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# Inclusion Relations for New Function Spaces on Riemann Surfaces

Rauno Aulaskari and Jouni Rättyä

*Abstract.* We introduce and study some new function spaces on Riemann surfaces. For certain parameter values these spaces coincide with the classical Dirichlet space, BMOA, or the recently defined $Q_p$ space. We establish inclusion relations that generalize earlier known inclusions between the above-mentioned spaces.

## 1 Introduction

Let $R$ be an open Riemann surface that possesses a Green's function, *i.e.*, $R \notin O_G$, and let $g_R(z, \alpha)$ denote the Green function on $R$ with logarithmic singularity at $\alpha \in R$. Let $A(R)$ denote the collection of all analytic functions on $R$. The classical Dirichlet space $AD(R)$ consists of those $F \in A(R)$ for which

$$\int_R |F'(z)|^2 \, dA(z) < \infty,$$

where $dA(z)$ is the element of the Lebesgue area measure on $R$. Following [7], we define $\mathrm{BMOA}(R)$ as the set of $F \in A(R)$ such that

$$\sup_{\alpha \in R} \int_R |F'(z)|^2 g_R(z, \alpha) \, dA(z) < \infty.$$

For $0 < p < \infty$, the space $Q_p(R)$, introduced in [2], consists of those $F \in A(R)$ for which

$$\sup_{\alpha \in R} \int_R |F'(z)|^2 g_R^p(z, \alpha) \, dA(z) < \infty.$$

Metzger [7] (see also [5]) showed that $\mathrm{BMOA}(R)$ contains $AD(R)$ analogously to the case of the unit disc. This result was sharpened in [2] by proving that $AD(R) \subset Q_p(R)$ for all $p > 0$; see also [1]. Notice that $Q_1(R) = \mathrm{BMOA}(R)$.

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We will generalize the above-mentioned definitions of function spaces in the following way. For $0 < p, q < \infty$, define

$$AD^q(R) = \left\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 dA(z) < \infty \right\},$$

$$H^q_{\mathrm{BMOA}}(R) = \left\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R(z,\alpha) dA(z) < \infty \right\},$$

$$H^q_{Q_p}(R) = \left\{ F \in A(R) : \sup_{\alpha \in R} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^p(z,\alpha) dA(z) < \infty \right\}.$$

Then $AD^2(R) = AD(R)$, $H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$ by [12] (see also [10]), and $H^2_{Q_p}(R) = Q_p(R)$ for all $0 < p < \infty$.

## 2 $AD^q(R) \subset \mathrm{BMOA}(R)$ for all $0 < q < \infty$

For $F \in A(R)$, $0 < q < \infty$ and $\alpha \in R$, let $H_{|F-F(\alpha)|^q}$ denote the least harmonic majorant of the subharmonic function $u(z) = |F(z) - F(\alpha)|^q$. We set $H_{|F-F(\alpha)|^q}(z) = \infty$ if $u$ admits no harmonic majorant. The following result follows by [12, Corollary 2.6]; see also [10, Proposition 1].

***Lemma A*** *Let $F \in A(R)$, $0 < q < \infty$ and $\alpha \in R$. Then*

$$H_{|F-F(\alpha)|^q}(\alpha) = \frac{q^2}{2\pi} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R(z,\alpha) dA(z).$$

An application of [6, Corollary 1] gives

$$\frac{1}{\pi} \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 dA(z) \geq \frac{2}{q} H_{|F-F(\alpha)|^q}(\alpha), \tag{2.1}$$

from which Lemma A yields

$$AD^q(R) \subset H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$$

for all $0 < q < \infty$.

## 3 $H^q_{Q_{p_1}}(R) \subset H^q_{Q_{p_2}}(R)$ for all $0 < p_1 < p_2 < \infty$

To prove this inclusion the following lemma is needed.

