

# ABSORPTION PROBABILITIES FOR GAUSSIAN POLYTOPES AND REGULAR SPHERICAL SIMPLICES

ZAKHAR KABLUCHKO,\* *Westfälische Wilhelms-Universität Münster*  
DMITRY ZAPOROZHETS,\*\* *St. Petersburg Department of Steklov Mathematical  
Institute*

## Abstract

The Gaussian polytope  $\mathcal{P}_{n,d}$  is the convex hull of  $n$  independent standard normally distributed points in  $\mathbb{R}^d$ . We derive explicit expressions for the probability that  $\mathcal{P}_{n,d}$  contains a fixed point  $x \in \mathbb{R}^d$  as a function of the Euclidean norm of  $x$ , and the probability that  $\mathcal{P}_{n,d}$  contains the point  $\sigma X$ , where  $\sigma \geq 0$  is constant and  $X$  is a standard normal vector independent of  $\mathcal{P}_{n,d}$ . As a by-product, we also compute the expected number of  $k$ -faces and the expected volume of  $\mathcal{P}_{n,d}$ , thus recovering the results of Affentranger and Schneider (*Discr. and Comput. Geometry*, 1992) and Efron (*Biometrika*, 1965), respectively. All formulas are in terms of the volumes of regular spherical simplices, which, in turn, can be expressed through the standard normal distribution function  $\Phi(z)$  and its complex version  $\Phi(iz)$ . The main tool used in the proofs is the conic version of the Crofton formula.

**Keywords:** Convex hull; random polytope; Gaussian polytope; Goodman–Pollack model; absorption probability; Wendel’s formula; regular simplex; spherical geometry; solid angle; convex cone; conic Crofton formula; average number of faces; error function; Schläfli’s function

2010 Mathematics Subject Classification: Primary 60D05; 52A22; 52B11  
Secondary 60G70; 52A20; 52A23

## 1. Statement of main results

### 1.1. Introduction

Let  $X_1, \dots, X_n$  be independent random vectors with standard Gaussian distribution on  $\mathbb{R}^d$ . The *Gaussian polytope*  $\mathcal{P}_{n,d}$  is defined as the convex hull of  $X_1, \dots, X_n$ , that is,

$$\mathcal{P}_{n,d} = \text{conv}(X_1, \dots, X_n) = \left\{ \sum_{i=1}^n \lambda_i X_i : \lambda_1, \dots, \lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The main aim of the present paper is to provide an explicit expression for the *absorption probability*, that is, the probability that  $\mathcal{P}_{n,d}$  contains a given deterministic point  $x \in \mathbb{R}^d$ . By rotational symmetry, the absorption probability depends only on the Euclidean norm  $|x|$ . It

Received 24 September 2018; revision received 18 December 2019.

\* Postal address: Orléans–Ring 10, 48149 Münster, Germany. Email: [zakhar.kabluchko@uni-muenster.de](mailto:zakhar.kabluchko@uni-muenster.de)

\*\* Postal address: Fontanka 27, 191011 St. Petersburg, Russia.

turns out that it is more convenient to pass to the complementary event and consider the *non-absorption probability*

$$f_{n,d}(|x|) := \mathbb{P}[x \notin \text{conv}(X_1, \dots, X_n)]. \quad (1.1)$$

A classical result of Wendel [31] (which is valid in a setting more general than the Gaussian one considered here; see also [28, Theorem 8.2.1]), states that

$$f_{n,d}(0) = \frac{1}{2^{n-1}} \left( \binom{n-1}{d-1} + \binom{n-1}{d-2} + \dots \right). \quad (1.2)$$

We shall compute  $f_{n,d}(r)$  for general  $r \geq 0$ . The main idea is that we shall make the point  $x$  random, with a rotationally symmetric Gaussian distribution and a certain variance  $\sigma^2 \geq 0$ . Namely, let  $X$  be a  $d$ -dimensional standard Gaussian random vector which is independent of  $X_1, \dots, X_n$ . We shall compute

$$p_{n,d}(\sigma^2) := \mathbb{P}[\sigma X \notin \text{conv}(X_1, \dots, X_n)]. \quad (1.3)$$

This probability can be related to a certain Laplace-type transform of  $f_{n,d}$ . After inverting the Laplace transform, we shall obtain a formula for  $f_{n,d}$ . This formula involves a certain function  $g_n(r)$  which expresses the volume of regular spherical simplices and which will be studied in detail below.

The probability  $p_{n,d}(\sigma^2)$  is closely related to the expected number of faces of the polytope  $\mathcal{P}_{n,d}$ . Let  $f_k(\mathcal{P}_{n,d})$  be the number of  $k$ -dimensional faces ( $k$ -faces) of  $\mathcal{P}_{n,d}$ . Exact formulas for  $\mathbb{E}f_k(\mathcal{P}_{n,d})$  were derived by Rényi and Sulanke [22, Section 4] (for  $d=2$ ), Efron [13] (for  $d=2, 3$ ), Raynaud [21] (for faces of maximal dimension, that is, for  $k=d-1$ ). Affentranger [2] proved an asymptotic formula valid for general  $d$  and  $k$ ; see also Carnal [10] for the case  $d=2$ . Baryshnikov and Vitale [5] showed that the expected number of  $k$ -faces of  $\mathcal{P}_{n,d}$  is the same as the expected number of  $k$ -faces of a random projection of the regular simplex with  $n$  vertices onto a uniformly chosen linear subspace of dimension  $d$  (the so-called *Goodman–Pollack model*). Finally, Affentranger and Schneider [3] expressed the expected number of  $k$ -faces of the random projection of any polytope in terms of the internal and external angles of that polytope. Combining the results of Affentranger and Schneider [3] and Baryshnikov and Vitale [5], one obtains an expression for  $\mathbb{E}f_k(\mathcal{P}_{n,d})$  in terms of the internal and external angles of the regular simplex.

Hug *et al.* [16] expressed some important functionals of the Gaussian polytope, including the expected number of  $k$ -faces, through probabilities of the form  $p_{n,d}(\sigma^2)$  and computed their asymptotics. As a by-product of our results, we shall provide explicit formulas for these functionals, thus recovering the results obtained in [3] and [16]. Recent surveys on random polytopes can be found in [15], [20], and [27].

## 1.2. Non-absorption probabilities

Our explicit formulas will be stated in terms of the functions  $g_n(r)$ ,  $n \in \mathbb{N}_0$ , defined by  $g_0(r) := 1$  and

$$g_n(r) = \mathbb{P}[\eta_1 < 0, \dots, \eta_n < 0], \quad r \geq -1/n, \quad n \in \mathbb{N}, \quad (1.4)$$

where  $(\eta_1, \dots, \eta_n)$  is a zero-mean Gaussian vector with

$$\text{Cov}(\eta_i, \eta_j) = \begin{cases} 1+r, & \text{if } i=j, \\ r, & \text{if } i \neq j. \end{cases} \quad (1.5)$$

The fact that (1.5) indeed defines a valid (i.e., positive semidefinite) covariance matrix for  $r \geq -1/n$  is easily verified using the inequality between the arithmetic and quadratic means.

Many known and some new properties of the function  $g_n$  (which is closely related to the Schlöfli function [8]) will be collected in Sections 1.3 and 1.4. At this point, we just state an explicit expression for  $g_n$  in terms of the standard normal distribution function  $\Phi$ . It is known that  $\Phi$  admits an analytic continuation to the entire complex plane. We shall need its values on the real and imaginary axes, namely

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt, \quad \Phi(iz) = \frac{1}{2} + \frac{i}{\sqrt{2\pi}} \int_0^z e^{t^2/2} dt, \quad z \in \mathbb{R}. \quad (1.6)$$

The reader more used to the error function  $\operatorname{erf}$  may transform everything by applying the formula  $\Phi(z) = 1/2 + \operatorname{erf}(z/\sqrt{2})$ . With this notation, an explicit formula for  $g_n$  reads as follows:

$$g_n(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi^n(\sqrt{r}x) e^{-x^2/2} dx \quad (1.7)$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x\sqrt{r}} e^{-z^2/2} dz \right)^n e^{-x^2/2} dx, & \text{if } r \geq 0, \\ \frac{2}{\sqrt{\pi}} \int_0^{\infty} \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{i}{\sqrt{\pi}} \int_0^{x\sqrt{-r}} e^{-z^2} dz \right)^n \right] e^{-x^2} dx, & \text{if } -1/n \leq r \leq 0, \end{cases} \quad (1.8)$$

where in (1.7) we agree that  $\sqrt{r} = i\sqrt{-r}$  if  $r < 0$ . The next theorem provides a formula for the probability that  $\sigma X \notin \mathcal{P}_{n,d}$ .

**Theorem 1.1.** *Let  $X, X_1, \dots, X_n$  be independent standard Gaussian random vectors in  $\mathbb{R}^d$ , where  $n \geq d+1$ . Then, for every  $\sigma \geq 0$ ,*

$$p_{n,d}(\sigma^2) = \mathbb{P}[\sigma X \notin \operatorname{conv}(X_1, \dots, X_n)] = 2(b_{n,d-1}(\sigma^2) + b_{n,d-3}(\sigma^2) + \dots), \quad (1.9)$$

where

$$b_{n,k}(\sigma^2) = \binom{n}{k} g_k \left( -\frac{\sigma^2}{1+k\sigma^2} \right) g_{n-k} \left( \frac{\sigma^2}{1+k\sigma^2} \right) \quad (1.10)$$

for  $k = 0, \dots, n$ , and  $b_{n,k} = 0$  for  $k \notin \{0, 1, \dots, n\}$ .

The proof of Theorem 1.1 will be given in Section 2. The main idea is to interpret  $p_{n,d}(\sigma^2)$  as the probability that a uniform random linear subspace intersects a certain  $n$ -dimensional convex cone  $C = C_n(\sigma^2)$ . By the conic Crofton formula (which will be recalled in Theorem 2.1 below), this intersection probability can be expressed in terms of the conic intrinsic volumes  $v_0(C), \dots, v_n(C)$  of  $C$ . At this point, we can forget about the original problem and concentrate on computing the conic intrinsic volumes, which is a purely geometric problem. It turns out that  $v_k(C) = b_{n,k}(\sigma^2)$ ; see Proposition 1.3 below.

**Example 1.1.** (*Wendel's formula.*) Let  $\sigma^2 = 0$ . By symmetry reasons it is clear (and will be stated in Proposition 1.4 (d)) that  $g_k(0) = 2^{-k}$ ,  $g_{n-k}(0) = 2^{-(n-k)}$ , and hence  $b_{n,k}(0) = 2^{-n} \binom{n}{k}$ . Theorem 1.1 simplifies to

$$\begin{aligned} \mathbb{P}[0 \notin \operatorname{conv}(X_1, \dots, X_n)] &= \frac{1}{2^{n-1}} \left( \binom{n}{d-1} + \binom{n}{d-3} + \dots \right) \\ &= \frac{1}{2^{n-1}} \left( \binom{n-1}{d-1} + \binom{n-1}{d-2} + \dots \right), \end{aligned} \quad (1.11)$$

where in the second line we used the defining property of the Pascal triangle. This recovers Wendel's formula (1.2) in the Gaussian case.

By conditioning on  $X$  in Theorem 1.1, we shall derive the following result.

**Corollary 1.1.** *The function  $f_{n,d}(|x|) = \mathbb{P}[x \notin \text{conv}(X_1, \dots, X_n)]$  satisfies*

$$\int_0^\infty f_{n,d}(\sqrt{2u}) u^{\frac{d}{2}-1} e^{-\lambda u} du = 2\Gamma(d/2)\lambda^{-\frac{d}{2}}(b_{n,d-1}(1/\lambda) + b_{n,d-3}(1/\lambda) + \dots)$$

for all  $\lambda > 0$ .

It is possible to invert the Laplace transform explicitly. Recall that  $\Phi$  is the standard normal distribution function.

