar{s}_i]$ for $\bar{s}_i \in \bigotimes_{\xi < \delta} 2^{<\omega}$ with $\text{dom}(\bar{s}_i) = \{\Delta_{\bar{y}_i, \bar{y}_j} : j < n, \bar{y}_j \neq \bar{y}_i\}$ and $\bar{s}_i$ minimal in the order of $\bigotimes_{\xi < \delta} 2^{<\omega}$ so that

$$\bar{y}_i \in [\bar{s}_i] \text{ and } \forall j < n (\bar{y}_j \neq \bar{y}_i \rightarrow \bar{y}_j \notin [\bar{s}_i]),$$

for every $i < n$.

Note that for any $\delta_0 \leq \delta$, if

$$\text{tp}(\bar{y}_0 \upharpoonright \delta_0, \dots, \bar{y}_{n-1} \upharpoonright \delta_0) := (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle V_i : i < n \rangle),$$

then $V_i = U_i \upharpoonright \delta_0$ for every $i < n$. Moreover, for any $\bar{y}'_0, \dots, \bar{y}'_{n-1} \in (2^\omega)^\delta$ with

$$\text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle),$$

we have that

$$\text{tp}(\bar{y}'_0 \upharpoonright \delta_0, \dots, \bar{y}'_{n-1} \upharpoonright \delta_0) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle V_i : i < n \rangle).$$

Any $\bar{y}_0, \dots, \bar{y}_{n-1}$, with $\text{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) := (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$, that are $\langle 2^\omega : i < \delta \rangle$ -mutually Cohen generic over M_1 as witnessed by $\xi_0 < \dots < \xi_k$, induce a $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ -generic and vice versa. Thus whenever τ is a $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ -name, we may write $\tau[\bar{y}_0, \dots, \bar{y}_{n-1}]$ for the evaluation of τ via the induced generic. It will not matter in what particular way we define the $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ -generic from given $\bar{y}_0, \dots, \bar{y}_{n-1}$. We may stipulate for

instance, that the generic induced by $\bar{y}_0, \dots, \bar{y}_{n-1}$ is $\langle \bar{z}_{l,j} : l < k, j < K_l \rangle$, where for each fixed $l < k$, $\langle \bar{z}_{l,j} : l < k, j < K_l \rangle$ enumerates $\{\bar{y}_i \restriction [\xi_l, \xi_{l+1}) : i < n\}$ in lexicographic order.

Let us get to the bulk of the proof. We will define a finite support iteration $\langle \mathbb{R}_\delta, \dot{S}_\delta : \delta \leq \alpha \rangle$ in M_1 , together with, for each $\delta \leq \alpha$, an $\mathbb{R}_\delta$ -name $\dot{X}_\delta$ for a closed subspace of $(2^\omega)^\delta$, where $\Vdash_{\mathbb{R}_{\delta_1}} \dot{X}_{\delta_0} = \dot{X}_{\delta_1} \restriction \delta_0$ for every $\delta_0 < \delta_1 \leq \alpha$. This uniquely determines the limit steps of the construction. Additionally we will make the following inductive assumptions $(1)_\delta$ and $(2)_\delta$ for all $\delta \leq \alpha$ and any $\mathbb{R}_\delta$ -generic G . Let $\bar{y}_0, \dots, \bar{y}_{n-1} \in \dot{X}_\delta[G]$ be arbitrary and $\text{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$. Then

- $(1)_\delta$ $\bar{y}_0, \dots, \bar{y}_{n-1}$ are strongly $\langle 2^\omega : i < \delta \rangle$ -mCg over M_1 and
- $(2)_\delta$ for any $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ -name $\dot{D} \in M_1$ for an open dense subset of a countable poset $\mathbb{Q} \in M_1$,

$$\bigcap \left\{ \dot{D}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta, \right. \\ \left. \text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \right\}$$

is open dense in $\mathbb{Q}$.

Having defined $\mathbb{R}_\delta$ and $\dot{X}_\delta$, for $\delta < \alpha$, we proceed as follows. Fix for now G an $\mathbb{R}_\delta$ -generic over M_1 and $X_\delta := \dot{X}_\delta[G]$. Then we define a forcing $\mathbb{S}_\delta \in M_1[G]$ which generically adds a continuous map $F : X_\delta \rightarrow \mathcal{T}$, so that for each $\bar{y} \in X_\delta$, $S_{\bar{y}} := F(\bar{y}) \leq_{\tilde{\rho}_{\bar{y}}} 2^{<\omega}$. In $M_1[G][F]$, we then define $X_{\delta+1} \subseteq (2^\omega)^{\delta+1}$ to be $\{\bar{y} \frown z : \bar{y} \in X_\delta, z \in [S_{\bar{y}}]\}$. The definition of $\mathbb{S}_\delta$ is as follows.

Work in $M_1[G]$. Since the map $\bar{y} \in (2^\omega)^\delta \mapsto \tilde{\rho}_{\bar{y}}$ is Borel and an element of M_1 and by $(1)_\delta$ any $\bar{y} \in X_\delta$ is Cohen generic over M_1 , it is continuous on X_δ . Since X_δ is compact we find a single weight $\tilde{\rho}$ on $2^{<\omega}$, so that $\tilde{\rho}_{\bar{y}} \subseteq \tilde{\rho}$ for every $\bar{y} \in X_\delta$. Let $\{O_s : s \in 2^{<\omega}\}$ be a basis of X_δ so that $O_s \subseteq O_t$ for $t \subseteq s$ and $O_s \cap O_t = \emptyset$ for $s \perp t$. This is possible since X_δ is homeomorphic to 2^ω . Let $\mathcal{FT}$ be the set of finite subtrees of $2^{<\omega}$. Then $\mathbb{S}_\delta$ consists of functions $h : 2^{\leq n} \rightarrow \mathcal{FT}$, for some $n \in \omega$, so that for every $s \subseteq t \in 2^{\leq n}$, $(h(s) \leq_{\tilde{\rho}} h(t))$. The extension relation is defined by function extension. Note that $\mathbb{S}_\delta$ is indeed a forcing poset with trivial condition $\emptyset$.

Given H , an $\mathbb{S}_\delta$ -generic over $M_1[G]$, we let $F : X_\delta \rightarrow \mathcal{T}$ be defined as

$$F(\bar{y}) := \bigcup_{\substack{s \in 2^{<\omega}, \bar{y} \in O_s \\ h \in H}} h(s).$$

CLAIM 4.22. *For every $\bar{y} \in X_\delta$, $F(\bar{y}) = S_{\bar{y}} \leq_{\tilde{\rho}} 2^{<\omega}$, in particular $S_{\bar{y}} \leq_{\tilde{\rho}_{\bar{y}}} 2^{<\omega}$. For any $\bar{y}_0, \bar{y}_1 \in X_\delta$, $[S_{\bar{y}_0}] \cap [S_{\bar{y}_1}] \neq \emptyset$. Any $z_0, \dots, z_{n-1} \in \bigcup_{\bar{y} \in X_\delta} [S_{\bar{y}}]$ are 2^ω -mutually Cohen generic over $M_1[G]$. And for any countable poset $\mathbb{Q} \in M_1$, any $m \in \omega$ and any dense open $E \subseteq (2^{<\omega})^n \times \mathbb{Q}$ in $M_1[G]$, there is $r \in \mathbb{Q}$ and $m_0 \geq m$ so that for any $z_0, \dots, z_{n-1} \in \bigcup_{\bar{y} \in X_\delta} [S_{\bar{y}}]$ where $z_0 \restriction m, \dots, z_{n-1} \restriction m$ are pairwise distinct, $((z_0 \restriction m_0, \dots, z_{n-1} \restriction m_0), r) \in E$.*