***Lemma 3.1*** *Let $R$ be an open Riemann surface that possesses a Green's function,* i.e., *$R \notin O_G$. Let $F \in A(R)$, and let $\alpha \in R$, $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$. Then*

$$\int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^{p_2}(z,\alpha) dA(z) \leq C \int_R |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^{p_1}(z,\alpha) dA(z),$$

*where*

$$C = \begin{cases} p_2(p_2-1)e^q q^{1-p_2}\Gamma(p_2-1)+p_2+1, & \text{if } 1 \le p_1 < p_2 < \infty, \\ \big(p_1((p_1-1)e^q q^{1-p_1}\Gamma(p_1-1,q)+1)\big)^{-1}, & \text{if } 0 < p_1 < p_2 \le 1. \end{cases}$$

**Proof** By considering a regular exhaustion of $R$, it is sufficient to prove the assertion in the case where $R$ is the interior of a compact bordered Riemann surface $\overline{R}$ and $F$ is analytic on $\overline{R}$.

Let $\alpha \in R$ and $R_{1,\alpha} = \{z \in R : g_R(z,\alpha) > 1\}$. Then clearly

$$\int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z) \le \int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z). \tag{3.1}$$

Let $\alpha, \alpha_j$, $j=1,\dots,m$, and $\beta_k$, $k=1,\dots,n$, be the distinct zeros of $F(z)-F(\alpha)$ in $R_{1,\alpha}$ and on $\partial R_{1,\alpha}$, respectively. For $\alpha, \alpha_j, \beta_k$, $j=1,\dots,m$ and $k=1,\dots,n$, we take the parameter discs $U(\alpha,\varepsilon)$ and $U(\alpha_j,\varepsilon)$ and the half discs $B(\beta_k,\varepsilon)$ such that they are mutually disjoint. Denote

$$R_{1,\alpha,\{\alpha_j\},\{\beta_k\}} = R_{1,\alpha} \setminus \left\{ U(\alpha,\varepsilon) \bigcup \cup_{j=1}^m U(\alpha_j,\varepsilon) \bigcup \cup_{k=1}^n B(\beta_k,\varepsilon) \right\}.$$

Green's formula yields

$$\int_{R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}} \big(g_R^{p_2}(z,\alpha)\triangle|F(z)-F(\alpha)|^q - |F(z)-F(\alpha)|^q \triangle g_R^{p_2}(z,\alpha)\big)\,dA(z) = \int_{\partial R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}} \left(|F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n} - g_R^{p_2}(z,\alpha)\frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\right) ds, \tag{3.2}$$

where $\triangle$ denotes the Laplacian, $\frac{\partial}{\partial n}$ denotes the differentiation in the inward normal direction, and $ds$ is the arc length element on $\partial R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}$. Lengthy but routine calculations show that

$$\triangle|F(z)-F(\alpha)|^q = q^2|F(z)-F(\alpha)|^{q-2}|F'(z)|^2$$

and

$$\triangle g_R^{p_2}(z,\alpha) = p_2(p_2-1)g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2,$$

where

$$P_\alpha(z) = g_R(z,\alpha) + i g_R^*(z,\alpha)$$

and $g_R^*(z,\alpha)$ is a harmonic conjugate of $g_R(z,\alpha)$. It is known that $g_R^*(z,\alpha)$ is locally defined up to an additive constant, and

$$\frac{\partial g_R^{p_2}(z,\alpha)}{\partial n} = p_2 \frac{\partial g_R(z,\alpha)}{\partial n}$$

for $z \in \partial R_{1,\alpha}$.