**Theorem 1.2.** *For all  $u > 0$  we have*

$$f_{n,d}(\sqrt{2u}) - f_{n,d}(0) = 2u^{1-(d/2)}(a_{n,d-1}(u) + a_{n,d-3}(u) + \dots),$$

where  $f_{n,d}(0)$  is given by Wendel's formula (1.2) and

$$a_{n,k}(u) = \binom{n}{k} \int_0^u e^{-vk} F'_{k,n-k}(v) (u-v)^{(d/2)-1} dv, \quad (1.12)$$

$$F_{k,n-k}(v) = \frac{1}{\pi} \int_0^v \left( \frac{\Phi^{n-k}(\sqrt{2w}) + \Phi^{n-k}(-\sqrt{2w})}{2\sqrt{w}} \right. \\ \left. \cdot \frac{\Phi^k(i\sqrt{2(v-w)}) + \Phi^k(-i\sqrt{2(v-w)})}{2\sqrt{v-w}} \right) dw. \quad (1.13)$$

### 1.3. Cones, solid angles, and intrinsic volumes

The function  $g_n$  has appeared (in many different parametrizations) in connection with internal and external angles of regular simplices and generalized orthants, but the results are somewhat scattered through the literature, and especially the properties of  $g_n(r)$  at negative values of  $r$  do not seem to be widely known. In the following two sections we shall provide an overview of what is known about  $g_n$ , state some new results, and fix the notation needed for the proof of Theorem 1.1.

A non-empty subset  $C \subset \mathbb{R}^N$  is called a *convex cone* if for every  $x, y \in C$  and  $\lambda, \mu > 0$  we have  $\lambda x + \mu y \in C$ . In the following we restrict our attention to *polyhedral* cones (or just cones, for short), which are defined as intersections of finitely many closed half-spaces whose boundaries contain the origin. The linear hull of  $C$ , i.e. the smallest linear space containing  $C$ , is denoted by  $L(C)$ . Letting  $Z$  be a standard Gaussian random vector on  $L(C)$ , the *solid angle* of the cone  $C$  is defined as

$$\alpha(C) = \mathbb{P}[Z \in C].$$

In fact, the same formula remains true if  $Z$  has any rotationally invariant distribution on  $L(C)$ . Note that we measure the solid angle with respect to the linear hull  $L(C)$  as the ambient space, so that the solid angle is never 0, even for cones with empty interior.

Denote the standard scalar product on  $\mathbb{R}^N$  by  $\langle \cdot, \cdot \rangle$  and let  $e_1, \dots, e_N$  be the standard basis of  $\mathbb{R}^N$ . Fix any  $r \geq -1/n$  and consider  $n$  vectors  $u_1, \dots, u_n$  in  $\mathbb{R}^N$ ,  $n \leq N$ , such that

$$\langle u_i, u_j \rangle = \begin{cases} 1+r & \text{if } i=j, \\ r & \text{if } i \neq j, \end{cases} \quad i, j \in \{1, \dots, n\}.$$

Denote the cone spanned by the vectors  $u_1, \dots, u_n$  by

$$C_n(r) := \text{pos}(u_1, \dots, u_n) := \{\alpha_1 u_1 + \dots + \alpha_n u_n : \alpha_1, \dots, \alpha_n \geq 0\}. \quad (1.14)$$

The specific choice of the vectors  $u_1, \dots, u_n$  as well as the dimension  $N$  of the ambient space will be of minor importance because we are interested in the isometry type of the cone only. For  $r = 0$ , the cone  $C_n(0)$  is isometric to the positive orthant  $\mathbb{R}_+^n$ . Vershik and Sporyshev [30] called  $C_n(r)$  the contracted (respectively, extended) orthant if  $r < 0$  (respectively,  $r > 0$ ). The extremal cases  $r \rightarrow \infty$  and  $r = -1/n$  correspond to a ray and a half-space, respectively.

**Proposition 1.1.** *For all  $r > -1/n$ , the solid angle of the cone  $C_n(r)$  is given by*

$$\alpha(C_n(r)) = g_n \left( -\frac{r}{1+nr} \right).$$

This fact can be used to relate  $g_n(r)$  to the volume of a regular spherical simplex. These volumes have been much studied since Schläfli [26].

**Theorem 1.3.** *Let  $S_n(\ell)$  be a regular spherical simplex, with  $n$  vertices and side length  $\ell$ , on the unit sphere  $\mathbb{S}^{n-1}$ . That is, the geodesic distance between any two vertices of the simplex is  $\ell \in (0, \arccos(-\frac{1}{n-1}))$ . Then the spherical volume of  $S_n(\ell)$  is given by*

$$\text{Vol}_{n-1}(S_n(\ell)) = \text{Vol}_{n-1}(\mathbb{S}^{n-1}) \cdot g_n \left( -\frac{\cos \ell}{1 + (n-1) \cos \ell} \right).$$

More concretely, writing  $r_* := -\frac{\cos \ell}{1+(n-1)\cos \ell}$ , we have

$$\text{Vol}_{n-1}(S_n(\ell)) = \frac{2}{\Gamma(\frac{n}{2})\sqrt{\pi}} \int_0^\infty \left( \int_{-\infty}^{x\sqrt{r_*}} e^{-z^2} dz \right)^n e^{-x^2} dx \quad \text{if } -\frac{1}{n-1} < \cos \ell \leq 0; \quad (1.15)$$

$$\text{Vol}_{n-1}(S_n(\ell)) = \frac{2}{\sqrt{\pi}} \int_0^\infty \text{Re} \left[ \left( \frac{1}{2} + \frac{i}{\sqrt{\pi}} \int_0^{x\sqrt{-r_*}} e^{-z^2} dz \right)^n \right] e^{-x^2} dx \quad \text{if } \cos \ell \geq 0. \quad (1.16)$$

*Proof.* Let  $u_1, \dots, u_n$  be as above. Observe that  $u_1/\sqrt{1+r}, \dots, u_n/\sqrt{1+r}$  can be viewed as vertices of  $S_n(\arccos \frac{r}{1+r})$ . So, choose  $r > -1/n$  such that  $\ell = \arccos \frac{r}{1+r}$ . Then  $r = \frac{\cos \ell}{1-\cos \ell}$  and

$$\frac{\text{Vol}_{n-1}(S_n(\ell))}{\text{Vol}_{n-1}(\mathbb{S}^{n-1})} = \alpha(C_n(r)) = g_n \left( -\frac{r}{1+nr} \right) = g_n \left( -\frac{\cos \ell}{1 + (n-1) \cos \ell} \right)$$

by Proposition 1.1.

Formula (1.15) can be found in the book of Böhm and Hertel [8, Satz 3, p. 283] or in the works of Ruben [25], and [24] and Hadwiger [14]. Note that [8] uses a different parametrization for  $S_n(\ell)$ ; see [8, Satz 2, p. 277] for the relation between both parametrizations. Observe also that the Schläfli function  $\mathbf{f}^{(n)}(\alpha)$  used in [8] is related to  $g_n$  via

$$\mathbf{f}^{(n)}(\alpha) = \frac{2^n}{n!} g_n \left( -\frac{\cos 2\alpha}{1 + \cos 2\alpha} \right),$$

as one can see by comparing [8, Satz 2, p. 279] with Theorem 1.3. The case  $\cos \ell > 0$  is missing in [8] and in many other references on the subject. Formula (1.16) was proved by Vershik and Sporyshev [30, Corollary 3, p. 192]; see also [9], [12], and [11] for asymptotic results.

To proceed, we need to recall some notions related to solid angles. A *polyhedral set* is an intersection of finitely many closed half-spaces (whose boundaries need not pass through the origin). If a polyhedral set is bounded, it is called a *polytope*. Polyhedral cones are special cases of polyhedral sets. Denote by  $\mathcal{F}_k(P)$  the set of  $k$ -dimensional faces of a polyhedral set  $P$ . The *tangent cone* at a face  $F \in \mathcal{F}_k(P)$  is defined by

$$T_F = T_F(P) = \{v \in \mathbb{R}^n : f_0 + \varepsilon v \in P \text{ for some } \varepsilon > 0\}, \quad (1.17)$$

where  $f_0$  is any point in the relative interior of  $F$ , i.e. the interior of  $F$  taken with respect to its affine hull. The *normal cone* at the face  $F \in \mathcal{F}_k(P)$  is defined as the polar of the tangent cone, that is,

$$N_F = N_F(P) = T_F^\circ(P) = \{w \in \mathbb{R}^n : \langle w, u \rangle \leq 0 \text{ for all } u \in T_F(P)\}. \quad (1.18)$$

For certain special values of  $r$  one can interpret  $g_n(r)$  as inner or normal solid angles at the faces of the regular simplex. The *inner* and *normal* (or *external*) solid angles of  $P$  at  $F$  are defined as the solid angles of the cones  $T_F(P)$  and  $N_F(P)$ , respectively.

**Proposition 1.2.** *Let  $\Delta_n := \text{conv}(e_1, \dots, e_n)$  be the  $(n-1)$ -dimensional regular simplex in  $\mathbb{R}^n$ .*

(a) *The normal solid angle at any  $(k-1)$ -dimensional face of  $\Delta_n$  equals*

$$g_{n-k}\left(\frac{1}{k}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{k}} e^{-z^2} dz \right)^{n-k} e^{-x^2} dx.$$

(b) *The inner solid angle at any  $(k-1)$ -dimensional face of  $\Delta_n$  equals*

$$g_{n-k}\left(-\frac{1}{n}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \text{Re} \left[ \left( \frac{1}{2} + \frac{i}{\sqrt{\pi}} \int_0^{x/\sqrt{n}} e^{z^2} dz \right)^{n-k} \right] e^{-x^2} dx.$$

Both parts were previously known; see [14], and [25] for part (a) and [23, Section 4] (where the method used was attributed to H. E. Daniels) as well as [30, Lemma 4] for part (b). A formula for the normal solid angles of crosspolytopes (which is similar to part (a)) was derived in [6].

The next proposition provides a geometric interpretation of  $b_{n,k}(\sigma^2)$ . The  $k$ th *conic intrinsic volume* of a cone  $C$  is given by

$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(F) \alpha(N_F(C)),$$

where we recall that  $\alpha(F)$  is the solid angle of the cone  $F$  measured with respect to the linear hull of  $F$ . See [4] for equivalent definitions and properties.

**Proposition 1.3.** *For every  $r > -1/n$  and  $k \in \{0, \dots, n\}$ , the  $k$ th conic intrinsic volume of the cone  $C_n(r)$  is given by*

$$v_k(C_n(r)) = b_{n,k}(r) = \binom{n}{k} g_k\left(-\frac{r}{1+kr}\right) g_{n-k}\left(\frac{r}{1+kr}\right).$$

**Remark 1.1.** As a consequence of the Gauss–Bonnet formula for conic intrinsic volumes (see [28, Theorem 6.5.5] or [4, Corollary 4.4]), we obtain the identities

$$\sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,2k}(r) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} b_{n,2k+1}(r) = \frac{1}{2}.$$

In particular, the numbers  $b_{n,0}(r), \dots, b_{n,n}(r)$  define a probability distribution on  $\{0, \dots, n\}$ , a fact which is not evident in view of the expression for  $g_n$  given in (1.7) and (1.8). For  $r = 0$  (in which case  $C_n(0)$  is the positive orthant  $\mathbb{R}_+^n$ ) this distribution reduces to the binomial one with parameters  $(n, 1/2)$ , because  $g_n(0) = 2^{-n}$  by Proposition 1.4 (d) below.

#### 1.4. Properties of $g_n$

Next we give a formula for  $g_n(r)$  which may be more convenient than its definition (1.4). Recall that  $\Phi$  denotes the standard normal distribution function (see (1.6)), viewed as an analytic function on the entire complex plane.

**Proposition 1.4.** *The function  $g_n : [-\frac{1}{n}, \infty) \rightarrow [0, 1]$  defined in (1.4) has the following properties.*

(a) For all  $n \in \mathbb{N}$  and  $r \geq -\frac{1}{n}$ ,

$$\begin{aligned} g_n(r) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi^n(\sqrt{r}x) e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (\Phi^n(\sqrt{r}x) + \Phi^n(-\sqrt{r}x)) e^{-x^2/2} dx, \end{aligned} \quad (1.19)$$

where, in the case of negative  $r$ , we use the convention  $\sqrt{r} = i\sqrt{-r}$ . In fact, the rightmost expression in (1.19) defines  $g_n$  as an analytic function on the half-plane  $\operatorname{Re} r > -\frac{1}{n}$ .