PROOF. We will make a genericity argument over $M_1[G]$. Let $h \in \mathbb{S}_\delta$ be arbitrary. Then it is easy to find $h' \leq h$, say with $\text{dom}(h') = 2^{\leq a_0}$, so that for every $s \in 2^{a_0}$ and every $t \in \text{term}(h(s))$, $|t| \geq m$. For the first claim, it suffices

through Lemma 4.14 to find $h'' \leq h'$, say with $\text{dom}(h'') = 2^{\leq a_1}$, $a_0 < a_1$, so that for every $s \in 2^{a_0}$ and $t \in 2^{a_1}$, with $s \subseteq t$, $h''(s) \triangleleft_{\bar{p}} h''(t)$. Finding h'' so that additionally $\text{term}(h''(t_0)) \cap \text{term}(h''(t_1)) = \emptyset$ for every $t_0 \neq t_1 \in 2^{a_1}$ proves the second claim. For the last two claims, given a fixed dense open subset $E \subseteq (2^{<\omega})^n \times \mathbb{Q}$ in $M_1[G]$, it suffices to find $r \in \mathbb{Q}$ and to ensure that for any pairwise distinct $s_0, \dots, s_{n-1} \in \bigcup_{s \in 2^{a_0}} \text{term}(h''(s))$ and $t_0 \supseteq s_0, \dots, t_{n-1} \supseteq s_{n-1}$ with $t_0, \dots, t_{n-1} \in \bigcup_{t \in 2^{a_1}} \text{term}(h''(t))$, $((t_0, \dots, t_{n-1}), r) \in E$. Then we may put $m_0 = \max\{|t| : t \in \bigcup_{s \in 2^{a_1}} \text{term}(h''(s))\}$. We may also assume wlog that $\mathbb{Q} = 2^{<\omega}$.

To find such h'' we apply Lemma 4.15 as in the proof of Proposition 4.16. More precisely, for every $s \in 2^{a_0}$, we find $T_s^0, T_s^1 \triangleright_{\bar{p}} h'(s)$, and we find $T \subseteq 2^{<\omega}$ finite, so that for any pairwise distinct $s_0, \dots, s_{n-1} \in \bigcup_{s \in 2^{a_0}} \text{term}(h'(s))$, any $t_0 \supseteq s_0, \dots, t_{n-1} \supseteq s_{n-1}$ with $t_0, \dots, t_{n-1} \in \bigcup_{s \in 2^{a_0}, i \in 2} \text{term}(T_s^i)$ and any $\sigma \in \text{term}(T)$, $((t_0, \dots, t_{n-1}), \sigma) \in E$ and $\text{term}(T_s^i) \cap \text{term}(T_t^j) = \emptyset$ for every $i, j \in 2$, $s, t \in 2^{a_0}$. Then simply define $h'' \leq h'$ with $\text{dom}(h'') = 2^{a_0+1}$, where $h''(s \cap i) = T_s^i$ for $s \in 2^{a_0}$, $i \in 2$. $\dashv$

The function F is obviously continuous and $X_{\delta+1}$ is a closed subset of $(2^\omega)^{\delta+1}$, with $X_{\delta+1} \upharpoonright \delta_0 = (X_{\delta+1} \upharpoonright \delta) \upharpoonright \delta_0 = X_\delta \upharpoonright \delta_0 = X_{\delta_0}$ for every $\delta_0 < \delta + 1$.

PROOF OF (1) $_{\delta+1}$ AND (2) $_{\delta+1}$. Let G be $\mathbb{R}_{\delta+1}$ generic over M_1 and $\bar{y}_0, \dots, \bar{y}_{n-1} \in \dot{X}_{\delta+1}[G] = X_{\delta+1}$ be arbitrary. By the inductive assumption we have that $\bar{y}_0 \upharpoonright \delta, \dots, \bar{y}_{n-1} \upharpoonright \delta$ are strongly $\langle 2^\omega : i < \delta \rangle$ -mCg over M_1 . By the above claim, whenever $\bar{y}_i \upharpoonright \delta \neq \bar{y}_j \upharpoonright \delta$, then $\bar{y}_i(\delta) \neq \bar{y}_j(\delta)$. Thus, for (1) $_{\delta+1}$, we only need to show that $\bar{y}_0, \dots, \bar{y}_{n-1}$ are mCg. Let $\text{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) = (\langle \xi_l : l < k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$, $\text{tp}(\bar{y}_0 \upharpoonright \delta, \dots, \bar{y}_{n-1} \upharpoonright \delta) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle U_i \upharpoonright \delta : i < n \rangle)$, and $n' = |\{y_i(\delta) : i < n\}| = K_{k-1}$. Then we may view a dense open subset of $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1}]} 2^{<\omega})^{K_l}$ as a $\prod_{l < k'} (\bigotimes_{\xi \in [\eta_l, \eta_{l+1}]} 2^{<\omega})^{M_l}$ -name for a dense open subset of $(2^{<\omega})^{n'}$. To this end, let $\dot{D} \in M_1$ be a $\prod_{l < k'} (\bigotimes_{\xi \in [\eta_l, \eta_{l+1}]} 2^{<\omega})^{M_l}$ name for a dense open subset of $(2^{<\omega})^{n'}$. Then we have, by (2) $_\delta$, that

$$\begin{aligned} \tilde{D} = \bigcap \Big\{ & \dot{D}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta, \\ & \text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle U_i \upharpoonright \delta : i < n \rangle) \Big\} \end{aligned}$$

is a dense open subset of $(2^{<\omega})^{n'}$ and $\tilde{D} \in M_1[G \upharpoonright \delta]$. By the above claim, $y_0(\delta), \dots, y_{n-1}(\delta)$ are mCg over $M_1[G \upharpoonright \delta]$ in 2^ω . Altogether, this shows that $\bar{y}_0, \dots, \bar{y}_{n-1}$ are $\langle 2^\omega : i < \delta + 1 \rangle$ -mCg over M_1 .

For (2) $_{\delta+1}$, let $\dot{D} \in M_1$ now be a $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1}]} 2^{<\omega})^{K_l}$ -name for a dense open subset of $\mathbb{Q}$. Consider a name $\dot{E}$ in M_1 for the dense open subset of $(2^{<\omega})^{n'} \times \mathbb{Q}$, where for any $\bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta$, with $\text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle U_i \upharpoonright \delta : i < n \rangle)$,

$$\begin{aligned} \dot{E}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] = \{ & (\bar{t}, r) : M_1[\bar{y}'_0, \dots, \bar{y}'_{n-1}] \models \\ & \bar{t} \Vdash r \in \dot{D}[\bar{y}'_0, \dots, \bar{y}'_{n-1}][\dot{z}_0, \dots, \dot{z}_{n'-1}] \}, \end{aligned}$$

where $(\dot{z}_0, \dots, \dot{z}_{n'-1})$ is a name for the $(2^{<\omega})^{n'}$ -generic. By $(2)_\delta$, we have that

$$\tilde{E} = \bigcap \left\{ \dot{E}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta, \right. \\ \left. \text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k \rangle, \langle U_i \restriction \delta : i < n \rangle) \right\}$$

is a dense open subset of $(2^{<\omega})^{n'} \times \mathbb{Q}$ and $\tilde{E} \in M_1[G \restriction \delta]$. Let $m \in \omega$ be large enough so that for any $i, j < n$, if $U_i \neq U_j$, then $\forall \bar{y}'_i \in U_i \cap X_{\delta+1}, \bar{y}'_j \in U_j \cap X_{\delta+1} (\bar{y}'_i(\delta) \restriction m \neq \bar{y}'_j(\delta) \restriction m)$. To see that such m exists, note that if $U_i \neq U_j$, then $U_i \cap X_{\delta+1}$ and $U_j \cap X_{\delta+1}$ are disjoint compact subsets of $X_{\delta+1}$. By the claim, there is $r \in \mathbb{Q}$ and $m_0 \geq m$ so that for any $z_0, \dots, z_{n'-1} \in \bigcup_{\bar{y} \in X_\delta} [S_{\bar{y}}]$, if $z_0 \restriction m, \dots, z_{n'-1} \restriction m$ are pairwise different, then $((z_0 \restriction m_0, \dots, z_{n'-1} \restriction m_0), r) \in \tilde{E}$. Altogether we find that

$$r \in \bigcap \left\{ \dot{D}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_{\delta+1}, \right. \\ \left. \text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \right\}.$$

Of course the same argument can be carried out below any condition in $\mathbb{Q}$, showing that this set is dense. That it is open is also clear since it is the intersection of open subsets of a partial order. $\dashv$

Now let $\delta \leq \alpha$ be a limit ordinal.