Let $H^1_{|F-F(\alpha)|^q}$ denote the least harmonic majorant of $|F(z)-F(\alpha)|^q$ on $R_{1,\alpha}$. It turns out that the function

$$\Phi_{1,\alpha}(z) := |(F(z)-F(\alpha))e^{P_\alpha(z)}|^q = |F(z)-F(\alpha)|^q e^{qg_R(z,\alpha)}$$

is subharmonic on $R_{1,\alpha}$ and

$$\Phi_{1,\alpha}(z) = e^q |F(z)-F(\alpha)|^q$$

for all $z \in \partial R_{1,\alpha}$. The maximum principle yields

$$|F(z)-F(\alpha)|^q \le e^q H^1_{|F-F(\alpha)|^q}(z) e^{-qg_R(z,\alpha)} \tag{3.3}$$

for all $z \in R_{1,\alpha}$.

Let $g_{R_{1,\alpha}}(z,\alpha)$ be the Green function of $R_{1,\alpha}$ with logarithmic singularity at $\alpha$. Then $\triangle g_{R_{1,\alpha}}(z,\alpha) = 0$ in $R_{1,\alpha,\{\alpha_j\},\{\beta_k\}}$ and $g_{R_{1,\alpha}}(z,\alpha) = 0$ for $z \in \partial R_{1,\alpha}$. By [12, 13], we have

$$\begin{aligned} H^1_{|F-F(\alpha)|^q}(\alpha) &= \frac{1}{2\pi}\int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_{R_{1,\alpha}}(z,\alpha)}{\partial n}\, ds \\ &= \frac{q^2}{2\pi}\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\, dA(z). \end{aligned} \tag{3.4}$$

To deal with the area integral in (3.4), denote $S_{t,\alpha} = \{z \in R : g_R(z,\alpha) = t\}$ for $t > 0$. If $z \in S_{t,\alpha}$, then $dt = \frac{\partial g_R(z,\alpha)}{\partial n} dn$. Letting $\varepsilon \to 0$ in (3.2) we see that all the integrals

$$\begin{gathered} \int_{\partial U(\alpha,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds, \qquad \int_{\partial U(\alpha_j,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds, \\ \int_{\partial B(\beta_k,\varepsilon)} |F(z)-F(\alpha)|^q \frac{\partial g_R^{p_2}(z,\alpha)}{\partial n}\, ds, \qquad \int_{\partial U(\alpha,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds, \\ \int_{\partial U(\alpha_j,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds, \qquad \int_{\partial B(\beta_k,\varepsilon)} g_R^{p_2}(z,\alpha) \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\, ds \end{gathered}$$

tend to zero for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Therefore the equality (3.2) becomes

(3.5)

$$
\begin{aligned}
I_{1,p_2,q}(\alpha) &= q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z) \\
&= p_2(p_2-1)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^q g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2\,dA(z) \\
&\qquad + p_2 \int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_R(z,\alpha)}{\partial n}\,ds - \int_{\partial R_{1,\alpha}} \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\,ds \\
&= p_2(p_2-1)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^q g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2\,dA(z) \\
&\quad + p_2 \int_{\partial R_{1,\alpha}} |F(z)-F(\alpha)|^q \frac{\partial g_R(z,\alpha)}{\partial n}\,ds \\
&\quad + q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z),
\end{aligned}
$$

where, by Green's formula,

$$
q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) = -\int_{\partial R_{1,\alpha}} \frac{\partial |F(z)-F(\alpha)|^q}{\partial n}\,ds.
$$

We first concentrate on the case $1 \le p_1 < p_2 < \infty$. By the formulae (3.3), (3.5), and (2.1), and by using the inequality $g_{R_{1,\alpha}}(z,\alpha) \le g_R(z,\alpha)$, $z \in R_{1,\alpha}$, we obtain