(b) For all  $n \in \{2, 3, \dots\}$  and  $r > -\frac{1}{n}$  we have

$$g'_n(r) = \frac{n(n-1)}{4\pi(r+1)\sqrt{2r+1}} g_{n-2}\left(\frac{r}{2r+1}\right).$$

(c)  $g_0(r) = 1$  (by definition) and  $g_1(r) = \frac{1}{2}$ .

(d) For every  $n \in \mathbb{N}$ , we have

$$g_n(0) = 2^{-n}, \quad g_n(1) = \frac{1}{n+1}, \quad \lim_{r \rightarrow +\infty} g_n(r) = \frac{1}{2}.$$

(e) For  $n \in \{2, 3, \dots\}$  we have  $g_n(-\frac{1}{n}) = 0$ .

(f)  $g_2(r) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{r}{1+r}$  and  $g_3(r) = \frac{1}{8} + \frac{3}{4\pi} \arcsin \frac{r}{1+r}$ .

(g) For every fixed  $n \in \mathbb{N}$  we have

$$g_n\left(-\frac{1}{n} + \varepsilon\right) \sim \frac{n^n \Gamma(n/2)}{2 \pi^{n/2} \Gamma(n) \sqrt{n}} \cdot \varepsilon^{(n-1)/2}, \quad \varepsilon \downarrow 0.$$

(h) For all  $n \in \mathbb{N}$ , the functions  $g_{2n}$  and  $g_{2n+1}$  admit extensions to analytic functions on some unramified cover of  $\mathbb{C} \setminus \{-1/k : k = 1, \dots, 2n\}$ .

**Remark 1.2.** Special values of  $g_n$  listed in Parts (d) and (e) were known to Schläfli [26, p. 267]; see also [8, pp. 285–286]. Part (b) is a consequence of the Schläfli differential relation; see Böhm and Hertel [8, Satz 2, p. 279]. For completeness, we shall provide a self-contained proof of Proposition 1.4 in Section 3.1.

**Remark 1.3.** Using the fact that  $\overline{\Phi(iz)} = \Phi(-iz)$  for  $z \in \mathbb{R}$ , one can state (1.19) in the case of real  $r \in [-\frac{1}{n}, 0]$  as follows:

$$\begin{aligned} g_n(r) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \operatorname{Re}(\Phi^n(i\sqrt{-r}x))e^{-x^2/2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{i}{\sqrt{\pi}} \int_0^{x\sqrt{-r}} e^{z^2} dz \right)^n \right] e^{-x^2} dx, \end{aligned}$$

a formula obtained by Vershik and Sporyshev [30, Corollary 3, p. 192].

**Remark 1.4.** Taking  $r = -1/n$  in (1.19), making the change of variables  $y = x/\sqrt{n}$ , and using that  $g_n(-1/n) = 0$  for  $n \geq 2$ , we obtain the curious identity

$$\int_{-\infty}^{+\infty} \left( \Phi(iy)e^{-y^2/2} \right)^n dy = 0, \quad n = 2, 3, \dots \quad (1.20)$$

Using induction and partial integration we shall extend this as follows.

**Proposition 1.5.** For all  $m \in \mathbb{N}_0$  and all  $n = m + 2, m + 3, \dots$  we have

$$\int_{-\infty}^{+\infty} y^m \left( \Phi(iy)e^{-y^2/2} \right)^n dy = 0. \quad (1.21)$$

Also, for all  $m \in \mathbb{N}_0$  we have

$$\int_0^{+\infty} y^m \left( \left( \Phi(iy)e^{-y^2/2} \right)^{m+1} + (-1)^m \left( \Phi(-iy)e^{-y^2/2} \right)^{m+1} \right) dy = \sqrt{\frac{\pi}{2}} \left( \frac{i}{\sqrt{2\pi}} \right)^m. \quad (1.22)$$

For  $n \leq m + 1$  the integral in (1.21) diverges since  $\Phi(iy) \sim \frac{1}{\sqrt{2\pi}iy} e^{y^2/2}$  as  $y \rightarrow \infty$ ,  $y \in \mathbb{R}$ ; see [1, Eq. 7.1.23, p. 298]. Equation (1.22) states a formula for the Cauchy principal value which is well defined for  $n = m + 1$ .

### 1.5. Expected number of $k$ -faces

Let  $f_k(\mathcal{P}_{n,d})$  be the number of  $k$ -dimensional faces of the Gaussian polytope  $\mathcal{P}_{n,d}$ . Recall the notation  $p_{n,d}(\sigma^2) = \mathbb{P}[\sigma X \notin \mathcal{P}_{n,d}]$ , where  $X$  is a standard normal vector in  $\mathbb{R}^d$  independent of  $\mathcal{P}_{n,d}$ . With the aid of the Blaschke–Petkantschin formula, Hug *et al.* [16, Theorem 3.2] showed that

$$\mathbb{E}f_k(\mathcal{P}_{n,d}) = \binom{n}{k+1} p_{n-k-1,d-k} \left( \frac{1}{k+1} \right). \quad (1.23)$$

Using this formula, they proved an asymptotic result of the form

$$\mathbb{E}f_k(\mathcal{P}_{n,d}) \underset{n \rightarrow \infty}{\sim} \bar{c}_{(k,d)} (\log n)^{(d-1)/2}, \quad (1.24)$$

where  $\bar{c}_{(k,d)}$  is an explicit constant only depending on  $d$  and  $k$ . With the aid of Theorem 1.1 one can derive the following explicit formula.



**Theorem 1.4.** For every  $k = 0, \dots, n-1$  we have

$$\mathbb{E}f_k(\mathcal{P}_{n,d}) = \frac{2 \cdot n!}{(k+1)!} \sum_{\substack{j=d-2i>k \\ i \in \mathbb{N}_0}} \frac{g_{j-1-k} \left(-\frac{1}{j}\right) g_{n-j} \left(\frac{1}{j}\right)}{(j-1-k)!(n-j)!}. \quad (1.25)$$

*Proof.* Combine Theorem 1.1 with (1.23).

**Remark 1.5.** The quantities  $g_{j-1-k}(-\frac{1}{j})$  and  $g_{n-j}(\frac{1}{j})$  appearing on the right-hand side of (1.25) are certain inner/normal solid angles of regular simplices; see Proposition 1.2. With this interpretation, the formula (1.25) is due to Affentranger and Schneider [3].

**Remark 1.6.** More generally, Hug *et al.* [16] considered also the functional

$$T_{0,b}^{d,k}(\mathcal{P}_{n,d}) = \sum_{F \in \mathcal{F}_k(\mathcal{P}_{n,d})} (\text{Vol}_k(F))^b, \quad b \geq 0,$$

which reduces to  $f_k(\mathcal{P}_{n,d})$  for  $b = 0$ . For  $k = d-1$  and  $b = 1$  one gets the surface area. Hug *et al.* [16] showed that

$$\mathbb{E}T_{0,b}^{d,k}(\mathcal{P}_{n,d}) = \mathbb{E}f_k(\mathcal{P}_{n,d}) \cdot \left(\frac{\sqrt{k+1}}{k!}\right)^b \prod_{j=1}^k \frac{\Gamma\left(\frac{d+b+1-j}{2}\right)}{\Gamma\left(\frac{d+1-j}{2}\right)}.$$

Thus, an explicit formula for  $\mathbb{E}T_{0,b}^{d,k}(\mathcal{P}_{n,d})$  can be obtained from Theorem 1.4.

The following fixed- $r$  asymptotics for  $g_n(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi^n(\sqrt{r}x) e^{-x^2/2} dx$  was derived in [21, pp. 44–45] and [29, Lemma 5].

**Proposition 1.6.** For any fixed  $r > 0$  we have

$$g_n(r) \underset{n \rightarrow \infty}{\sim} \Gamma(1/r) r^{-1/2} n^{-1/r} (4\pi \log n)^{\frac{1}{2r} - \frac{1}{2}}.$$

This can be used to compute the large- $n$  asymptotics of the probability that  $\sigma X \notin \mathcal{P}_{n,d}$ .

**Corollary 1.2.** Fix any  $\sigma^2 > 0$  and write  $r = \frac{\sigma^2}{1+(d-1)\sigma^2}$ . Then

$$p_{n,d}(\sigma^2) = \mathbb{P}[\sigma X \notin \mathcal{P}_{n,d}] \underset{n \rightarrow \infty}{\sim} \frac{n^{-1/\sigma^2}}{(d-1)!} g_{d-1}(-r) \Gamma\left(\frac{1}{r}\right) r^{-1/2} (4\pi \log n)^{\frac{1}{2r} - \frac{1}{2}}.$$

*Proof.* It follows from Proposition 1.6 and (1.10) that for every fixed  $k \in \mathbb{N}_0$  and  $\sigma^2 > 0$ ,

$$b_{n,k}(\sigma^2) \underset{n \rightarrow \infty}{\sim} \Gamma\left(k + \sigma^{-2}\right) \frac{\sqrt{k + \sigma^{-2}}}{k!} g_k\left(-\frac{\sigma^2}{1 + k\sigma^2}\right) n^{-\frac{1}{\sigma^2}} (4\pi \log n)^{\frac{1}{2\sigma^2} + \frac{k-1}{2}}.$$

In particular,  $b_{n,d-m}(\sigma^2) = o(b_{n,d-1}(\sigma^2))$  for all  $m \geq 2$  and as  $n \rightarrow \infty$ . Theorem 1.1 yields  $p_{n,d}(\sigma^2) \sim 2b_{n,d-1}(\sigma^2)$ , from which the required asymptotics follows.

**Remark 1.7.** Applying Corollary 1.2 to the right-hand side of (1.23) we deduce the asymptotic formula

$$\begin{aligned}\mathbb{E}f_k(\mathcal{P}_{n,d}) &= \binom{n}{k+1} p_{n-k-1,d-k} \left( \frac{1}{k+1} \right) \\ &\underset{n \rightarrow \infty}{\sim} \frac{2}{\sqrt{d}} \binom{d}{k+1} g_{d-1-k} \left( -\frac{1}{d} \right) (4\pi \log n)^{(d-1)/2},\end{aligned}$$

which recovers a result of Affentranger [2] (see also [3], [5], and [16]) stated in (1.2).

### 1.6. Expected volume

Let us derive from our results the following formula for the expected volume of the Gaussian polytope, due to Efron [13]:

$$\mathbb{E} \text{Vol}_d(\mathcal{P}_{n,d}) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \cdot \frac{n!}{d!(n-d-1)!} \int_{-\infty}^{\infty} \Phi^{n-d-1}(t) \varphi^{d+1}(t) dt. \quad (1.26)$$

In fact, Efron [13] proved the formula for  $d = 2$  and stated it for general  $d$ ; another proof (valid for arbitrary  $d$ ) can be found in [19].