PROOF OF $(1)_\delta$ AND $(2)_\delta$. Let G be $\mathbb{R}_\delta$ -generic over M_1 , $\bar{y}_0, \dots, \bar{y}_{n-1} \in \dot{X}_\delta[G] = X_\delta$, this time wlog pairwise distinct, and $\text{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$. We will make a genericity argument over M_1 to show $(1)_\delta$ and $(2)_\delta$. To this end, let $D_0 \subseteq \prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ be dense open, $D_0 \in M_1$, and let $\dot{D}_1 \in M_1$ be a $\prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ -name for a dense open subset of $\mathbb{Q}$. Then consider the dense open subset $D_2 \subseteq \prod_{l < k} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l} \times \mathbb{Q}$ in M_1 , where

$$D_2 = \{(r_0, r_1) : r_0 \in D_0 \wedge r_0 \Vdash r_1 \in \dot{D}_1\}.$$

Also let $\bar{h}_0 \in G$ be an arbitrary condition so that

$$\bar{h}_0 \Vdash \forall i < n (U_i \cap \dot{X}_\delta \neq \emptyset).$$

Then there is $\delta_0 < \delta$ so that $\text{supp}(\bar{h}_0), \xi_{k-1} + 1 \subseteq \delta_0$. We may equally well view D_2 as a $\prod_{l < k-1} (\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l} \times (\bigotimes_{\xi \in [\xi_{k-1}, \delta_0)} 2^{<\omega})^{K_{k-1}}$ -name $\dot{E} \in M_1$ for a dense open subset

$$E \subseteq \left(\bigotimes_{\xi \in [\delta_0, \xi_k)} 2^{<\omega} \right)^{K_{k-1}} \times \mathbb{Q} = \left(\bigotimes_{\xi \in [\delta_0, \delta)} 2^{<\omega} \right)^n \times \mathbb{Q}.$$

We follow again from $(2)_{\delta_0}$ that the set $\tilde{E} \in M_1[G \restriction \mathbb{R}_{\delta_0}]$, where

$$\tilde{E} = \bigcap \left\{ \dot{E}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_{\delta_0}, \right. \\ \left. \text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_0 < \dots < \xi_{k-1} < \delta_0 \rangle, \langle K_l : l < k \rangle, \langle U_i \restriction \delta_0 : i < n \rangle) \right\}$$

is dense open. Let $((\bar{t}_0, \dots, \bar{t}_{n-1}), r) \in \tilde{E}$ be arbitrary and $\bar{h}_1 \in G \cap \mathbb{R}_{\delta_0}$, $\bar{h}_1 \leq \bar{h}_0$, so that $\bar{h}_1 \Vdash ((\bar{t}_0, \dots, \bar{t}_{n-1}), r) \in \tilde{E}$.

Let us show by induction on $\xi \in [\delta_0, \delta)$, $\xi > \sup(\bigcup_{i < n} \text{dom}(\bar{t}_i))$, that there is a condition $\bar{h}_2 \in \mathbb{R}_\xi$, $\bar{h}_2 \leq \bar{h}_1$, so that

$$\begin{aligned} \bar{h}_2 \Vdash \forall \bar{y}'_0, \dots, \bar{y}'_{n-1} \in \dot{X}_\delta (\text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \\ \rightarrow \bar{y}'_0 \in [\bar{t}_0] \wedge \dots \wedge \bar{y}'_{n-1} \in [\bar{t}_{n-1}]) \end{aligned}$$

and in particular, if $\bar{h}_2 \in G$, then for all $\bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta$ with $\text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$, the generic corresponding to $\bar{y}'_0, \dots, \bar{y}'_{n-1}$ hits D_0 , and $r \in \dot{D}_1[\bar{y}'_0, \dots, \bar{y}'_{n-1}]$. Since $\bar{h}_0 \in G$ was arbitrary, genericity finishes the argument.

The limit step of the induction follows directly from the earlier steps since if $\text{dom}(\bar{t}_i) \subseteq \xi$, with ξ limit, then there is $\eta < \xi$ so that $\text{dom}(\bar{t}_i) \subseteq \eta$. So let us consider step $\xi + 1$. Then there is, by the inductive assumption, $\bar{h}'_2 \in \mathbb{R}_\xi$, $\bar{h}'_2 \leq \bar{h}_1$, so that

$$\begin{aligned} \bar{h}'_2 \Vdash \forall \bar{y}'_0, \dots, \bar{y}'_{n-1} \in \dot{X}_\delta (\text{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \\ \rightarrow (\bar{y}'_0 \in [\bar{t}_0 \restriction \xi] \wedge \dots \wedge \bar{y}'_{n-1} \in [\bar{t}_{n-1} \restriction \xi])). \end{aligned}$$

Now extend $\bar{h}'_2$ to $\bar{h}''_2$ in $\mathbb{R}_\xi$, so that there is $m \in \omega$ such that for every $s \in 2^m$ and every $i < n$, either $\bar{h}''_2 \Vdash \dot{O}_s \subseteq U_i \restriction \xi$ or $\bar{h}''_2 \Vdash \dot{O}_s \cap (U_i \restriction \xi) = \emptyset$, where $\langle \dot{O}_s : s \in 2^{<\omega} \rangle$ is a name for the base of $\dot{X}_\xi$ used to define $\dot{S}_\xi$. The reason why this is possible is that in any extension by $\mathbb{R}_\xi$ and for every $i < n$, by compactness of $X_\xi \cap (U_i \restriction \xi)$, there is a finite set $a \subseteq 2^{<\omega}$ so that $X_\xi \cap (U_i \restriction \xi) = \bigcup_{s \in a} O_s$. Let us define $h : 2^{\leq m} \rightarrow \mathcal{FT}$, where

$$h(s) = \begin{cases} \emptyset, & \text{if } \forall i < n (\bar{h}''_2 \Vdash \dot{O}_s \cap U_i \restriction \xi = \emptyset), \\ \{t \in 2^{<\omega} : t \subseteq t_i(\xi)\}, & \text{if } \bar{h}''_2 \Vdash \dot{O}_s \subseteq U_i \restriction \xi \text{ and } i < n. \end{cases}$$

Note that h is well-defined as $(U_i \restriction \xi) \cap (U_j \restriction \xi) = \emptyset$ for every $i \neq j < n$. Since $\emptyset \trianglelefteq_\rho T$ and $T \trianglelefteq_\rho T$ for any weight ρ and any finite tree T , we have that $\bar{h}''_2 \Vdash h \in \dot{S}_\xi$ and $\bar{h}_2 = \bar{h}''_2 \hat{\smallfrown} h \in \mathbb{R}_{\xi+1}$ is as required. $\dashv$