(3.6)

$$
\begin{aligned}
I_{1,p_2,q}(\alpha) &\le p_2(p_2-1)e^q \int_{R_{1,\alpha}} H^1_{|F-F(\alpha)|^q}(z) g_R^{p_2-2}(z,\alpha)|P'_\alpha(z)|^2 e^{-q g_R(z,\alpha)}\,dA(z) \\
&\quad + 2\pi p_2 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) \\
&\le p_2(p_2-1)e^q \int_1^\infty \left( \int_{S_{t,\alpha}} H^1_{|F-F(\alpha)|^q}(z) \frac{\partial g_R(z,\alpha)}{\partial n}\,ds \right) g_R^{p_2-2}(z,\alpha) e^{-q g_R(z,\alpha)}\,dt \\
&\quad + p_2 q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\,dA(z) \\
&\quad + q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z) \\
&\le 2\pi p_2(p_2-1)e^q H^1_{|F-F(\alpha)|^q}(\alpha) \int_1^\infty t^{p_2-2}e^{-qt}\,dt \\
&\quad + p_2 q^2 \int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)
\end{aligned}
$$

$$+ q^2 \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R(z,\alpha)\, dA(z)$$

$$\leq p_2(p_2-1) q^2 e^q \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\, dA(z)$$

$$\cdot \frac{1}{q^{p_2-1}} \int_q^\infty u^{p_2-2} e^{-u}\, du$$

$$+ q^2(p_2+1) \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^{p_1}(z,\alpha)\, dA(z)$$

$$\leq q^2 \big( p_2(p_2-1) e^q q^{1-p_2} \Gamma(p_2-1) + p_2 + 1 \big)$$

$$\cdot \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^{p_1}(z,\alpha)\, dA(z),$$

where $\Gamma(p_2-1) = \int_0^\infty u^{p_2-2} e^{-u}\, du$ is the gamma function. By combining (3.1) and (3.6) we obtain the desired inequality for $1 \leq p_1 < p_2 < \infty$.

Let now $0 < p_1 < p_2 \leq 1$. Then the estimate (3.3) gives

(3.7)

$$I_{1,p_1,q}(\alpha) \geq p_1(p_1-1) e^q \int_{R_{1,\alpha}} H^1_{|F-F(\alpha)|^q}(z) e^{-q g_R(z,\alpha)} g_R^{p_1-2}(z,\alpha) |P'_\alpha(z)|^2\, dA(z)$$

$$+ 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2 \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2\, dA(z)$$

$$= 2\pi p_1(p_1-1) e^q H^1_{|F-F(\alpha)|^q}(\alpha) \int_1^\infty t^{p_1-2} e^{-qt}\, dt$$

$$+ 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha) + q^2 \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2\, dA(z)$$

$$= 2\pi p_1 H^1_{|F-F(\alpha)|^q}(\alpha) \big( (p_1-1) e^q q^{1-p_1} \Gamma(p_1-1,q) + 1 \big)$$

$$+ q^2 \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2\, dA(z),$$

where $\Gamma(p_1-1,q) = \int_q^\infty u^{p_1-2} e^{-u}\, du$ is the incomplete gamma function. We note that

$$A(p_1,q) = (p_1-1) e^q q^{1-p_1} \Gamma(p_1-1,q) + 1 > 0,$$

and hence by dividing by $q^2$ in (3.7) we obtain

$$\int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_R^{p_1}(z,\alpha)\, dA(z) \tag{3.8}$$

$$\geq p_1 A(p_1,q) \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2 g_{R_{1,\alpha}}(z,\alpha)\, dA(z)$$

$$+ \int_{R_{1,\alpha}} |F(z) - F(\alpha)|^{q-2} |F'(z)|^2\, dA(z).$$

Since $g_{R_{1,\alpha}}(z,\alpha) = g_R(z,\alpha) - 1$ for $z \in R_{1,\alpha}$, (3.8) yields

$$\begin{aligned}(3.9)\qquad \int_{R_{1,\alpha}} &|F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_1}(z,\alpha)\,dA(z)\\ &\geq p_1 A(p_1,q)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R(z,\alpha)\,dA(z)\\ &\quad + \big(1-p_1A(p_1,q)\big)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\geq p_1A(p_1,q)\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^{p_2}(z,\alpha)\,dA(z).\end{aligned}$$