Since the surface measure of the unit sphere in  $\mathbb{R}^d$  equals  $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ , we can express the expected volume of the Gaussian polytope as follows:

$$\begin{aligned}\mathbb{E} \text{Vol}_d(\mathcal{P}_{n,d}) &= \int_0^\infty (1 - f_{n,d}(r)) \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} dr \\ &= \frac{(2\pi)^{d/2}}{\Gamma(d/2)} \int_0^\infty (1 - f_{n,d}(\sqrt{2u})) u^{(d/2)-1} du,\end{aligned}$$

where we have made the change of variables  $r = \sqrt{2u}$ . On the other hand, by Corollary 1.1 together with the identity  $\sum_{k=0}^n b_{n,k}(1/\lambda) = 1$  (see Remark 1.1), we can write

$$\int_0^\infty (1 - f_{n,d}(\sqrt{2u})) u^{(d/2)-1} e^{-\lambda u} du = 2\Gamma(d/2) \lambda^{-d/2} (b_{n,d+1}(1/\lambda) + b_{n,d+3}(1/\lambda) + \dots)$$

for all  $\lambda > 0$ . Hence, by the monotone convergence theorem,

$$\mathbb{E} \text{Vol}_d(\mathcal{P}_{n,d}) = 2^{(d/2)+1} \pi^{d/2} \lim_{\lambda \downarrow 0} \lambda^{-d/2} (b_{n,d+1}(1/\lambda) + b_{n,d+3}(1/\lambda) + \dots). \quad (1.27)$$

Recall from (1.10) that for every fixed  $k \in \{0, \dots, n\}$ ,

$$b_{n,k}(1/\lambda) = \binom{n}{k} g_k \left( -\frac{1}{\lambda + k} \right) g_{n-k} \left( \frac{1}{\lambda + k} \right).$$

Clearly,

$$\lim_{\lambda \downarrow 0} g_{n-k} \left( \frac{1}{\lambda + k} \right) = g_{n-k} \left( \frac{1}{k} \right),$$

whereas Proposition 1.4 (g) yields

$$g_k \left( -\frac{1}{\lambda + k} \right) = g_k \left( -\frac{1}{k} + \frac{\lambda(1 + o(1))}{k^2} \right) \sim \frac{\Gamma(k/2)\sqrt{k}}{2\pi^{k/2}\Gamma(k)} \lambda^{(k-1)/2} \text{ as } \lambda \downarrow 0.$$

Taking everything together, we obtain

$$b_{n,k}(1/\lambda) \sim \binom{n}{k} \frac{\Gamma(k/2)\sqrt{k}}{2\pi^{k/2}\Gamma(k)} g_{n-k} \left(\frac{1}{k}\right) \lambda^{(k-1)/2} \text{ as } \lambda \downarrow 0.$$

So, in the sum on the right-hand side of (1.27) the term  $b_{n,d+1}(1/\lambda) \sim \text{const} \cdot \lambda^{d/2}$  dominates and we obtain

$$\begin{aligned} \mathbb{E} \text{Vol}_d(\mathcal{P}_{n,d}) &= (2\pi)^{d/2} \binom{n}{d+1} \frac{\Gamma(\frac{d+1}{2})\sqrt{d+1}}{\pi^{\frac{d+1}{2}}\Gamma(d+1)} g_{n-d-1} \left(\frac{1}{d+1}\right) \\ &= \binom{n}{d+1} \frac{\sqrt{d+1}}{2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)} g_{n-d-1} \left(\frac{1}{d+1}\right), \end{aligned}$$

where we used the Legendre duplication formula. Recalling formula (1.19) for  $g_n(r)$ , and performing some simple transformations, we arrive at (1.26). Proposition 1.6 yields the following asymptotic formula due to Affentranger [2, Theorem 4]:

$$\mathbb{E} \text{Vol}_d(\mathcal{P}_{n,d}) \sim \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} (2 \log n)^{d/2}$$

as  $n \rightarrow \infty$ , while  $d$  stays fixed. A more refined asymptotics was derived in [19].

### 1.7. Low-dimensional examples

Let  $d = 2$ . Theorem 1.1 simplifies to

$$\begin{aligned} \mathbb{P}[\sigma X \notin \text{conv}(X_1, \dots, X_n)] &= 2b_{n,1}(\sigma^2) = 2ng_1 \left(-\frac{\sigma^2}{1+\sigma^2}\right) g_{n-1} \left(\frac{\sigma^2}{1+\sigma^2}\right) \\ &= ng_{n-1} \left(\frac{\sigma^2}{1+\sigma^2}\right). \end{aligned}$$

We obtain the formula

$$\mathbb{P}[\sigma X \notin \text{conv}(X_1, \dots, X_n)] = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi^{n-1} \left(\frac{\sigma x}{\sqrt{1+\sigma^2}}\right) e^{-x^2/2} dx.$$

The next theorem gives an explicit formula for the non-absorption probability in the case  $d = 2$ .

**Theorem 1.5.** *Let  $\xi, \xi_1, \dots, \xi_n$  be standard normal random variables and let  $W$  be a random variable with the arcsine density  $\frac{1}{\pi}(1-y^2)^{-1/2}dy$  on  $[-1, 1]$ , all variables being independent. Define  $M_k = \max\{\xi_1, \dots, \xi_k\}$ . Then, for all  $u \geq 0$ ,*

$$\begin{aligned} f_{n,2}(\sqrt{2u}) &= \mathbb{P}[M_n^2 + \xi^2 \leq 2u] + \frac{d}{du} \mathbb{P}[M_n^2 + \xi^2 \leq 2u] \\ &= \mathbb{P}[M_n^2 + \xi^2 \leq 2u] + ne^{-u} \mathbb{P}[M_{n-1} \leq \sqrt{2u}W]. \end{aligned}$$

That is,  $f_{n,2}(\sqrt{2u})$  is the sum of the distribution function and the density of the random variable  $\frac{1}{2}(M_n^2 + \xi^2)$  at  $u$ .

Two-dimensional absorption probabilities were studied in a very general setting by Jewell and Romano [17], and [18], but their method does not seem to yield an explicit formula like that in Theorem 1.5.

Consider now the case  $d = 3$ . Using first Theorem 1.1, (1.10), and then Proposition 1.4, we arrive at

$$\begin{aligned} \mathbb{P}[\sigma X \notin \text{conv}(X_1, \dots, X_n)] \\ &= 2b_{n,2}(\sigma^2) + 2b_{n,0}(\sigma^2) \\ &= n(n-1)g_2\left(-\frac{\sigma^2}{1+2\sigma^2}\right)g_{n-2}\left(\frac{\sigma^2}{1+2\sigma^2}\right) + 2g_n(\sigma^2) \\ &= \frac{n(n-1)}{\sqrt{2\pi}}\left(\frac{1}{4} - \frac{1}{2\pi}\arcsin\frac{\sigma^2}{1+\sigma^2}\right)\int_{-\infty}^{+\infty}\Phi^{n-2}\left(\frac{\sigma x}{\sqrt{1+2\sigma^2}}\right)e^{-x^2/2}dx \\ &\quad + \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Phi^n(\sigma x)e^{-x^2/2}dx, \end{aligned}$$

but we were unable to invert the Laplace transform to obtain a formula for  $f_{n,3}$  similar to that of Theorem 1.5. Similarly, for  $d = 4$  we obtain

$$\begin{aligned} \mathbb{P}[\sigma X \notin \text{conv}(X_1, \dots, X_n)] \\ &= 2b_{n,3}(\sigma^2) + 2b_{n,1}(\sigma^2) \\ &= \frac{n(n-1)(n-2)}{3}g_3\left(-\frac{\sigma^2}{1+3\sigma^2}\right)g_{n-3}\left(\frac{\sigma^2}{1+3\sigma^2}\right) + ng_{n-1}\left(\frac{\sigma^2}{1+\sigma^2}\right) \\ &= \frac{n(n-1)(n-2)}{3\sqrt{2\pi}}\left(\frac{1}{8} - \frac{3}{4\pi}\arcsin\frac{\sigma^2}{1+2\sigma^2}\right)\int_{-\infty}^{+\infty}\Phi^{n-3}\left(\frac{\sigma x}{\sqrt{1+3\sigma^2}}\right)e^{-x^2/2}dx \\ &\quad + \frac{n}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Phi^{n-1}\left(\frac{\sigma x}{\sqrt{1+\sigma^2}}\right)e^{-x^2/2}dx. \end{aligned}$$

Taking  $\sigma = 1$  in Theorem 1.1 yields the *probability content* of the Gaussian polytope, which is defined as

$$C_{n,d} := \mathbb{P}[X \in \text{Conv}(X_1, \dots, X_n)] = 1 - \mathbb{P}[X \notin \text{Conv}(X_1, \dots, X_n)].$$

For  $d = 2, 3, 4$  we obtain the formulas

$$\begin{aligned} C_{n,2} &= 1 - \frac{n}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Phi^{n-1}\left(\frac{x}{\sqrt{2}}\right)e^{-x^2/2}dx, \\ C_{n,3} &= 1 - \frac{n(n-1)}{6\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Phi^{n-2}\left(\frac{x}{\sqrt{2}}\right)e^{-x^2/2}dx - \frac{2}{n+1}, \\ C_{n,4} &= C_{n,2} - \frac{n(n-1)(n-2)}{3\sqrt{2\pi}}\left(\frac{1}{8} - \frac{3}{4\pi}\arcsin\frac{1}{3}\right)\int_{-\infty}^{+\infty}\Phi^{n-3}\left(\frac{x}{2}\right)e^{-x^2/2}dx. \end{aligned}$$

The formulas for  $d = 2, 3$  were obtained by Efron [13], in Equations (7.5) and (7.6) on p. 341.

### 1.8. Absorption probability in the Goodman–Pollack model

Let  $v_1, \dots, v_{n+1}$  be the vertices of an  $n$ -dimensional regular simplex inscribed into the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ . That is,  $|v_i| = 1$  for all  $1 \leq i \leq n+1$  and  $\rho := \langle v_i, v_j \rangle = -1/n$  for all  $1 \leq i < j \leq n+1$ . Let  $O$  be a random orthogonal matrix sampled according to the Haar probability measure on the orthogonal group  $O(n)$ . Consider the randomly rotated regular simplex with vertices  $Ov_1, \dots, Ov_{n+1}$  and project it onto some fixed  $d$ -dimensional linear subspace  $V_d \subset \mathbb{R}^n$ . The choice of  $V_d$  is irrelevant, so that we shall assume that  $V_d \equiv \mathbb{R}^d$  is spanned by the first  $d$  vectors of the standard orthonormal basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$ . Denote orthogonal projection onto  $V_d$  by  $\Pi$ . The resulting random polytope

$$\mathcal{Q}_{n+1,d} := \text{Conv}(\Pi Ov_1, \dots, \Pi Ov_{n+1}) \subset \mathbb{R}^d$$

is said to be distributed according to the *Goodman–Pollack model*. Affentranger and Schneider [3] and Baryshnikov and Vitale [5] observed that the Gaussian polytope  $\mathcal{P}_{n+1,d}$  is closely related to the Goodman–Pollack polytope  $\mathcal{Q}_{n+1,d}$ . In particular, Baryshnikov and Vitale [5] showed that all functionals which remain invariant under affine transformations of the polytope (like the number of  $k$ -faces) have the same distribution in both models. We are interested in the non-absorption probability in the Goodman–Pollack model, that is,

$$f_{n,d}^*(|x|) := \mathbb{P}[x \notin \mathcal{Q}_{n,d}], \quad x \in \mathbb{R}^d.$$

Clearly, this functional is not invariant with respect to the affine transformations of  $\mathcal{Q}_{n,d}$ . We cannot compute the non-absorption probability explicitly, but it is possible to evaluate a certain integral transform of  $f_{n,d}^*$ .

**Theorem 1.6.** *For every  $\sigma^2 \geq 0$  we have*

$$\frac{1}{B\left(\frac{d}{2}, \frac{n-d}{2}\right)} \int_0^\infty \frac{u^{d-1}}{(1+u^2)^{n/2}} f_{n,d}^* \left( \sqrt{\frac{n\sigma^2 + 1}{n-1}} \cdot u \right) du = b_{n,d-1}(\sigma^2) + b_{n,d-3}(\sigma^2) + \dots$$

Here,  $B(\cdot, \cdot)$  denotes the Euler beta function. The proof of Theorem 1.6 will be given in Section 5.

**Remark 1.8.** It is also possible to consider random projections  $\tilde{\mathcal{Q}}_{n,d}$  of the random orthogonal transformation of the regular simplex  $\text{Conv}(e_1, \dots, e_n)$  inscribed into the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ . For the non-absorption probability  $\tilde{f}_{n,d}^*(|x|) := \mathbb{P}[x \notin \tilde{\mathcal{Q}}_{n,d}]$ ,  $x \in \mathbb{R}^d$ , one can obtain

$$\frac{1}{B\left(\frac{d}{2}, \frac{n-d+1}{2}\right)} \int_0^\infty \frac{u^{d-1} \tilde{f}_{n,d}^*(\sigma u)}{(1+u^2)^{(n+1)/2}} du = b_{n,d-1}(\sigma^2) + b_{n,d-3}(\sigma^2) + \dots$$

by a slight simplification of the proof of Theorem 1.6; see Remark 5.1.

