THESIS ABSTRACT

CLAUDIO AGOSTINI. ***Generalized descriptive set theory at uncountable cardinals & actions of monoids in combinatorics***. Università degli Studi di Torino 2022. Supervised by Luca Motto Ros. MSC: Primary 03E15, 05D10, Secondary 54E99, 03C98.

Abstract

The thesis is divided into two parts. The first one focuses on generalized descriptive set theory, and the second one on combinatorics, model theory, and Ramsey theory.

Generalized descriptive set theory (GDST) is a natural extension of (classical) descriptive set theory (DST) where countable is replaced by uncountable. But the framework of GDST is narrow if compared to that of DST, as so far GDST has mostly concentrated on the study of the generalized Baire space ${}^{\kappa}\kappa$, rather than considering arbitrary "Polish-like" spaces or standard κ-Borel spaces. Also, GDST is usually developed for cardinals satisfying $\kappa^{<\kappa} = \kappa$, which implies that κ must be regular.

The goal of the first part of the thesis is to fill these gaps, studying classes of spaces that could take the role of Polish spaces in the generalized context under the weak assumption $2^{<\kappa} = \kappa$, which allows one to include singular cardinals and can consistently hold at every cardinal (e.g., in models of $\mathsf{ZFC} + \mathsf{GCH}$).

In Chapter 1, we begin by considering the case when κ is regular. We consider several candidates for "Polish-like" spaces that have been proposed in the literature (e.g., $\mathbb{G}$-Polish spaces and SC_κ-spaces), and introduce a new one ($f\mathrm{SC}_\kappa$-spaces). We show that all these classes are nicely organized in four groups, with two clear dividing lines between them: κ-additivity, which can be interpreted as a strong analog of zero-dimensionality, and the degree of completeness (Figure 1).

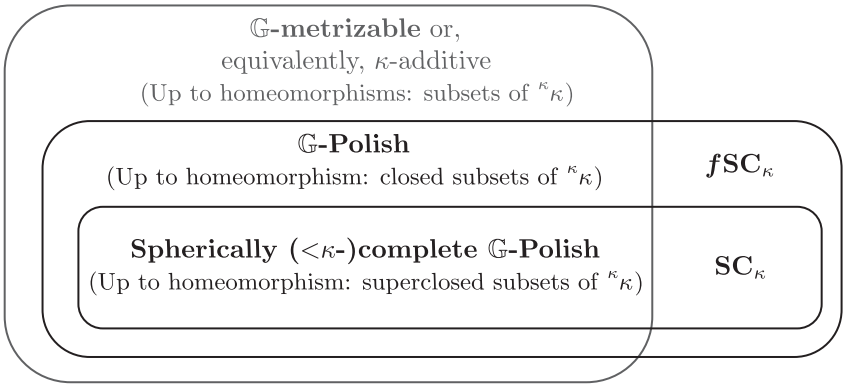

FIGURE 1. Relationships among Polish-like classes of regular Hausdorff spaces of weight $\leq \kappa$, for a totally ordered Abelian group $\mathbb{G}$ of degree $\deg(\mathbb{G}) = \kappa > \omega$.

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All the proposed classes give rise to the same class of spaces up to κ-Borel isomorphism, providing a natural setup to work with. Then, various results from classical DST about Polish, Borel, and standard Borel spaces are extended to this context.

Chapter 2 extends the previous analysis to embrace singular cardinals too. In particular, it contains an in-depth study of the generalizations and characterizations of metrizability necessary in the singular case. The main result on this topic is a new metrization theorem in terms of topological games that holds for both classical metrizability and $\mathbb{G}$-metrizability.

Chapter 3 features various examples of spaces in the classes considered above, and a study of linearly ordered topological spaces (LOTS) and generalized ordered spaces (GO-spaces) in the context of GDST.

The second part of the thesis deals with a recently discovered notion in combinatorics. In 2019, Solecki introduced the classes of Ramsey monoids and $\mathbb{Y}$-controllable monoids to collect and extend different theorems in combinatorics, like Hindman's Finite Sum Theorem, Carlson's Theorem, Gowers' FIN_κ Theorem, and Furstenberg–Katznelson's Ramsey Theorem. Then, he provided a necessary condition and some sufficient conditions for a finite monoid to be Ramsey or $\mathbb{Y}$-controllable.

Chapters 4 and 5 aim to continue the work started by Solecki on these and other related classes of monoids. We improve the necessary conditions and the sufficient conditions provided by Solecki, reaching in particular a full characterization of Ramsey monoids. This further extends results like Carlson's Theorem and Gowers' FIN_κ Theorem, but it also sets a precise limit on when it is possible to obtain similar statements.

We also give examples of classes of $\mathbb{Y}$-controllable monoids that do not satisfy some of the sufficient conditions, suggesting possible strategies to improve the results we provided. Then, we show that in certain particular classes of $\mathbb{Y}$-controllable monoids with stronger properties, the remaining sufficient conditions become necessary as well.

In Chapter 5, we also study local versions of the classes of Ramsey and $\mathbb{Y}$-controllable monoids that are better suited for infinite monoids.

The thesis contains material from joint works with Luca Motto Ros, Philipp Schlicht, and Eugenio Colla.

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