This finishes the definition of $\mathbb{R}_\alpha$ and $\dot{X}_\alpha$. Finally let G be $\mathbb{R}_\alpha$ -generic over M_1 and $X_\alpha = \dot{X}_\alpha[G]$. Now let us define $\bar{q} \leq \bar{p}$ recursively so that for every $\delta \leq \alpha$,

$$\forall \bar{x} \in [\bar{q}] (\Phi(\bar{x} \restriction C) \restriction \delta \in X_\alpha \restriction \delta).$$

If $\beta \notin C$ we let $\dot{q}(\beta)$ be a name for the trivial condition $2^{<\omega}$, say, e.g., $\dot{q}(\beta) = \dot{p}(\beta)$. If $\beta \in C$, say $\beta = \iota(\delta)$, we define $\dot{q}(\beta)$ to be a name for the tree generated by

$$\eta_{T_{\bar{x}_G \restriction (C \cap \beta)}}^{-1}(S_{\bar{y}}),$$

where $\bar{x}_G$ is the generic sequence added by $\mathbb{P}_\lambda$ and $\bar{y} = \Phi(\bar{x}_G \restriction C) \restriction \delta$. This ensures that $\bar{q} \restriction \beta \Vdash \dot{q}(\beta) \in \mathbb{Q}_\beta \wedge \dot{q}(\beta) \leq \dot{p}(\beta)$. Inductively we see that $\bar{q} \restriction \beta \hat{\smallfrown} \bar{p} \restriction (\lambda \setminus \beta) \Vdash \Phi(\bar{x}_G \restriction C) \restriction \delta \in X_\alpha \restriction \delta$. Having defined $\bar{q}$, it is also easy to check that it is a good master condition over M_0 , with $[\bar{q}] = \Phi^{-1}(X_\alpha) \times (2^\omega)^{\lambda \setminus C}$. Since for every $\bar{x} \in [\bar{q}]$, $\Phi(\bar{x} \restriction C) \in X_\alpha$ and by $(1)_\alpha$, $\bar{q}$ is as required. $\dashv$

PROPOSITION 4.23. *Let $E \subseteq [X]^{<\omega}$ be an analytic hypergraph on X , say E is the projection of a closed set $F \subseteq [X]^{<\omega} \times \omega^\omega$, and let $f: [\bar{p}] \restriction C \rightarrow X$ be continuous so that $\bar{p} \Vdash \dot{y} = f(\bar{x}_G \restriction C)$ (cf. Lemma 2.3). Then there is a good master condition $\bar{q} \leq \bar{p}$, with $[\bar{q}] \restriction C$ a closed subset of $[\bar{p}] \restriction C$ and $[\bar{q}] = ([\bar{q}] \restriction C) \times (2^\omega)^{\lambda \setminus C}$, a compact E -independent set $Y \subseteq X$, $N \in \omega$ and continuous functions $\phi: [\bar{q}] \restriction C \rightarrow [Y]^{<N}$, $w: [\bar{q}] \restriction C \rightarrow \omega^\omega$, so that*

- (i) *either $f''([\bar{q}] \restriction C) \subseteq Y$, thus $\bar{q} \Vdash \dot{y} \in Y$,*
- (ii) *or $\forall \bar{x} \in [\bar{q}] \restriction C ((f(\bar{x}) \cup \{f(\bar{x})\}, w(\bar{x})) \in F)$, thus $\bar{q} \Vdash \{\dot{y}\} \cup Y$ is not E -independent.*

PROOF. On $(2^\omega)^\alpha$ let us define the analytic hypergraph $\tilde{E}$, where

$$\{\bar{y}_0, \dots, \bar{y}_{n-1}\} \in \tilde{E} \leftrightarrow \{f(\Phi^{-1}(\bar{y}_0)), \dots, f(\Phi^{-1}(\bar{y}_{n-1}))\} \in E.$$

By Main Lemma 3.14, there is a countable model M and $\bar{s} \in \bigotimes_{i < \alpha} 2^{<\omega}$ so that either

- (1) *for any $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly $\langle 2^\omega : i < \alpha \rangle$ -mCg over M , $\{\bar{y}_0, \dots, \bar{y}_{n-1}\}$ is E -independent,*

or for some $N \in \omega$,

- (2) *there are $\phi_0, \dots, \phi_{N-1}: (2^\omega)^\alpha \rightarrow (2^\omega)^\alpha$ continuous so that for any $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg over M , $\{\phi_j(\bar{y}_i) : j < N, i < n\}$ is E -independent but $\{\bar{y}_0\} \cup \{\phi_j(\bar{y}_0) : j < N\} \in E$.*

Let M_1 be a countable elementary model with $M_0, M, \bar{p}, \mathbb{P}_\lambda \in M_1$ and apply Lemma 4.21 to get the condition $\bar{q} \leq \bar{p}$. In case (1), let $Y := f''([\bar{q}] \restriction C)$. Then (i) is satisfied. To see that Y is E -independent let $\bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{q}]$ be arbitrary and suppose that $\{f(\bar{x}_0 \restriction C), \dots, f(\bar{x}_{n-1} \restriction C)\} \in E$. By definition of $\tilde{E}$ this implies that $\{\Phi(\bar{x}_0 \restriction C), \dots, \Phi(\bar{x}_{n-1} \restriction C)\} \in \tilde{E}$ but this is a contradiction to (1) and the conclusion of Lemma 4.21. In case (2), by elementarity, the ϕ_j are in M_1 and there is a continuous function $\tilde{w} \in M_1$, with domain some dense G_δ subset of $(2^\omega)^\alpha$, so that $\bar{s} \Vdash \{f(\bar{z}), \phi_j(\bar{z}) : j < N\}, \tilde{w}(\bar{z}) \in F$, where $\bar{z}$ is a name for the Cohen generic. Let $\phi(\bar{x}) = \{f(\Phi^{-1}(\phi_j(\Phi(\bar{x})))) : j < N\}$, $w(\bar{x}) = \tilde{w}(\Phi(\bar{x}))$ for $\bar{x} \in [\bar{q}] \restriction C$ and $Y := \bigcup_{\bar{x} \in [\bar{q}] \restriction C} \phi(\bar{x})$. Since $\Phi(\bar{x})$ is generic over M_1 , we indeed have that $(\phi(\bar{x}), w(\bar{x})) \in F$ for every $\bar{x} \in [\bar{q}] \restriction C$. Seeing that Y is E -independent is as before. $\dashv$

§5. Main results and applications.

THEOREM 5.1. ($V=L$) *Let $\mathbb{P}$ be a countable support iteration of Sacks or splitting forcing of arbitrary length. Let X be a Polish space and $E \subseteq [X]^{<\omega}$ be an analytic hypergraph. Then there is a Δ_2^1 maximal E -independent set in $V^\mathbb{P}$. If $X = 2^\omega$ or $X = \omega^\omega$, $r \in 2^\omega$, and $E \in \Sigma_1^1(r)$, then we can find a $\Delta_2^1(r)$ such set.*

PROOF. We will only concentrate on the case $X = 2^\omega$ since the rest follows easily from the fact that there is a Borel isomorphism from 2^ω to any uncountable Polish space X , and if $X = \omega^\omega$ that isomorphism is (lightface) Δ_1^1 . If X is countable, then the statement is trivial. Also, let us only consider splitting forcing. The proof for Sacks forcing is the same.