## 2. Proof of Theorem 1.1

### 2.1. Reduction to intrinsic volumes

We can replace  $X$  by  $-X$  because by the symmetry of the Gaussian distribution

$$\mathbb{P}[\sigma X \notin \text{Conv}(X_1, \dots, X_n)] = \mathbb{P}[-\sigma X \notin \text{Conv}(X_1, \dots, X_n)].$$

Clearly,  $-\sigma X \notin \text{Conv}(X_1, \dots, X_n)$  if and only if  $0 \notin \text{Conv}(X_1 + \sigma X, \dots, X_n + \sigma X)$ . This, in turn, is equivalent to the following condition:

$$\alpha_1 X_1 + \dots + \alpha_n X_n + (\alpha_1 + \dots + \alpha_n) \sigma X = 0, \quad \alpha_1, \dots, \alpha_n \geq 0 \implies \alpha_1 = \dots = \alpha_n = 0.$$

To interpret this geometrically, we consider the following convex cone in the space  $\mathbb{R}^{n+1}$ :

$$C := \{(\alpha_1, \dots, \alpha_n, (\alpha_1 + \dots + \alpha_n)\sigma) : \alpha_1, \dots, \alpha_n \geq 0\}. \quad (2.1)$$

This cone is spanned by  $e_1 + \sigma e_{n+1}, \dots, e_n + \sigma e_{n+1}$  (where  $e_1, \dots, e_{n+1}$  is the standard basis of  $\mathbb{R}^{n+1}$ ) and is therefore isometric to the cone  $C_n(\sigma^2)$  introduced in (1.14). Let also  $U$  be a random linear subspace of  $\mathbb{R}^{n+1}$  given by

$$U = \{(y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} : y_1 X_1 + \dots + y_n X_n + y_{n+1} X = 0\}.$$

Observe that since  $X_1, \dots, X_n, X$  are independent, identically distributed (i.i.d.) standard Gaussian random vectors on  $\mathbb{R}^d$ , where  $d < n$ , the linear space  $U$  almost surely (a.s.) has codimension  $d$  and is uniformly distributed on the corresponding linear Grassmannian. The above discussion shows that

$$\mathbb{P}[\sigma X \notin \text{Conv}(X_1, \dots, X_n)] = \mathbb{P}[U \cap C = \{0\}]. \quad (2.2)$$

The next result, known as the conic Crofton formula [28, pp. 261–262] or [4, Corollary 5.2], is of major importance for us.

**Theorem 2.1.** *Let  $C \subset \mathbb{R}^N$  be a convex polyhedral cone which is not a linear subspace. If  $U \subset \mathbb{R}^N$  is a uniformly distributed linear subspace of codimension  $d$ , then*

$$\mathbb{P}[C \cap U = \{0\}] = 2(v_{d-1}(C) + v_{d-3}(C) + \dots),$$

where  $v_0(C), \dots, v_N(C)$  are the conic intrinsic volumes of  $C$  given by

$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(F) \alpha(N_F(C)). \quad (2.3)$$

Combining (2.2) with the conic Crofton formula, we obtain

$$\mathbb{P}[\sigma X \notin \text{Conv}(X_1, \dots, X_n)] = \mathbb{P}[U \cap C = \{0\}] = 2(v_{d-1}(C) + v_{d-3}(C) + \dots).$$

In the following, we shall show that the number of  $k$ -faces of  $C = C_n(\sigma^2)$  is  $\binom{n}{k}$ , and for every  $k$ -face  $F \in \mathcal{F}_k(C)$  we have

$$\alpha(F) = g_k \left( -\frac{\sigma^2}{1 + k\sigma^2} \right), \quad \alpha(N_F(C)) = g_{n-k} \left( \frac{\sigma^2}{1 + k\sigma^2} \right), \quad (2.4)$$

where  $g_k(r)$  is as in Section 1.2. This will prove Theorem 1.1.

## 2.2. The polar cone

The *polar cone* of a convex cone  $D \subset \mathbb{R}^N$  is defined by

$$D^\circ = \{x \in \mathbb{R}^N : \langle x, y \rangle \leq 0 \text{ for all } y \in D\}.$$

**Proposition 2.1.** *Let  $r > -1/n$ . The polar cone of  $C_n(r)$  taken with respect to the ambient space  $L(C_n(r))$  is isometric to  $C_n(-\frac{r}{1+nr})$ . That is to say, there is an orthogonal transformation  $O: \mathbb{R}^N \rightarrow \mathbb{R}^N$  such that*

$$O(C_n^\circ(r) \cap L(C_n(r))) = C_n\left(-\frac{r}{1+nr}\right).$$

*Proof.* Since  $D^\circ = D$  and since the transformation  $r \mapsto -\frac{r}{1+nr}$  is an involution, it suffices to prove the proposition for  $r = \sigma^2 \geq 0$ . Since we work in the linear hull of the cone, there is no restriction of generality in assuming that it has the form  $C = C_n(\sigma^2)$  given in (2.1). Thus,  $C$  is spanned by the vectors  $u_1, \dots, u_n$  given by  $u_i = e_i + \sigma e_{n+1} \in \mathbb{R}^{n+1}$ ,  $1 \leq i \leq n$ . The linear space spanned by  $u_1, \dots, u_n$  is

$$L(C) = \{(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{R}^{n+1} : \alpha_{n+1} = \sigma(\alpha_1 + \dots + \alpha_n)\}.$$

The polar cone of  $C = C_n(\sigma^2)$  taken with respect to  $L(C)$  as the ambient space is

$$C^\circ \cap L(C) = \{(\alpha_1, \dots, \alpha_{n+1}) \in L(C) : \alpha_1 + \sigma\alpha_{n+1} \leq 0, \dots, \alpha_n + \sigma\alpha_{n+1} \leq 0\}.$$

The lineality space of a cone  $D$  is defined as  $D \cap (-D)$ . The lineality space of  $C^\circ \cap L(C)$  is trivial, namely

$$\{(\alpha_1, \dots, \alpha_{n+1}) \in L(C) : \alpha_1 + \sigma\alpha_{n+1} = 0, \dots, \alpha_n + \sigma\alpha_{n+1} = 0\} = \{0\}.$$

It follows that the cone  $C^\circ \cap L(C)$  is spanned by its one-dimensional faces. These are obtained by turning all inequalities in the definition of the cone into equalities, except one. For example, one of the one-dimensional faces is given by

$$R_1 = \{(\alpha_1, \dots, \alpha_{n+1}) \in L(C) : \alpha_1 + \sigma\alpha_{n+1} \leq 0, \alpha_2 + \sigma\alpha_{n+1} = 0, \dots, \alpha_n + \sigma\alpha_{n+1} = 0\}.$$

Taking  $\alpha_{n+1} = \sigma/(1 + n\sigma^2)$ , we obtain that  $R_1$  is a ray spanned by the vector

$$\left(1 - \frac{\sigma^2}{1 + n\sigma^2}, -\frac{\sigma^2}{1 + n\sigma^2}, \dots, -\frac{\sigma^2}{1 + n\sigma^2}, \frac{\sigma}{1 + n\sigma^2}\right),$$

where the value of the first coordinate was computed using the linear relation in the definition of  $L(C)$ . Thus, the cone  $C^\circ \cap L(C)$  is spanned by the vectors

$$v_i := e_i - \frac{\sigma^2}{1 + n\sigma^2}(e_1 + \dots + e_n) + \frac{\sigma}{1 + n\sigma^2}e_{n+1}, \quad i = 1, \dots, n. \quad (2.5)$$

It is easy to verify that

$$\langle v_i, v_j \rangle = \begin{cases} 1 - \frac{\sigma^2}{1 + n\sigma^2} & \text{if } i = j, \\ -\frac{\sigma^2}{1 + n\sigma^2} & \text{if } i \neq j. \end{cases} \quad (2.6)$$

Thus, the cone spanned by  $v_1, \dots, v_n$  is isometric to  $C_n(-\frac{\sigma^2}{1 + n\sigma^2})$ .

### 2.3. Proof of Proposition 1.1

We prove that

$$\alpha(C_n(r)) = g_n \left( -\frac{r}{1+nr} \right).$$

Consider the cone  $D \subset \mathbb{R}^N$  spanned by the vectors  $v_1, \dots, v_n$  such that

$$\langle v_i, v_j \rangle = \begin{cases} 1 - \frac{r}{1+nr} & \text{if } i=j, \\ -\frac{r}{1+nr} & \text{if } i \neq j. \end{cases}$$

Then  $D$  is isometric to  $C_n(-\frac{r}{1+nr})$ . The polar cone is given by

$$D^\circ = \{x \in \mathbb{R}^N : \langle x, v_1 \rangle \leq 0, \dots, \langle x, v_n \rangle \leq 0\}.$$

Let  $\xi$  be a standard normal random vector on  $\mathbb{R}^N$ . Then the solid angle of  $D^\circ$  is given by

$$\alpha(D^\circ) = \mathbb{P}[\xi \in D^\circ] = \mathbb{P}[\langle \xi, v_1 \rangle \leq 0, \dots, \langle \xi, v_n \rangle \leq 0].$$

Introducing the random variables  $\eta_i := \langle \xi, v_i \rangle$ ,  $i = 1, \dots, n$ , we observe that the random vector  $(\eta_1, \dots, \eta_n)$  is zero-mean Gaussian with covariances given by

$$\text{Cov}(\eta_i, \eta_j) = \langle v_i, v_j \rangle = \begin{cases} 1 - \frac{r}{1+nr} & \text{if } i=j, \\ -\frac{r}{1+nr} & \text{if } i \neq j. \end{cases} \quad (2.7)$$

Hence,  $\alpha(D^\circ) = \mathbb{P}[\eta_1 \leq 0, \dots, \eta_n \leq 0] = g_n \left( -\frac{r}{1+nr} \right)$  by definition of  $g_n$ . On the other hand,  $D^\circ$  is the direct sum of  $L(D) \cap D^\circ$  and the orthogonal complement of  $L(D)$ . It follows that

$$\alpha(L(D) \cap D^\circ) = \alpha(D^\circ) = g_n \left( -\frac{r}{1+nr} \right).$$

By Proposition 2.1,  $L(D) \cap D^\circ$  is isometric to  $C_n(r)$ , thus completing the proof.  $\square$

### 2.4. Internal and normal angles: proof of (2.4)

Recall that  $C$  is a cone given by (2.1) and that the linear hull of  $C$  is a codimension-1 linear subspace of  $\mathbb{R}^n$  given by

$$L(C) = \{(\alpha_1, \dots, \alpha_n, (\alpha_1 + \dots + \alpha_n)\sigma) : \alpha_1, \dots, \alpha_n \in \mathbb{R}\}. \quad (2.8)$$

Inside  $L(C)$ , the convex cone  $C$  is defined by the inequalities  $\alpha_1 \geq 0, \dots, \alpha_n \geq 0$ . The  $k$ -faces of  $C$  are obtained by turning  $n-k$  of these inequalities into equalities; therefore the number of  $k$ -faces is  $\binom{n}{k}$ . Without restriction of generality, we consider a  $k$ -face  $F$  of the form

$$F = \{(\alpha_1, \dots, \alpha_k, \underbrace{0, \dots, 0}_{n-k}, (\alpha_1 + \dots + \alpha_k)\sigma) : \alpha_1, \dots, \alpha_k \geq 0\}. \quad (2.9)$$

Since  $F$  is isometric to  $C_k(\sigma^2)$ , Proposition 1.1 yields the following formula for its solid angle:

$$\alpha(F) = \alpha(C_k(\sigma^2)) = g_k \left( -\frac{\sigma^2}{1+k\sigma^2} \right).$$



To compute  $\alpha(N_F(C))$ , observe that by the polar correspondence,  $N_F(C) \cap L(C)$  is some  $(n-k)$ -dimensional face of the polar cone  $C^\circ \cap L(C)$ . The latter cone is isometric to  $C_n(-\frac{\sigma^2}{1+n\sigma^2})$  by Proposition 2.1. Since all  $(n-k)$ -dimensional faces of  $C_n(-\frac{\sigma^2}{1+n\sigma^2})$  are isometric to  $C_{n-k}(-\frac{\sigma^2}{1+n\sigma^2})$ , we can apply Proposition 1.1 to obtain that

$$\begin{aligned}\alpha(N_F(C)) &= \alpha(N_F(C) \cap L(C)) = \alpha\left(C_{n-k}\left(-\frac{\sigma^2}{1+n\sigma^2}\right)\right) \\ &= g_{n-k}\left(-\frac{-\frac{\sigma^2}{1+n\sigma^2}}{1-\frac{(n-k)\sigma^2}{1+n\sigma^2}}\right) = g_{n-k}\left(\frac{\sigma^2}{1+k\sigma^2}\right).\end{aligned}$$

This completes the proof of (2.4) and of Theorem 1.1.  $\square$

## 2.5. Proof of Proposition 1.2

By symmetry, we may consider the face of the form  $F = \text{Conv}(e_1, \dots, e_k)$ . It follows from (1.17) that the tangent cone is given by

$$T_F(\Delta_n) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_1 + \dots + \alpha_n = 0, \alpha_{k+1} \geq 0, \dots, \alpha_n \geq 0\}.$$

Thus,  $T_F(\Delta_n)$  is a direct orthogonal sum of the linear subspace  $L_{n,k}$  given by  $\alpha_1 + \dots + \alpha_k = 0, \alpha_{k+1} = \dots = \alpha_n = 0$  (which is the lineality space of the cone) and the cone  $D_{n,k} = \text{pos}(u_1, \dots, u_{n-k})$  spanned by the  $n-k$  vectors

$$u_i := -(e_1 + \dots + e_k)/k + e_{i+k}, \quad i = 1, \dots, n-k.$$

The scalar products of these vectors are given by

$$\langle u_i, u_j \rangle = \begin{cases} 1 + \frac{1}{k} & \text{if } i = j, \\ \frac{1}{k} & \text{if } i \neq j. \end{cases}$$

Hence, the cone  $D_{n,k}$  is isometric to  $C_{n-k}(1/k)$ . From Proposition 1.1 we deduce that the solid angle of  $T_F(\Delta_n)$  is  $g_{n-k}(-1/n)$ .