The last inequality follows from the fact that $1-p_1A(p_1,q) > 0$. The desired inequality for $0 < p_1 < p_2 \leq 1$ follows by combining (3.1) and (3.9). ■

***Theorem 3.2*** *Let $R$ be a Riemann surface such that $R \notin Q_G$, and let $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$. Then the following inclusion holds:*

$$H^q_{Q_{p_1}}(R) \subset H^q_{Q_{p_2}}(R).$$

**Proof** If either $0 < p_1 < p_2 \leq 1$ or $1 \leq p_1 < p_2 < \infty$, then the assertion follows directly from Lemma 3.1. If $0 < p_1 \leq 1 < p_2 < \infty$, then Lemma 3.1 gives

$$H^q_{Q_{p_1}}(R) \subset H^q_{\mathrm{BMOA}}(R) \subset H^q_{Q_{p_2}}(R)$$

for all $0 < q < \infty$. ■

## 4 $AD^q(R) \subset H^q_{Q_p}(R)$ for all $0 < p, q < \infty$

In Section 2, we noted that the inclusion $AD^q(R) \subset H^q_{\mathrm{BMOA}}(R) = \mathrm{BMOA}(R)$ holds for all $0 < q < \infty$. This fact is sharpened in this section by showing the following result.

***Theorem 4.1*** *$AD^q(R) \subset H^q_{Q_p}(R)$ for all $0 < p, q < \infty$.*

**Proof** Theorem 3.2 implies that $\mathrm{BMOA}(R) \subset H^q_{Q_p}(R)$ for all $1 \leq p < \infty$ and $0 < q < \infty$. Combining this with the inclusion $AD^q(R) \subset \mathrm{BMOA}(R)$, $0 < q < \infty$, we deduce

$$(4.1)\qquad AD^q(R) \subset H^q_{Q_p}(R)$$

for all $1 \leq p < \infty$ and $0 < q < \infty$.

Now let $0 < p < 1$. Recall that $R_{1,\alpha} = \{z \in R : g_R(z,\alpha) > 1\}$. By (3.5),

$$\begin{aligned}(4.2)\qquad q^2\int_{R_{1,\alpha}} &|F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z) \leq\\ &2\pi p H^1_{|F-F(\alpha)|^q}(\alpha) + q^2\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z),\end{aligned}$$

because $p-1<0$. Suppose now that $F \in AD^q(R)$. Then there exists $M_1 > 0$ such that

$$\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) \le \int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z) \le M_1 < \infty$$

for all $\alpha \in R$. By Section 2 we know that $F \in \mathrm{BMOA}(R)$. Hence, by Lemma A, there exists $M_2 > 0$ such that

$$H^1_{|F-F(\alpha)|^q}(\alpha) \le H_{|F-F(\alpha)|^q}(\alpha) \le M_2 < \infty \tag{4.3}$$

for all $\alpha \in R$. By (4.2) and (4.3), we deduce

$$\begin{aligned}\int_{R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z) &\le \frac{1}{q^2}\left(2\pi p M_2 + q^2 M_1\right)\\ &= M_1 + \frac{2\pi p}{q^2}M_2\end{aligned} \tag{4.4}$$

for all $\alpha \in R$. On the other hand, we immediately see that

$$\begin{aligned}&\int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z)\\ &\le \int_{R\setminus R_{1,\alpha}} |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\le \int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2\,dA(z)\\ &\le M_1\end{aligned} \tag{4.5}$$

for all $\alpha \in R$. Combining (4.4) and (4.5) we obtain

$$\sup_{\alpha\in R}\int_R |F(z)-F(\alpha)|^{q-2}|F'(z)|^2 g_R^p(z,\alpha)\,dA(z) \le 2M_1 + \frac{2\pi p}{q^2}M_2$$

for all $0<p<1$ and $0<q<\infty$. Thus $F \in H^q_{Q_p}(R)$ for all $0<p<1$ and $0<q<\infty$. This together with (4.1) completes the proof. ■