First let us mention some well-known facts and introduce some notation. Recall that a set $Y \subseteq 2^\omega$ is $\Sigma_2^1(x)$ -definable if and only if it is $\Sigma_1(x)$ -definable over $H(\omega_1)$ (see, e.g., [18, Lemma 25.25]). Also recall that there is a Σ_1^1 set $A \subseteq 2^\omega \times 2^\omega$ that is universal for analytic sets, i.e., for every analytic $B \subseteq 2^\omega$, there is some $x \in 2^\omega$ so that $B = A_x$, where $A_x = \{y \in 2^\omega : (x, y) \in A\}$. In the same way, there is a universal Π_1^0 set $F \subseteq 2^\omega \times [2^\omega]^{<\omega} \times \omega^\omega$ [19, Theorems 22.3 and 26.1]. For any $x \in 2^\omega$, let E_x be the analytic hypergraph on 2^ω consisting of $a \in [2^\omega]^{<\omega} \setminus \{\emptyset\}$ so that there is $b \in [A_x]^{<\omega}$ with $a \cup b \in E$. Then there is $y \in 2^\omega$ so that E_x is the projection of F_y . Moreover, it is standard to note, from the way A and F are defined, that for every x , $y = e(x, r)$ for some fixed recursive function e . Whenever $\alpha < \omega_1$ and $Z \subseteq (2^\omega)^\alpha$ is closed, it can be coded naturally by the set $S \subseteq \bigotimes_{i < \alpha} 2^{<\omega}$, where

$$S = \{(\bar{x} \upharpoonright a) \upharpoonright n : \bar{x} \in Z, a \in [\alpha]^{<\omega}, n \in \omega\}$$

and we write $Z = Z_S$. Similarly, any continuous function $f : Z \rightarrow \omega^\omega$ can be coded by a function $\zeta : S \rightarrow \omega^{<\omega}$, where

$$f(\bar{x}) = \bigcup_{\bar{s} \in S, \bar{x} \in [\bar{s}]} \zeta(\bar{s})$$

and we write $f = f_\zeta$. For any $\beta < \alpha$ and $\bar{x} \in Z \upharpoonright \beta$, let us write $T_{\bar{x}, Z} = \{s \in 2^{<\omega} : \exists \bar{z} \in Z (\bar{z} \upharpoonright \delta = \bar{x} \wedge s \subseteq z(\delta))\}$. The set Ψ_0 of pairs (α, S) , where S codes a closed set $Z \subseteq (2^\omega)^\alpha$ so that for every $\beta < \alpha$ and $\bar{x} \in Z \upharpoonright \beta$, $T_{\bar{x}, Z} \in \mathbb{SP}$ is then Δ_1 over $H(\omega_1)$. This follows since the set of such S is Π_1^1 , seen as a subset of $\mathcal{P}(\bigotimes_{i < \alpha} 2^{<\omega})$, uniformly on α . Similarly, the set Ψ_1 of triples (α, S, ζ) , where $(\alpha, S) \in \Psi_0$ and ζ codes a continuous function $f : Z_S \rightarrow \omega^\omega$, is Δ_1 .

Now let $\langle \alpha_\xi, S_\xi, \zeta_\xi : \xi < \omega_1 \rangle$ be a Δ_1 -definable enumeration of all triples $(\alpha, S, \zeta) \in \Psi_1$. This is possible since we assume $V = L$ (cf. [18, Theorem 25.26]). Let us recursively construct a sequence $\langle x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi : \xi < \omega_1 \rangle$, where for each $\xi < \omega_1$,

- (1) $\bigcup_{\xi' < \xi} A_{x_{\xi'}} = A_{y_\xi}$ and $A_{y_\xi} \cup A_{x_\xi}$ is E -independent,
- (2) $\bar{\eta}_\xi = \langle \eta_{\xi, j} : j < N \rangle$ for some $N \in \omega$,
- (3) $T_\xi \subseteq S_\xi$, $(\alpha_\xi, T_\xi, \eta_{\xi, j}) \in \Psi_1$ for every $j < N$ and $(\alpha_\xi, T_\xi, \theta_\xi) \in \Psi_1$,
- (4) either $\forall \bar{x} \in Z_{T_\xi} (f_{\zeta_\xi}(\bar{x}) \in A_{x_\xi})$ or $\forall \bar{x} \in Z_{T_\xi} (\forall n < N (f_{\eta_{\xi, n}}(\bar{x}) \in A_{x_\xi}) \wedge \{f_{\eta_{\xi, n}}(\bar{x}), f_{\zeta_\xi}(\bar{x}) : n < N\}, f_{\theta_\xi}(\bar{x})) \in F_{e(y_\xi, r)})$,

and $(x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$ is $<_L$ -least such that (1)–(4), where $<_L$ is the Δ_1 -good global well-order of L . That $<_L$ is Δ_1 -good means that for every $z \in L$, the set $\{z' : z' <_L z\}$ is $\Delta_1(z)$ uniformly on the parameter z . In particular, quantifying over this set does not increase the complexity of a Σ_n -formula. Note that (1)–(4) are all $\Delta_1(r)$ in the given variables. For example, the second part of (1) is uniformly $\Pi_1^1(r)$ in the variables x_ξ, y_ξ , similarly for (4).

CLAIM 5.2. *For every $\xi < \omega_1$, $(x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$ exists.*

PROOF. Assume we succeeded in constructing the sequence up to ξ . Then there is y_ξ so that $\bigcup_{\xi' < \xi} A_{x_{\xi'}} = A_{y_\xi}$. By Lemma 2.7, there is a good master condition $\bar{r} \in \mathbb{P}_{\alpha_\xi}$ so that $[\bar{r}] \subseteq Z_{S_\xi}$, where $\mathbb{P}_{\alpha_\xi}$ is the α_ξ -long csi of splitting forcing. Then f_{ζ_ξ} corresponds to a $\mathbb{P}_{\alpha_\xi}$ -name $\dot{y}$ so that $\bar{r} \Vdash \dot{y} = f_{\zeta_\xi}(\bar{x}_G)$. Let M_0 be a countable elementary model with $\dot{y}, \mathbb{P}_{\alpha_\xi}, \bar{r} \in M_0$ and $\bar{p} \leq \bar{r}$ a good master condition over M_0 .

Let $C := \alpha_\xi$ and consider the results of the last section. By Proposition 4.23 applied to E_{y_ξ} and $X = 2^\omega$, there is $\bar{q} \leq \bar{p}$, a compact E_{y_ξ} independent set Y_ξ , $N \in \omega$ and continuous functions $\phi: [\bar{q}] \rightarrow [Y_\xi]^{<N}$, $w: [\bar{q}] \rightarrow \omega^\omega$ such that

- (i) either $f''_{\zeta_\xi}([\bar{q}]) \subseteq Y_\xi$,
- (ii) or $\forall \bar{x} \in [\bar{q}]((\phi(\bar{x}) \cup \{f_{\zeta_\xi}(\bar{x})\}, w(\bar{x})) \in F)$.

Let x_ξ , T_ξ , $\bar{\eta}_\xi = \langle \eta_{\xi,j} : j < N \rangle$ and θ_ξ be such that $A_{x_\xi} = Y_\xi$, $Z_{T_\xi} = [\bar{q}]$, $\{f_{\eta_{\xi,j}}(\bar{x}) : j < N\} = \phi(\bar{x})$ for every $\bar{x} \in [\bar{q}]$, and $f_{\theta_\xi} = w$. Then $(x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$ is as required. $\dashv$

Let $Y = \bigcup_{\xi < \omega_1} A_{x_\xi}$. Then Y is $\Sigma_1(r)$ -definable over $H(\omega_1)$, namely $x \in Y$ iff there is a sequence $\langle x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi : \xi \leq \alpha < \omega_1 \rangle$ so that for every $\xi \leq \alpha$, (1)–(4), for every $(x, y, T, \bar{\eta}, \theta) <_L (x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$, not (1)–(4), and $x \in A_{x_\alpha}$.