The normal cone  $N_F(\Delta_n) = T_F^\circ(\Delta_n)$  is the direct orthogonal sum of the line  $\{\alpha_1 = \dots = \alpha_n\}$  and the polar cone of  $D_{n,k}$  taken with respect to the ambient space  $L(D_{n,k})$ . The latter cone is isometric to  $C_{n-k}(-1/n)$  by Proposition 2.1. From Proposition 1.1 we deduce that the solid angle of  $N_F(\Delta_n)$  equals  $g_{n-k}(1/k)$ .  $\square$

## 2.6. Proof of Proposition 1.3

If  $r = \sigma^2 \geq 0$ , then the proof follows immediately from (2.4). Let  $r \in (-\frac{1}{n}, 0)$ . For a cone  $D \subset \mathbb{R}^N$  we have the relation  $v_k(D) = v_{N-k}(D^\circ)$ ; see [4, Section 2.2]. Applying this relation to the cone  $C_n(r)$  in the ambient space  $L(C_n(r))$  and recalling Proposition 2.1, we obtain

$$v_k(C_n(r)) = v_{n-k}\left(C_n\left(-\frac{r}{1+nr}\right)\right) = b_{n,n-k}\left(-\frac{r}{1+nr}\right),$$

where the last step follows from the already established part of Proposition 1.3 and the fact that  $-\frac{r}{1+nr} > 0$ . Using the definition of  $b_{n,n-k}$ , we obtain

$$\begin{aligned} b_{n,n-k} \left( -\frac{r}{1+nr} \right) &= \binom{n}{n-k} g_{n-k} \left( -\frac{\frac{r}{1+nr}}{1 - \frac{(n-k)r}{1+nr}} \right) g_k \left( \frac{-\frac{r}{1+nr}}{1 - \frac{(n-k)r}{1+nr}} \right) \\ &= \binom{n}{k} g_k \left( -\frac{r}{1+kr} \right) g_{n-k} \left( \frac{r}{1+kr} \right) = b_{n,k}(r), \end{aligned}$$

which proves the claimed formula.  $\square$

### 3. Properties of $g_n$

#### 3.1. Proof of Proposition 1.4

In the following let  $(\eta_1, \dots, \eta_n)$  be a zero-mean Gaussian vector whose covariance matrix  $\Sigma = (r_{ij})_{i,j=1}^n$  is given by

$$r_{ij} = \text{Cov}(\eta_i, \eta_j) = \begin{cases} 1+r & \text{if } i=j, \\ r & \text{if } i \neq j. \end{cases}$$

Using the inequality between the arithmetic and quadratic means, it is easy to check that this matrix is positive semidefinite for  $r \geq -1/n$ . Recall that by definition

$$g_n(r) = \mathbb{P}[\eta_1 < 0, \dots, \eta_n < 0] = \mathbb{P}[\eta_1 > 0, \dots, \eta_n > 0], \quad r \geq -1/n. \quad (3.1)$$

*Proof of (a).* Let  $r > -1/n$  be real. (The case  $r = -1/n$  can be then deduced from the continuity of  $g_n$  at  $-1/n$ .) It is straightforward to check that  $\det \Sigma = 1 + nr > 0$  and that the inverse matrix  $\Sigma^{-1} = (s_{ij})_{i,j=1}^n$  is given by

$$s_{ij} = \begin{cases} 1 - \frac{r}{1+nr} & \text{if } i=j, \\ -\frac{r}{1+nr} & \text{if } i \neq j. \end{cases}$$

Using (3.1) and the formula for the multivariate Gaussian density, we obtain, for all  $r > -1/n$ ,

$$g_n(r) = \frac{1}{(2\pi)^{n/2} \sqrt{1+nr}} \int_{(0,\infty)^n} \exp \left\{ \frac{r}{2(1+nr)} \left( \sum_{j=1}^n x_j \right)^2 - \frac{1}{2} \sum_{j=1}^n x_j^2 \right\} dx_1 \dots dx_n. \quad (3.2)$$

The integral converges for complex  $r$  satisfying  $\text{Re} \frac{r}{1+nr} < \frac{1}{n}$ , which is equivalent to  $\text{Re } r > -\frac{1}{n}$ . Indeed, by the inequality  $(\sum_{j=1}^n x_j)^2 \leq n \sum_{j=1}^n x_j^2$ , we have

$$\text{Re} \left( \frac{r}{2(1+nr)} \left( \sum_{j=1}^n x_j \right)^2 - \frac{1}{2} \sum_{j=1}^n x_j^2 \right) \leq \frac{1}{2} \left( \text{Re} \frac{r}{(1+nr)} - \frac{1}{n} \right) \left( \sum_{j=1}^n x_j \right)^2.$$

Hence the right-hand side of (3.2) defines an analytic function of  $r$  in the half-plane  $\operatorname{Re} r > -\frac{1}{n}$ . In particular,  $g_n(r)$  has an analytic continuation to this half-plane.

Next we prove that for all  $r > 0$ ,

$$g_n(r) = \frac{1}{\sqrt{2\pi}} \int_0^\infty (\Phi^n(\sqrt{rx}) + \Phi^n(-\sqrt{rx})) e^{-x^2/2} dx. \quad (3.3)$$

Let  $\xi, \xi_1, \dots, \xi_n$  be i.i.d. standard Gaussian random variables. We have a distributional representation  $\eta_k = \xi_k - \sqrt{r}\xi$ ,  $k = 1, \dots, n$ . It follows that

$$\begin{aligned} g_n(r) &= \mathbb{P}[\eta_1 < 0, \dots, \eta_n < 0] \\ &= \mathbb{P}[\xi_1 < \sqrt{r}\xi, \dots, \xi_n < \sqrt{r}\xi] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi^n(\sqrt{rx}) e^{-x^2/2} dx, \end{aligned} \quad (3.4)$$

which yields (3.3) after splitting the integral.

It remains to prove that the right-hand side of (3.3) is an analytic function of  $r$  in the half-plane  $\operatorname{Re} r > -\frac{1}{n}$ , which would imply that (3.3) holds in this half-plane by the uniqueness of the analytic continuation. First of all, observe that for every fixed  $x > 0$ , the expression  $\Phi^n(zx) + \Phi^n(-zx)$  is an analytic function of  $z \in \mathbb{C}$  which remains invariant under the substitution  $z \mapsto -z$ . Hence it can be written as an everywhere convergent Taylor series in even powers of  $z$ . It follows that for every fixed  $x > 0$ , the expression  $(\Phi^n(\sqrt{rx}) + \Phi^n(-\sqrt{rx}))$  defines an analytic function of  $r \in \mathbb{C}$ . To prove the analyticity of the integral on the right-hand side of (3.3), we argue as follows. We have the estimate

$$|\Phi(z)| \leq C \max\{1, |e^{-z^2/2}|\} = C \max\{1, e^{-\operatorname{Re}(z^2)/2}\}, \quad z \in \mathbb{C}.$$

It follows that

$$|\Phi^n(\sqrt{rx}) + \Phi^n(-\sqrt{rx})| e^{-x^2/2} \leq 2C \max\{e^{-x^2/2}, e^{-\operatorname{Re}(1+nr)x^2/2}\}.$$

By the dominated convergence theorem, the right-hand side of (3.3) is a continuous function of  $r$  on the half-plane  $\operatorname{Re} r > -\frac{1}{n}$ . Moreover, by Fubini's theorem and by the analyticity of  $(\Phi^n(\sqrt{rx}) + \Phi^n(-\sqrt{rx}))$ , the integral of the right-hand side of (3.3) along any triangular contour vanishes. By Morera's theorem, the right-hand side of (3.3) is an analytic function in the half-plane  $\operatorname{Re} r > -\frac{1}{n}$ . By the uniqueness principle for analytic functions, (3.3) must hold in this half-plane.

*Proof of (b).* By analyticity, it suffices to consider  $r > 0$ . Differentiating under the sign of the integral in (3.4), we obtain

$$\begin{aligned} g'_n(r) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} n\Phi^{n-1}(\sqrt{rx}) \frac{1}{\sqrt{2\pi}} e^{-rx^2/2} \frac{x}{2\sqrt{r}} e^{-x^2/2} dx \\ &= \frac{n}{4\pi\sqrt{r}} \int_{-\infty}^{+\infty} \Phi^{n-1}(\sqrt{rx}) x e^{-(r+1)x^2/2} dx. \end{aligned}$$

Writing  $xe^{-(r+1)x^2/2}dx = -\frac{1}{r+1}de^{-(r+1)x^2/2}$  and integrating by parts yields

$$\begin{aligned} g'_n(r) &= \frac{n}{4\pi\sqrt{r}} \int_{-\infty}^{+\infty} (n-1)\Phi^{n-2}(\sqrt{r}x) \frac{1}{\sqrt{2\pi}} e^{-rx^2/2} \sqrt{r} \cdot \frac{1}{r+1} e^{-(r+1)x^2/2} dx \\ &= \frac{n(n-1)}{4\pi(r+1)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi^{n-2}(\sqrt{r}x) e^{-(2r+1)x^2/2} dx. \end{aligned}$$

Finally, making the change of variables  $y := \sqrt{2r+1}x$ , we arrive at

$$\begin{aligned} g'_n(r) &= \frac{n(n-1)}{4\pi(r+1)\sqrt{2r+1}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi^{n-2}\left(\sqrt{\frac{r}{2r+1}}y\right) e^{-y^2/2} dy \\ &= \frac{n(n-1)}{4\pi(r+1)\sqrt{2r+1}} g_{n-2}\left(\frac{r}{2r+1}\right). \end{aligned}$$

*Proof of (c).* By definition,  $g_1(r) = \mathbb{P}[\eta_1 < 0] = \frac{1}{2}$ , since  $\eta_1$  is centered Gaussian.

*Proof of (d) and (e).* In the case  $r = 0$ , the random variables  $\eta_1, \dots, \eta_n$  are independent standard Gaussian and hence

$$g_n(0) = \mathbb{P}[\eta_1 < 0, \dots, \eta_n < 0] = 2^{-n}.$$

In the case  $r = 1$ , we have a distributional representation  $\eta_i = \xi_i - \xi$ , where  $\xi, \xi_1, \dots, \xi_n$  are independent standard Gaussian. Hence

$$g_n(1) = \mathbb{P}[\xi_1 - \xi < 0, \dots, \xi_n - \xi < 0] = \mathbb{P}[\max\{\xi_1, \dots, \xi_n\} < \xi] = \frac{1}{n+1}$$

because any of the values  $\xi, \xi_1, \dots, \xi_n$  can be the maximum with the same probability.

To prove that  $\lim_{r \rightarrow +\infty} g_n(r) = \frac{1}{2}$ , use (3.3) together with the relation

$$\lim_{r \rightarrow +\infty} (\Phi^n(\sqrt{r}x) + \Phi^n(-\sqrt{r}x))e^{-x^2/2} = e^{-x^2/2}, \quad x \geq 0,$$

and the dominated convergence theorem.