## 5 $H^q_{Q_p}(R) \subset \mathcal{B}(R)$ for all $0<p,q<\infty$

Let $\lambda_R(\alpha)$ be the density of the hyperbolic distance (Poincaré metric) on a hyperbolic Riemann surface $R$. The Bloch space is defined as

$$\mathcal{B}(R) := \left\{ F \in A(R) : \sup_{\alpha\in R}\frac{|F'(\alpha)|}{\lambda_R(\alpha)} < \infty \right\}.$$

The purpose of this section is to show the maximal property of $\mathcal{B}(R)$ with respect to the spaces $H^q_{Q_p}(R)$. In the case of the unit disc, an analogous result follows by a work of Rubel and Timoney [9].

***Theorem 5.1*** *$H^q_{Q_p}(R) \subset \mathcal{B}(R)$ for all $0 < p, q < \infty$.*

**Proof** Let $\pi\colon \mathbb{D} \to R$ be a universal covering map of the unit disc $\mathbb{D}$ to the Riemann surface $R$. Let $\Omega$ denote the fundamental polygon of the Fuchsian group $\Gamma$. If $\alpha \in R$ and $a \in \Omega$ satisfy $\pi(a) = \alpha$, then we may take the Green function of the Riemann surface $\mathbb{D}/\Gamma$ by setting $g_\Gamma(z,a) = g_R(\pi(z),\alpha)$. By a result of Myrberg [11, p. 522], we know that

$$g_\Gamma(z,a) = \sum_{\gamma\in\Gamma} g_{\mathbb{D}}\big(z,\gamma(a)\big),$$

where $g_{\mathbb{D}}(z,a)$ is the Green function of $\mathbb{D}$ with logarithmic singularity at $a$. Therefore we may define the space $H^q_{Q_p}(\mathbb{D}/\Gamma) = H^q_{Q_p}(R)$ in the sense that $f \in H^q_{Q_p}(\mathbb{D}/\Gamma)$ if $f$ is analytic in $\mathbb{D}$ and $f = F\circ\pi$, where $F \in H^q_{Q_p}(R)$. With a similar understanding, $\mathcal{B}(\mathbb{D}/\Gamma) = \mathcal{B}(R)$.

First let $1 \le p < \infty$. Suppose now that $f \in H^q_{Q_p}(\mathbb{D}/\Gamma)$, but $f \notin \mathcal{B}(\mathbb{D}/\Gamma)$. Then [3, Lemma] or [8] implies that there exist a sequence of points $\{a_n\}$ in $\mathbb{D}$ and a sequence of positive numbers $\{\rho_n\}$ such that $\rho_n/(1-|a_n|) \to 0$, as $n \to \infty$, and $\{f(a_n+\rho_n\xi)-f(a_n)\}$ converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant analytic function $f_0(\xi)$. Here, without loss of generality, we may assume that $a_n \in \Omega$ for each $n \in \mathbb{N}$. Note that in general this is not possible, but the reasoning in (5.1) below shows that we may do so. Now, for $\delta > 0$, set $K = K(\delta) = \{\xi \in \mathbb{C} : |\xi| \le \delta\}$. Denote $\varphi_n(\xi) = a_n + \rho_n\xi$ and $g_n(\xi) = f(\varphi_n(\xi)) - f(\varphi_n(0)) = f(a_n+\rho_n\xi) - f(a_n)$. Then

$$|g_n(\xi)|^{q-2} \to |f_0(\xi)|^{q-2} \ge \delta_1 > 0 \quad\text{and}\quad |g_n'(\xi)|^2 \to |f_0'(\xi)|^2 \ge \delta_2 > 0$$

uniformly in

$$K_1 = K \setminus \left(\cup_{j=1}^n D(\xi_j,\varepsilon) \cup \cup_{i=1}^m D(\eta_i,\varepsilon)\right),$$