CLAIM 5.3. *In $V^\mathbb{P}$, the reinterpretation of Y is maximal E -independent.*

PROOF. Let $\bar{p} \in \mathbb{P}$ and $\dot{y} \in M_0$ be a $\mathbb{P}$ -name for an element of 2^ω , $M_0 \ni \mathbb{P}$, $\bar{p}$ a countable elementary model. Then let $\bar{q} \leq \bar{p}$ be a good master condition over M_0 and C countable, $f: [\bar{q}] \upharpoonright C \rightarrow 2^\omega$ continuous according to Lemma 2.3. Now $(2^\omega)^C$ is canonically homeomorphic to $(2^\omega)^\alpha$, $\alpha = \text{otp}(C)$, via the map $\Phi: (2^\omega)^C \rightarrow (2^\omega)^\alpha$. Then we find some $\xi < \omega_1$ so that $\alpha_\xi = \alpha$, $\Phi''([\bar{q}] \upharpoonright C) = Z_{S_\xi}$, and $f_{\zeta_\xi} \circ \Phi = f$. On the other hand, $\Phi^{-1}(Z_{T_\xi})$ is a subset of $[\bar{q}] \upharpoonright C$ conforming to the assumptions of Lemma 2.7. Thus we get $\bar{r} \leq \bar{q}$ so that $[\bar{r}] \upharpoonright C \subseteq \Phi^{-1}(Z_{T_\xi})$. According to (4), either $\bar{r} \Vdash \dot{y} \in A_{x_\xi}$ or $\bar{r} \Vdash \{\dot{y}\} \cup A_{x_\xi} \cup A_{y_\xi}$ is not E -independent. Thus we cannot have that $\bar{p} \Vdash \dot{y} \notin Y \wedge \{\dot{y}\} \cup Y$ is E -independent. This finishes the proof of the claim, as $\bar{p}$ and $\dot{y}$ were arbitrary. $\dashv$

To see that Y is $\Delta_2^1(r)$ in $V^\mathbb{P}$ it suffices to observe that any $\Sigma_2^1(r)$ set that is maximal E -independent is already $\Pi_2^1(r)$. $\dashv$

A priori, Theorem 5.1 only works for hypergraphs that are defined in the ground model. But note that there is a universal analytic hypergraph on $2^\omega \times 2^\omega$, whereby we can follow the more general statement of Theorem 1.2.

THEOREM 5.4. *After forcing with the ω_2 -length countable support iteration of $\mathbb{SP}$ over L , there is a Δ_2^1 ultrafilter, a Π_1^1 maximal independent family, and a Δ_2^1 Hamel basis, and in particular, $\mathfrak{i}_B = \mathfrak{i}_{cl} = \mathfrak{u}_B = \mathfrak{u}_{cl} = \omega_1 < \mathfrak{r} = \mathfrak{i} = \mathfrak{u} = \omega_2$.*

PROOF. Apply Theorem 5.1 to E_u , E_i , and E_h from the introduction. To see that $\mathfrak{i}_{cl} = \mathfrak{u}_{cl} = \omega_1$ note that every analytic set is the union of $\mathfrak{d}$ many compact sets and that $\mathfrak{d} = \omega_1$, since $\mathbb{SP}$ is ω^ω -bounding. $\dashv$

THEOREM 5.5. *($V=L$) Let $\mathbb{P}$ be either Sacks or splitting forcing and $k \in \omega$. Let X be a Polish space and $E \subseteq [X]^{<\omega}$ be an analytic hypergraph. Then there is a Δ_2^1 maximal E -independent set in $V^{\mathbb{P}^k}$.*

PROOF. This is similar to the proof of Theorem 5.1, using Main Lemma 3.4 and Proposition 4.16 to get an analogue of Proposition 4.23. $\dashv$

Lastly, we are going to prove Theorem 1.4.

LEMMA 5.6. *Let $X \subseteq [\omega]^\omega$ be closed so that $\forall x, y \in X(|x \cap y| = \omega)$. Then X is σ -compact.*

PROOF. If not, then by Hurewicz's Theorem (see [19, Theorem 7.10]), there is a superperfect tree $T \subseteq \omega^{<\omega}$ so that $[T] \subseteq X$, identifying elements of $[\omega]^\omega$ with their increasing enumeration, as usual. But then it is easy to recursively construct increasing sequences $\langle s_n : n \in \omega \rangle$, $\langle t_n : n \in \omega \rangle$ in T so that $s_0 = t_0 = \text{stem}(T)$, for every $n \in \omega$, t_n and s_n are infinite-splitting nodes in T , and $s_{2n+1}(|s_{2n}|) > t_{2n+1}(|t_{2n+1}| - 1)$, $t_{2n+2}(|t_{2n}|) > s_{2n+1}(|s_{2n+1}| - 1)$. Then, letting $x = \bigcup_{n \in \omega} s_n$ and $y = \bigcup_{n \in \omega} t_n$, $x \cap y \subseteq |s_0|$, viewing x, y as elements of $[\omega]^\omega$. This contradicts that $x, y \in X$. $\dashv$

The proof of Theorem 1.4 is a modification of Shelah's proof that $\mathfrak{d} \leq \mathfrak{i}$.

PROOF OF THEOREM 1.4. Let $\langle C_\alpha : \alpha < \kappa \rangle$ be compact independent families so that $\mathcal{I} = \bigcup_{\alpha < \kappa} C_\alpha$ is maximal independent and $\kappa < \mathfrak{d}$ and assume without loss of generality that $\{C_\alpha : \alpha < \kappa\}$ is closed under finite unions. Here, we will identify elements of $[\omega]^\omega$ with their characteristic function in 2^ω at several places and it should always be clear from context which representation we consider at the moment.

CLAIM 5.7. *There are $\langle x_n : n \in \omega \rangle$ pairwise distinct in $\mathcal{I}$ so that $\{x_n : n \in \omega\} \cap C_\alpha$ is finite for every $\alpha < \kappa$.*

PROOF. The closure of $\mathcal{I}$ is not independent. Thus there is $x \in \bar{\mathcal{I}} \setminus \mathcal{I}$. Now we pick $\langle x_n : n \in \omega \rangle \subseteq \mathcal{I}$ converging to x . Since C_α is closed, whenever for infinitely many n , $x_n \in C_\alpha$, then also $x \in C_\alpha$ which is impossible. $\dashv$

Fix a sequence $\langle x_n : n \in \omega \rangle$ as above. And let $a_\alpha = \{n \in \omega : x_n \in C_\alpha\} \in [\omega]^{<\omega}$. We will say that x is a Boolean combination of a set $X \subseteq [\omega]^\omega$, if there are finite disjoint $Y, Z \subseteq X$ so that $x = (\bigcap_{y \in Y} y) \cap (\bigcap_{z \in Z} \omega \setminus z)$.

CLAIM 5.8. *For any $\alpha < \kappa$ there is $f_\alpha : \omega \rightarrow \omega$ so that for any $K \in [C_\alpha \setminus \{x_n : n \in a_\alpha\}]^{<\omega}$, for all but finitely many $k \in \omega$, and for any Boolean combination x of $K \cup \{x_0, \dots, x_k\}$, $x \cap [k, f_\alpha(k)) \neq \emptyset$.*

PROOF. We define $f_\alpha(k)$ as follows. For every $l \leq k$, we define a collection of basic open subsets of $(2^\omega)^l$, $\mathcal{O}_{0,l} := \{[\bar{s}] : \bar{s} \in (2^{<\omega})^l \wedge \forall i < l (|s_i| > k) \wedge (\exists i < l, n \in a_\alpha (s_i \subseteq x_n) \vee \exists i < j < l (s_i \not\subseteq s_j))\}$. Further we call any $[\bar{s}] \notin \mathcal{O}_{0,l}$ good if for any $F, G \subseteq l$ with $F \cap G = \emptyset$ and for any Boolean combination x of $\{x_0, \dots, x_k\}$, there is $k' > k$ so that for every $i \in F$, $s_i(k') = 1$, for every $i \in G$, $s_i(k') = 0$, and $x(k') = 1$. Let $\mathcal{O}_{1,l}$ be the collection of all good $[\bar{s}]$. We see that $\bigcup_{l \leq k} (\mathcal{O}_{0,l} \cup \mathcal{O}_{1,l})$ is an open cover of $C_\alpha \cup (C_\alpha)^2 \cup \dots \cup (C_\alpha)^k$. Thus it has a finite subcover $\mathcal{O}'$. Now let $f_\alpha(k) := \max\{|t| : \exists [\bar{s}] \in \mathcal{O}' \exists i < k (t = s_i)\}$.