Finally, in the case  $r = -1/n$  we have the linear relation  $\eta_1 + \dots + \eta_n = 0$  (which can be verified by showing that the variance of  $\eta_1 + \dots + \eta_n$  vanishes); hence  $g_n(-\frac{1}{n}) = 0$ .

*Proof of (f).* Let  $n = 2$ . Introduce the variables  $\eta_1^* := \eta_1/\sqrt{r+1}$  and  $\eta_2^* := \eta_2/\sqrt{r+1}$  which have joint Gaussian law with unit variances and covariance  $r/(1+r)$ . It follows that

$$g_2(r) = \mathbb{P}[\eta_1 < 0, \eta_2 < 0] = \mathbb{P}[\eta_1^* < 0, \eta_2^* < 0] = \frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{r}{1+r},$$

by the well-known Sheppard formula for the quadrant probability of a bivariate Gaussian density; see [7, p. 121] and the references therein.

An alternative proof of this identity is based on Part (b). We give only the proof for  $g_3$ , since the proof for  $g_2$  is similar. By Part (b),  $g_3$  satisfies the differential equation  $g'_3(r) = \frac{3}{4\pi} \frac{1}{(r+1)\sqrt{2r+1}}$  together with the initial condition  $g_3(0) = 1/8$ . It is easy to check that  $g_3(r) = \frac{1}{8} + \frac{3}{4\pi} \arcsin \frac{r}{1+r}$  is the solution.

*Proof of (g).* Using (3.2) with  $r = -\frac{1}{n} + \varepsilon$  and then introducing the variables  $y_i := x_i/\sqrt{\varepsilon}$  yields

$$\begin{aligned} g_n\left(-\frac{1}{n} + \varepsilon\right) &= \frac{1}{(2\pi)^{n/2}\sqrt{n\varepsilon}} \int_{(0,\infty)^n} \exp\left\{\frac{-\frac{1}{n} + \varepsilon}{2\varepsilon n} \left(\sum_{i=1}^n x_i\right)^2 - \frac{1}{2} \sum_{i=1}^n x_i^2\right\} dx_1 \dots dx_n \\ &= \frac{\varepsilon^{n/2}}{(2\pi)^{n/2}\sqrt{n\varepsilon}} \int_{(0,\infty)^n} \exp\left\{\frac{-\frac{1}{n} + \varepsilon}{2n} \left(\sum_{i=1}^n y_i\right)^2 - \frac{\varepsilon}{2} \sum_{i=1}^n y_i^2\right\} dy_1 \dots dy_n \\ &\sim \frac{\varepsilon^{(n-1)/2}}{(2\pi)^{n/2}\sqrt{n\varepsilon}} \int_{(0,\infty)^n} \exp\left\{-\frac{1}{2n^2} \left(\sum_{i=1}^n y_i\right)^2\right\} dy_1 \dots dy_n \end{aligned}$$

as  $\varepsilon \downarrow 0$ . The volume of the simplex  $\{y_1, \dots, y_n \geq 0, y_1 + \dots + y_n \leq s\}$  is  $s^n/n!$ . Hence the integral on the right-hand side equals

$$\int_0^\infty e^{-s^2/(2n^2)} d\left(\frac{s^n}{n!}\right) = \int_0^\infty e^{-s^2/(2n^2)} \frac{s^{n-1} ds}{(n-1)!} = 2^{\frac{n}{2}-1} n^n \Gamma(n/2) / \Gamma(n),$$

which completes the proof of (g).

*Proof of (h).* The functions  $g_0(r) = 1$  and  $g_1(r) = 1/2$  are defined on the whole complex plane. Assume, by induction, that  $g_{2n-2}$  and  $g_{2n-1}$  are defined as multivalued analytic functions everywhere outside the set  $\{-1/k: k = 1, \dots, 2n-2\}$ . In order to define  $g_{2n}$  and  $g_{2n+1}$  we use the differential equations from Part (b):

$$\begin{aligned} g'_{2n}(r) &= \frac{n(2n-1)}{2\pi(r+1)\sqrt{2r+1}} g_{2n-2}\left(\frac{r}{2r+1}\right), \\ g'_{2n+1}(r) &= \frac{n(2n+1)}{2\pi(r+1)\sqrt{2r+1}} g_{2n-1}\left(\frac{r}{2r+1}\right). \end{aligned}$$

It is easy to check that  $\frac{r}{2r+1} \in \{-1/k: k = 1, \dots, 2n-2\}$  if and only if  $r \in \{-\frac{1}{3}, -\frac{1}{4}, \dots, -\frac{1}{2n}\}$ . Hence the right-hand sides of the differential equations are defined as analytic functions on some unramified cover of  $\mathbb{C} \setminus \{-1/k: k = 1, \dots, 2n\}$ , and we can define  $g_{2n}$  and  $g_{2n+1}$  by path integration.

### 3.2. Proof of Proposition 1.5

For  $m \in \mathbb{N}_0$  and  $n = m+2, m+3, \dots$  write

$$I(m, n) := \int_{-\infty}^{+\infty} y^m \left( \Phi(iy) e^{-y^2/2} \right)^n dy.$$

We have to show that  $I(m, n) = 0$ . From (1.20) we already know that  $I(0, n) = 0$  for all  $n = 2, 3, \dots$ , which is the basis of our induction. Define also  $I(-1, n) = 0$  for  $n \in \mathbb{N}$ . It follows from (1.6) that

$$\frac{d}{dy} \Phi(iy) = \frac{i}{\sqrt{2\pi}} e^{y^2/2}.$$

Performing partial integration, we can write

$$\begin{aligned} I(m, n) &= -\frac{i\sqrt{2\pi}}{n+1} \int_{-\infty}^{+\infty} y^m e^{-(n+1)y^2/2} d\Phi^{n+1}(iy) \\ &= \frac{i\sqrt{2\pi}}{n+1} \int_{-\infty}^{+\infty} \Phi^{n+1}(iy) \left( my^{m-1} e^{-(n+1)y^2/2} - (n+1)y^{m+1} e^{-(n+1)y^2/2} \right) dy \\ &= \frac{i\sqrt{2\pi}}{n+1} (mI(m-1, n+1) - (n+1)I(m+1, n+1)). \end{aligned}$$

Observe that this identity is true also for  $m=0$ . Assuming by induction that we proved that  $I(l, n)=0$  for all  $l=0, 1, \dots, m$  and  $n=l+2, l+3, \dots$ , we obtain from the above identity that  $I(m+1, n+1)=0$ .

Let now  $n=m+1$ , in which case the integral diverges and we have to pass to the Cauchy principal value. Write

$$I(m, m+1) := \int_0^{+\infty} y^m \left( \left( \Phi(iy)e^{-y^2/2} \right)^{m+1} + (-1)^m \left( \Phi(-iy)e^{-y^2/2} \right)^{m+1} \right) dy.$$

We need to prove that

$$I(m, m+1) = \sqrt{\frac{\pi}{2}} \left( \frac{i}{\sqrt{2\pi}} \right)^m.$$

To treat the case  $m=0$ , we observe that  $\Phi(iy) + \Phi(-iy) = 1$  for  $y \in \mathbb{R}$  (see (1.6)), whence

$$I(0, 1) = \int_0^{+\infty} \left( \Phi(iy)e^{-y^2/2} + \Phi(-iy)e^{-y^2/2} \right) dy = \int_0^{+\infty} e^{-y^2/2} dy = \sqrt{\frac{\pi}{2}}.$$

We proceed by induction. Observe that

$$\begin{aligned} &\frac{d}{dy} \left( \Phi^{m+2}(iy) + (-1)^{m+1} \Phi^{m+2}(-iy) \right) \\ &= (m+2) \left( \Phi^{m+1}(iy) + (-1)^m \Phi^{m+1}(-iy) \right) \frac{i}{\sqrt{2\pi}} e^{y^2/2}. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} I(m, m+1) &= -\frac{i\sqrt{2\pi}}{m+2} \int_0^{+\infty} y^m e^{-(m+2)y^2/2} d \left( \Phi^{m+2}(iy) + (-1)^{m+1} \Phi^{m+2}(-iy) \right) \\ &= \frac{i\sqrt{2\pi}}{m+2} \int_0^{+\infty} \left( \Phi^{m+2}(iy) + (-1)^{m+1} \Phi^{m+2}(-iy) \right) \left( my^{m-1} - (m+2)y^{m+1} \right) e^{-(m+2)y^2/2} dy \\ &= \frac{i\sqrt{2\pi}}{m+2} (mI(m-1, m+2) - (m+2)I(m+1, m+2)). \end{aligned}$$

But we already know that  $mI(m-1, m+2)=0$ , whence  $I(m, m+1) = -i\sqrt{2\pi}I(m+1, m+2)$  for all  $m \in \mathbb{N}_0$ , and the claim follows by induction.  $\square$

#### 4. Inverting the Laplace transforms

##### 4.1. Proof of Corollary 1.1

Conditioning on the event  $|X| = r$  and noting that  $|X|$  has a  $\chi$ -distribution with  $d$  degrees of freedom, we can write

$$\begin{aligned}\mathbb{P}[\sigma X \notin \text{Conv}(X_1, \dots, X_n)] &= \int_0^\infty f_{n,d}(\sigma r) \frac{2^{1-(d/2)}}{\Gamma(d/2)} r^{d-1} e^{-r^2/2} dr \\ &= \frac{1}{\Gamma(d/2)\sigma^d} \int_0^\infty f_{n,d}(\sqrt{2u}) u^{(d/2)-1} e^{-u/\sigma^2} du,\end{aligned}$$

where we made the change of variables  $\sigma r = \sqrt{2u}$ ,  $dr = \sigma^{-1} du / \sqrt{2u}$ . Taking  $\lambda := 1/\sigma^2$  and applying Theorem 1.1, we obtain

$$\frac{\lambda^{d/2}}{\Gamma(d/2)} \int_0^\infty f_{n,d}(\sqrt{2u}) u^{(d/2)-1} e^{-\lambda u} du = 2(b_{n,d-1}(1/\lambda) + b_{n,d-3}(1/\lambda) + \dots),$$

which proves the theorem.  $\square$

##### 4.2. Proof of Theorem 1.2

From Corollary 1.1 and Wendel's formula (1.11) we know that

$$\begin{aligned}\int_0^\infty (f_{n,d}(\sqrt{2u}) - f_{n,d}(0)) u^{(d/2)-1} e^{-\lambda u} du \\ = 2\Gamma(d/2)\lambda^{-d/2} \left( b_{n,d-1}(1/\lambda) - \frac{1}{2^n} \binom{n}{d-1} + b_{n,d-3}(1/\lambda) - \frac{1}{2^n} \binom{n}{d-3} + \dots \right).\end{aligned}$$

By the uniqueness of the Laplace transform, it suffices to show that

$$\int_0^\infty a_{n,k}(u) e^{-\lambda u} du = \Gamma(d/2)\lambda^{-d/2} \left( b_{n,k}(1/\lambda) - \frac{1}{2^n} \binom{n}{k} \right).$$

Recalling the formulas for  $a_{n,k}(u)$  and  $b_{n,k}(1/\lambda)$  (see (1.10), (1.12)), we rewrite this as

$$\begin{aligned}\int_0^\infty e^{-\lambda u} \frac{1}{\Gamma(d/2)} \left( \int_0^u e^{-vk} F'_{k,n-k}(v) (u-v)^{(d/2)-1} dv \right) du \\ = \lambda^{-d/2} \left( g_k \left( -\frac{1}{\lambda+k} \right) g_{n-k} \left( \frac{1}{\lambda+k} \right) - \frac{1}{2^n} \right).\end{aligned}$$

The inner integral on the left-hand side is the fractional Riemann–Liouville integral of order  $d/2$  of the function  $e^{-vk} F'_{k,n-k}(v)$ . Recall that the fractional integral of order  $\alpha \geq 0$  is defined by

$$J_\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_0^u f(v) (u-v)^{\alpha-1} dv,$$

and its Laplace transform is just  $\lambda^{-\alpha}$  times the Laplace transform of  $f$ :

$$\int_0^\infty J_\alpha f(u) e^{-\lambda u} du = \lambda^{-\alpha} \int_0^\infty f(u) e^{-\lambda u} du.$$