where $D(\xi_j,\varepsilon) = \{\xi : |\xi-\xi_j| < \varepsilon\} \subset K$ and $D(\eta_i,\varepsilon) = \{\xi : |\xi-\eta_i| < \varepsilon\}$, $\eta_i \in \partial K$, for all $j = 1,\dots,n$ and $i = 1,\dots,m$. Here, for $0 < q < \infty$, the points $\xi_j$, $j = 1,\dots,n$, are the zeros and poles of $f_0$ in $K = \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$, and the points $\eta_i$, $i = 1,\dots,m$, are the zeros and poles of $f_0$ in $\partial K$. We take $\varepsilon > 0$ so small that all the discs $D(\xi_j,\varepsilon)$ and $D(\eta_i,\varepsilon)$ are pairwise disjoint. Now

$$\begin{aligned}\log\left|\frac{1-\overline{\varphi_n(0)}\varphi_n(\xi)}{\varphi_n(\xi)-\varphi_n(0)}\right| &= \log\left|\frac{1-\overline{a_n}(a_n+\rho_n\xi)}{a_n+\rho_n\xi-a_n}\right| \\ &= \log\left|\frac{1-|a_n|}{\rho_n}\frac{1+|a_n|}{\xi}-\overline{a_n}\right| \to \infty,\end{aligned}$$

as $n \to \infty$, for all $\xi \in K_1$. On the other hand, by the assumption,

$$
\begin{aligned}
(5.1) \qquad & \int_{K_1} |g_n(\xi)|^{q-2} |g_n'(\xi)|^2 g_{\mathbb{D}}^p(\varphi_n(\xi), \varphi_n(0))\, dA(\xi) \\
&= \int_{\varphi_n(K_1)} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 g_{\mathbb{D}}^p(z, a_n)\, dA(z) \\
&\leq \int_{\mathbb{D}} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 g_{\mathbb{D}}^p(z, a_n)\, dA(z) \\
&= \sum_{\gamma \in \Gamma} \int_{\Omega} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 g_{\mathbb{D}}^p(\gamma(z), a_n)\, dA(z) \\
&= \int_{\Omega} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 \Big( \sum_{\gamma \in \Gamma} g_{\mathbb{D}}^p(\gamma(z), a_n) \Big)\, dA(z) \\
&\leq \int_{\Omega} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 \Big( \sum_{\gamma \in \Gamma} g_{\mathbb{D}}(\gamma(z), a_n) \Big)^p\, dA(z) \\
&= \int_{\Omega} |f(z) - f(a_n)|^{q-2} |f'(z)|^2 g_{\Gamma}^p(z, a_n)\, dA(z) \leq C < \infty
\end{aligned}
$$

for all $n \in \mathbb{N}$. But this is a contradiction, since the left-hand side of (5.1) tends to infinity as $n \to \infty$. Thus $H^q_{Q_p}(\mathbb{D}/\Gamma) \subset \mathcal{B}(\mathbb{D}/\Gamma)$ for all $1 \leq p < \infty$ and $0 < q < \infty$. The assertion follows from the nesting property in Theorem 3.2. ∎

## 6 $H^q_{Q_p}(R) \neq \mathcal{B}(R)$

Using the same idea as in the proof of [4, Theorem 4.2] we can prove that there exists a Riemann surface $R$ such that $H^q_{Q_p}(R) \neq \mathcal{B}(R)$. Since the proof is almost identical to the original one, we omit the details.

***Theorem 6.1*** *For every $0 < p, q < \infty$ there exists a Riemann surface $R$ such that $H^q_{Q_p}(R) \neq \mathcal{B}(R)$.*

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*University of Eastern Finland, Department of Physics and Mathematics, 80101 Joensuu, Finland*
*e-mail*: rauno.aulaskari@uef.fi jouni.rattya@uef.fi