Now we want to show that f_α is as required. Let $(y_0, \dots, y_{l-1}) \in (C_\alpha \setminus \{x_n : n \in a_\alpha\})^l$ be arbitrary, $y_0, \dots, y_{l-1}$ pairwise distinct, and $k \geq l$ so that $y_i \upharpoonright k \neq x_n \upharpoonright k$ for all $i < l$, $n \in a_\alpha$ and $y_i \upharpoonright k \neq y_j \upharpoonright k$ for all $i < j < l$. In the definition of $f_\alpha(k)$, we have the finite cover $\mathcal{O}'$ of $(C_\alpha)^l$ and thus $(y_0, \dots, y_{l-1}) \in [\bar{s}]$ for some $[\bar{s}] \in \mathcal{O}'$. We see that $[\bar{s}] \in \mathcal{O}_{0,l}$ is impossible as we chose k large enough so that for no $i < l$, $n \in a_\alpha$, $s_i \subseteq x_n$, and for every $i < j < l$, $s_i \perp s_j$. Thus $[\bar{s}] \in \mathcal{O}_{1,l}$. But then, by the definition of $\mathcal{O}_{1,l}$, $f_\alpha(k)$ is as required. $\dashv$

As $\kappa < \mathfrak{d}$ we find $f \in \omega^\omega$ so that f is unbounded over $\{f_\alpha : \alpha < \kappa\}$. Let $x_n^0 := x_n$ and $x_n^1 := \omega \setminus x_n$ for every $n \in \omega$. For any $g \in 2^\omega$ and $n \in \omega$ we define $y_{n,g} := \bigcap_{m \leq n} x_m^{g(m)}$. Further define $y_g = \bigcup_{n \in \omega} y_{n,g} \cap f(n)$. Note $y_{n,g} \subseteq y_{m,g}$ for $m \leq n$ and that $y_g \subseteq^* y_{n,g}$ for all $n \in \omega$.

CLAIM 5.9. *For any $g \in 2^\omega$, y_g has infinite intersection with any Boolean combination of $\bigcup_{\alpha < \kappa} C_\alpha \setminus \{x_n : n \in \omega\}$.*

PROOF. Let $\{y_0, \dots, y_{l-1}\} \in [C_\alpha \setminus \{x_n : n \in a_\alpha\}]^l$ for some $l \in \omega$, $\alpha < \kappa$ be arbitrary. Here, recall that $\{C_\alpha : \alpha < \kappa\}$ is closed under finite unions. We have that there is some $k_0 \in \omega$ so that for every $k \geq k_0$, any Boolean combination y of $\{y_0, \dots, y_{l-1}\}$ and x of $\{x_n : n \leq k\}$, $x \cap y \cap [k, f_\alpha(k)) \neq \emptyset$. Let y be an arbitrary Boolean combination of $\{y_0, \dots, y_{l-1}\}$ and $m \in \omega$. Then there is $k > m, k_0$ so that $f(k) > f_\alpha(k)$. But then we have that $y_{k,g}$ is a Boolean combination of $\{x_0, \dots, x_k\}$ and thus $y_{k,g} \cap y \cap [k, f(k)) \neq \emptyset$. In particular, this shows that $y \cap y_g \not\subseteq m$ and unfixing m , $|y \cap y_g| = \omega$. $\dashv$

Now let Q_0, Q_1 be disjoint countable dense subsets of 2^ω . We see that $|y_g \cap y_h| < \omega$ for $h \neq g \in 2^\omega$. Thus the family $\{y_g : g \in Q_0 \cup Q_1\}$ is countable almost disjoint and we can find $y'_g =^* y_g$, for every $g \in Q_0 \cup Q_1$, so that $\{y'_g : g \in Q_0 \cup Q_1\}$ is pairwise disjoint. Let $y = \bigcup_{g \in Q_0} y'_g$. We claim that any Boolean combination x of sets in $\mathcal{I}$ has infinite intersection with y and $\omega \setminus y$. To see this, assume without loss of generality that x is of the form $\tilde{x} \cap x_0^{g(0)} \cap \dots \cap x_k^{g(k)}$, where $\tilde{x}$ is a Boolean combination of sets in $\mathcal{I} \setminus \{x_n : n \in \omega\}$ and $g \in 2^\omega$. As Q_0 is dense there is some $h \in Q_0$ such that $h \upharpoonright (k+1) = g \upharpoonright (k+1)$. Thus we have that $y'_h \subseteq^* x_0^{g(0)} \cap \dots \cap x_k^{g(k)}$ but also $y'_h \cap \tilde{x}$ is infinite by the claim above. In particular we have that $y \cap x$ is infinite. The complement of y is handled by replacing Q_0 with Q_1 . We now have a contradiction to $\mathcal{I}$ being maximal. $\dashv$

§6. Concluding remarks. Our focus in this paper was on Sacks and splitting forcing but it is clear that the method presented is more general. We mostly used that our forcing has Axiom A with continuous reading of names and that it is a weighted tree forcing (Definition 4.9), both in a definable way. For instance, the more general versions of splitting forcing given by Shelah in [27] fall into this class. It would be interesting to know for what other tree forcings Theorem 5.1 holds true. In [26], the authors showed that after adding a single Miller real over L , every (two-dimensional) graph on a Polish space has a Δ_1^1 maximal independent set. It is very plausible that this can be extended to the countable support iteration. One line of attack might be to use a similar method to ours, where Cohen genericity is replaced by other kinds of genericity. For instance, the following was shown by Spinas in [28] (compare this with Proposition 4.16):

FACT. *Let M be a countable model, then there is a superperfect tree T so that for any $x \neq y \in [T]$, (x, y) is $\mathbb{M}^2$ generic over M , where $\mathbb{M}$ denotes Miller forcing.*

On the other hand, it is impossible to have that any three $x, y, z \in [T]$ are mutually generic. This follows from a fact due to Velickovic and Woodin (see [33, Theorem 1]) that there is a Borel function $h: (\omega^\omega)^3 \rightarrow 2^\omega$, such that for any superperfect T , $h''([T]^3) = 2^\omega$. Also, $\mathbb{M}^3$ always adds a Cohen real (see, e.g., the last paragraph in [4]). This means that Theorem 5.5 can't hold for Miller forcing and $k \geq 3$, even for just equivalence relations. Namely, after adding a Cohen real, there can't be any E_0 -transversal that is definable with parameters from the ground model. This doesn't rule out though that the iteration might work. Let us ask the following question.

QUESTION 1. *Does Theorem 5.1 hold true for Miller forcing?*

A positive result would yield a model in which $i_B < i_{cl}$, as per $\mathfrak{d} \leq i_{cl}$. No result of this kind has been obtained so far.

Another common way to iterate Sacks or splitting forcing is to use the countable support product. The argument in Remark 3.21 can be used to see that no definable E_1 -transversals can exist in an extension by a (uncountably long) countable support product of Sacks or splitting forcing. This raises the question for which hypergraph Theorem 5.1 applies to countable support products.

QUESTION 2. *Is there a nice characterization of hypergraphs for which Theorem 5.1 holds when using countable support products?*

An interesting application of our method that appears in the authors thesis, is that P -points exist after iterating splitting forcing over a model of CH. To the best of our knowledge this is different to any other method of P -point existence in the literature.

Other applications are related to questions about families of reals and the existence of a well-order of the continuum. For example, it is known that after adding ω_1 many Sacks reals, there is no well-order of the reals in $L(\mathbb{R})$. On the other hand, if we start with $V = L$, a Δ_2^1 Hamel base exists in the extension. In particular, a Hamel base will exist in $L(\mathbb{R})$. This gives a new solution to a question by Pincus and Prikry [22], which asks whether a Hamel basis can exist without a well-order of the reals. This has only been solved recently (see [2]). Our results solve this problem not just for Hamel bases but for a big class of families of reals. In an upcoming paper [24] we will consider further applications to questions related to the Axiom of Choice.

§7. Acknowledgments. I would like to thank the Austrian Science Fund, FWF, for generous support through START Project Y1012-N35. I would also like to thank Jörg Brendle, David Chodounský, and David Schritterser for many fruitful discussions.

REFERENCES

- [1] J. E. BAUMGARTNER and R. LAVER, *Iterated perfect-set forcing*. *Annals of Mathematical Logic*, vol. 17 (1979), no. 3, pp. 271–288.
- [2] M. BERIASHVILI, R. SCHINDLER, L. WU, and L. YU, *Hamel bases and well-ordering the continuum*. *Proceedings of the American Mathematical Society*, vol. 146(2018), pp. 3565–3573.
- [3] M. BOCKSTEIN, *Un théorème de séparabilité pour les produits topologiques*. *Fundamenta Mathematicae*, vol. 35 (1948), pp. 242–246.
- [4] J. BRENDLE, *Mutual generics and perfect free subsets*. *Acta Mathematica Hungarica*, vol. 82 (1999), pp. 143–161.
- [5] J. BRENDLE, V. FISCHER, and Y. KHOMSKII, *Definable maximal independent families*. *Proceedings of the American Mathematical Society*, vol. 147 (2019), no. 8, pp. 3547–3557.
- [6] J. BRENDLE and Y. KHOMSKII, *Mad families constructed from perfect almost disjoint families*, this JOURNAL, vol. 78 (2013), no. 4, pp. 1164–1180.
- [7] B. L. BUDINAS, *The selector principle for analytic equivalence relations does not imply the existence of an A_2 well ordering of the continuum*. *Mathematics of the USSR-Sbornik*, vol. 48 (1984), pp. 159–172.
- [8] P. ERDŐS, K. KUNEN, and R. MAULDIN, *Some additive properties of sets of real numbers*. *Fundamenta Mathematicae*, vol. 113(1981), no. 3, pp. 187–199.
- [9] V. FISCHER, S. D. FRIEDMAN, and Y. KHOMSKII, *Co-analytic mad families and definable wellorders*. *Archive for Mathematical Logic*, vol. 52(2013), nos. 7–8, pp. 809–822.

- [10] V. FISCHER, S. D. FRIEDMAN, and L. ZDOMSKYY, *Projective wellorders and mad families with large continuum*. *Annals of Pure and Applied Logic*, vol. 162(2011), no. 11, pp. 853–862.
- [11] V. FISCHER and J. SCHILHAN, *Definable towers*. *Fundamenta Mathematicae*, vol. 256(2022), pp. 221–241.
- [12] V. FISCHER and D. SCHRITTESSER, *A Sacks indestructible co-analytic maximal eventually different family*. *Fundamenta Mathematicae*, vol. 252 (2021), pp. 179–201.
- [13] V. FISCHER, D. SCHRITTESSER, and A. TÖRNQUIST, *A co-analytic Cohen-indestructible maximal cofinitary group*, this JOURNAL, vol. 82(2017), no. 2, pp. 629–647.
- [14] V. FISCHER and A. TÖRNQUIST, *A co-analytic maximal set of orthogonal measures*, this JOURNAL, vol. 75(2010), no. 4, pp. 1403–1414.
- [15] K. GÖDEL, *The Consistency of the Continuum Hypothesis*. Princeton University Press, Princeton, 1940.
- [16] M. GOLSHANI, *An introduction to forcing*, preprint, 2019, [arXiv:1503.08092](https://arxiv.org/abs/1503.08092).
- [17] P. HEIN and O. SPINAS, *Antichains of perfect and splitting trees*. *Archive for Mathematical Logic*, vol. 59(2019), nos. 3–4, pp. 367–388.
- [18] T. JECH, *Set Theory*. Springer Science & Business Media, Berlin–Heidelberg, 2013.
- [19] A. S. KECHRIS, *Classical Descriptive Set Theory*. Graduate Texts in Mathematics, vol. 156, Springer, New York, 1995.
- [20] G. LAGUZZI, H. MILDENBERGER, and B. STUBER-ROUSSELLE, *On splitting trees* (2020).
- [21] A. W. MILLER, *Infinite combinatorics and definability*. *Annals of Pure and Applied Logic*, vol. 41(1989), no. 2, pp. 179–203.
- [22] D. PINCUS and K. PRIKRY, *Lusin sets and well ordering the continuum*. *Proceedings of the American Mathematical Society*, vol. 49(1975), no. 2, pp. 429–435.
- [23] J. SCHILHAN, *Coanalytic ultrafilter bases*. *Archive for Mathematical Logic* (2021), <https://doi.org/10.1007/s00153-021-00801-7>.
- [24] ———, *Maximal sets without choice*, in preparation.
- [25] D. SCHRITTESSER, *Definable discrete sets with large continuum*, preprint, 2017, [arxiv:1610.03331](https://arxiv.org/abs/1610.03331).
- [26] D. SCHRITTESSER and A. TÖRNQUIST, *Definable maximal discrete sets in forcing extensions*. *Mathematical Research Letters*, vol. 25(2018), no. 5, pp. 1591–1612.
- [27] S. SHELAH, *Vive la Différence I: Nonisomorphism of ultrapowers of countable models*, *Set Theory of the Continuum* (H. Judah, W. Just, and H. Woodin, editors), Springer, New York, 1992, pp. 357–405.
- [28] O. SPINAS, *Ramsey and freeness properties of polish planes*. *Proceedings of the London Mathematical Society*, vol. 82(2001), no. 1, pp. 31–63.
- [29] ———, *Analytic countably splitting families*, this JOURNAL, vol. 69(2004), no. 1, pp. 101–117.
- [30] ———, *Splitting squares*. *Israel Journal of Mathematics*, vol. 162(2007), no. 1, pp. 57–73.
- [31] S. M. SRIVASTAVA, *Selection theorems for G_δ -valued multifunctions*. *Transactions of the American Mathematical Society*, vol. 254(1979), pp. 283–293.
- [32] J. E. VAUGHAN, *Small uncountable cardinals and topology*, *Open Problems in Topology*, North-Holland, Amsterdam, 1990, pp. 195–218, With an appendix by S. Shelah.
- [33] B. VELICKOVIC and W. HUGH WOODIN, *Complexity of reals in inner models of set theory*. *Annals of Pure and Applied Logic*, vol. 92(1998), no. 3, pp. 283–295.
- [34] Z. VIDNYÁNSZKY, *Transfinite inductions producing coanalytic sets*. *Fundamenta Mathematicae*, vol. 224(2014), no. 2, pp. 155–174.

SCHOOL OF MATHEMATICS
 UNIVERSITY OF LEEDS
 LEEDS LS2 9JT, UK
 E-mail: j.schilhan@leeds.ac.uk