Using this property, we deduce that it is sufficient to prove that

$$\int_0^\infty e^{-\lambda u} e^{-uk} F'_{k,n-k}(u) du = g_k \left( -\frac{1}{\lambda+k} \right) g_{n-k} \left( \frac{1}{\lambda+k} \right) - \frac{1}{2^n}.$$

Writing  $\mu := \lambda + k$ , we rewrite this as

$$\int_0^\infty e^{-\mu u} F'_{k,n-k}(u) du = g_k(-1/\mu) g_{n-k}(1/\mu) - \frac{1}{2^n}.$$

Observe that  $\lim_{u \downarrow 0} F_{k,n-k}(u) = 2^{-n}$ . This follows from (1.13) by observing that

$$\Phi^{n-k}(0) + \Phi^{n-k}(0) = \frac{2}{2^{n-k}}, \quad \operatorname{Re} \Phi^k(0) = \frac{1}{2^k}, \quad \int_0^1 \frac{dw}{\sqrt{w(1-w)}} = \pi$$

and using dominated convergence. Using partial integration, we can write the above as

$$\mu \int_0^\infty e^{-\mu u} F_{k,n-k}(u) du = g_k(-1/\mu) g_{n-k}(1/\mu). \quad (4.1)$$

Recall from (1.13) that  $F_{k,n-k}(u) = \int_0^u h_{n-k}^{(1)}(w) h_k^{(2)}(u-w) dw$ , where

$$h_{n-k}^{(1)}(w) = \frac{\Phi^{n-k}(\sqrt{2w}) + \Phi^{n-k}(-\sqrt{2w})}{2\sqrt{\pi w}}, \quad h_k^{(2)}(w) = \frac{\Phi^k(i\sqrt{2w}) + \Phi^k(-i\sqrt{2w})}{2\sqrt{\pi w}}.$$

Let us compute the Laplace transforms of  $h_{n-k}^{(1)}$  and  $h_k^{(2)}$ :

$$\begin{aligned} \int_0^\infty e^{-\mu w} h_{n-k}^{(1)}(w) dw &= \int_0^\infty e^{-\mu w} \left( \Phi^{n-k}(\sqrt{2w}) + \Phi^{n-k}(-\sqrt{2w}) \right) \frac{dw}{2\sqrt{\pi w}} \\ &= \frac{1}{\sqrt{2\pi\mu}} \int_0^\infty e^{-y^2/2} \left( \Phi^{n-k} \left( \frac{y}{\sqrt{\mu}} \right) + \Phi^{n-k} \left( -\frac{y}{\sqrt{\mu}} \right) \right) dy \\ &= \mu^{-1/2} g_{n-k}(1/\mu), \end{aligned}$$

where we made the change of variables  $w = y^2/(2\mu)$  in the second step and recalled (1.19) in the last step. Arguing in an analogous way, we obtain

$$\begin{aligned} \int_0^\infty e^{-\mu w} h_k^{(2)}(w) dw &= \int_0^\infty e^{-\mu w} \left( \Phi^k(i\sqrt{2w}) + \Phi^k(-i\sqrt{2w}) \right) \frac{dw}{2\sqrt{\pi w}} \\ &= \frac{1}{\sqrt{2\pi\mu}} \int_0^\infty e^{-y^2/2} \left( \Phi^k \left( i \frac{y}{\sqrt{\mu}} \right) + \Phi^k \left( -i \frac{y}{\sqrt{\mu}} \right) \right) dy \\ &= \mu^{-1/2} g_k(-1/\mu). \end{aligned}$$

Since the Laplace transform of the convolution is the product of the Laplace transforms, we arrive at (4.1), which completes the proof.  $\square$



### 4.3. Proof of Theorem 1.5

Corollary 1.1 with  $d = 2$  states that for all  $\lambda > 0$ ,

$$\int_0^\infty f_{n,2}(\sqrt{2u})e^{-\lambda u}du = \frac{2}{\lambda}b_{n,1}(1/\lambda) = \frac{n}{\lambda}g_{n-1}\left(\frac{1}{1+\lambda}\right),$$

where we used (1.10) and the fact that  $g_1(r) = 1/2$ . Recalling the formula for  $g_{n-1}$  (see (1.19)), we arrive at

$$\begin{aligned} \int_0^\infty f_{n,2}(\sqrt{2u})e^{-\lambda u}du &= \frac{n}{\lambda\sqrt{2\pi}} \int_0^\infty \left( \Phi^{n-1}\left(\frac{x}{\sqrt{1+\lambda}}\right) + \Phi^{n-1}\left(-\frac{x}{\sqrt{1+\lambda}}\right) \right) e^{-x^2/2} dx \\ &= \frac{\sqrt{1+\lambda}}{\lambda} \cdot \frac{n}{\sqrt{2\pi}} \int_0^\infty \frac{\Phi^{n-1}(\sqrt{2z}) + \Phi^{n-1}(-\sqrt{2z})}{\sqrt{2z}} e^{-z} e^{-z\lambda} dz \\ &= \frac{\sqrt{1+\lambda}}{\lambda} \cdot \int_0^\infty f_{M_n^2/2}(z) e^{-z\lambda} dz, \end{aligned}$$

where we used the change of variables  $x/\sqrt{1+\lambda} = \sqrt{2z}$ , and  $f_{M_n^2/2}$  denotes the density of the random variable  $\frac{1}{2}M_n^2$ , namely

$$f_{M_n^2/2}(z) = \frac{d}{dz} \left( \Phi^n(\sqrt{2z}) - \Phi^n(-\sqrt{2z}) \right) = \frac{n}{\sqrt{2\pi}} \frac{\Phi^{n-1}(\sqrt{2z}) + \Phi^{n-1}(-\sqrt{2z})}{\sqrt{2z}} e^{-z},$$

where  $z > 0$ . The inverse Laplace transform of  $\sqrt{1+\lambda}/\lambda$  is given by  $(2\Phi(\sqrt{2z}) - 1) + e^{-z}/\sqrt{\pi z}$ . Observe that the first summand is just the distribution function  $F_{\xi^2/2}$  of  $\frac{1}{2}\xi^2$  (with  $\xi$  standard normal), whereas the second summand is the density  $f_{\xi^2/2} = F'_{\xi^2/2}$  of the same random variable. Since the inverse Laplace transform of a product is a convolution of the inverse Laplace transforms, we arrive at

$$\begin{aligned} f_{n,2}(\sqrt{2u}) &= \int_0^u f_{M_n^2/2}(z) F_{\xi^2/2}(u-z) dz + \int_0^u f_{M_n^2/2}(z) f_{\xi^2/2}(u-z) dz \\ &= \mathbb{P}[M_n^2 + \xi^2 \leq 2u] + \frac{d}{du} \mathbb{P}[M_n^2 + \xi^2 \leq 2u], \end{aligned}$$

which proves the first formula in Theorem 1.5. To verify the second formula, observe that by the change of variables  $x := \sqrt{2z}$  and then  $y := x/\sqrt{2u}$ ,

$$\begin{aligned} \int_0^u f_{M_n^2/2}(z) f_{\xi^2/2}(u-z) dz &= \frac{n}{\sqrt{2\pi}} \int_0^u \frac{\Phi^{n-1}(\sqrt{2z}) + \Phi^{n-1}(-\sqrt{2z})}{\sqrt{2z}} e^{-z} \frac{e^{-(u-z)}}{\sqrt{\pi}\sqrt{u-z}} dz \\ &= ne^{-u} \int_0^{\sqrt{2u}} \frac{\Phi^{n-1}(x) + \Phi^{n-1}(-x)}{\pi\sqrt{2u-x^2}} dx \\ &= ne^{-u} \int_{-\sqrt{2u}}^{\sqrt{2u}} \frac{\Phi^{n-1}(x)}{\pi\sqrt{2u-x^2}} dx \\ &= ne^{-u} \int_{-1}^1 \frac{\Phi^{n-1}(\sqrt{2u}y)}{\pi\sqrt{1-y^2}} dy. \end{aligned}$$

The integral equals  $\mathbb{P}[M_{n-1} \leq \sqrt{2u}W]$  since  $\Phi^{n-1}$  is the distribution function of  $M_{n-1}$ .  $\square$

## 5. Proof of Theorem 1.6

The proof relies strongly on the ideas of Baryshnikov and Vitale [5] combined with the Bartlett decomposition of the Gaussian matrix and Theorem 1.1.

*Step 1: Bartlett decomposition of a Gaussian matrix.* Let  $O$  be a random orthogonal matrix distributed according to the Haar probability measure on the group  $O(n)$ . Independently of  $O$ , let  $L$  be a random lower-triangular  $n \times n$  matrix with a.s. positive entries on the diagonal and the following distribution. The entries of  $L$  are independent; the squared diagonal entries have  $\chi^2$ -distributions with  $n, n-1, \dots, 1$  degrees of freedom, whereas the entries below the diagonal are standard normal. Define an  $n \times n$  matrix  $G$  by

$$G = LO.$$

It is known (Bartlett decomposition) that the entries of  $G$  are independent standard Gaussian random variables.

*Step 2: Relating the Goodman–Pollack model to the Gaussian polytope.* Consider an  $n \times (n+1)$  matrix  $S$  whose columns are the vectors  $v_1, \dots, v_{n+1}$ . Recall also that  $\Pi$  is the  $d \times n$  matrix of the projection from  $\mathbb{R}^n$  onto the linear subspace  $\mathbb{R}^d$  spanned by  $e_1, \dots, e_d$ . Note that  $\Pi$  consists of an identity matrix  $I_d$  extended by a  $d \times (n-d)$  zero matrix. Following Baryshnikov and Vitale [5], consider the  $d \times (n+1)$  matrix

$$\tilde{S} := \Pi GS = \Pi LOS = \tilde{L} \Pi OS, \quad (5.1)$$

where  $\tilde{L}$  is a lower-triangular  $d \times d$  matrix obtained from  $L$  by removing all rows and columns except the first  $d$  ones. The last equality follows from the simple identity  $\Pi L = \tilde{L} \Pi$ . It follows from the corresponding properties of  $L$  that the matrix  $\tilde{L}$  is lower-triangular, its entries are independent, the squared diagonal entries have  $\chi^2$ -distributions with  $n, n-1, \dots, n-d$  degrees of freedom, and the entries below the diagonal are standard Gaussian.

Observe that the matrices  $\tilde{L}$  and  $\Pi OS$  are stochastically independent, and the columns of the latter matrix are the vectors  $\Pi O v_1, \dots, \Pi O v_{n+1}$ , whose convex hull is the Goodman–Pollack polytope  $\mathcal{Q}_{n+1,d}$ .

Denote the columns of the matrix  $\tilde{S}$  by  $Y_{\bullet,1}, \dots, Y_{\bullet,n+1}$ . Let us write  $Y_{\bullet,j} = (Y_{1,j}, \dots, Y_{d,j})^\top$ ,  $1 \leq j \leq n+1$ . It follows from the formula  $\tilde{S} = \Pi GS$  that the random variables  $Y_{i,j}$ ,  $1 \leq i \leq d$ ,  $1 \leq j \leq n+1$ , are jointly Gaussian with mean zero. Let us compute their covariances. Writing  $G = (g_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n}$  with independent standard Gaussian entries  $g_{i,j}$ , we observe that  $\Pi G = (g_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n}$ , and hence

$$\begin{aligned} \mathbb{E}[Y_{i_1,j_1} Y_{i_2,j_2}] &= \mathbb{E} \left[ \langle (g_{i_1,1}, \dots, g_{i_1,n})^\top, v_{j_1} \rangle \langle (g_{i_2,1}, \dots, g_{i_2,n})^\top, v_{j_2} \rangle \right] \\ &= \begin{cases} 0 & \text{if } i_1 \neq i_2, \\ \langle v_{j_1}, v_{j_2} \rangle & \text{if } i_1 = i_2. \end{cases} \end{aligned}$$

By the properties of the regular simplex we have  $\langle v_{j_1}, v_{j_2} \rangle = -1/n < 0$  provided that  $j_1 \neq j_2$ . Let  $W$  be a standard Gaussian random vector on  $\mathbb{R}^d$  independent of  $L$  and  $O$  (and hence of  $G$  and  $\tilde{S}$ ). It follows that

$$\sqrt{\frac{n}{n+1}} Y_{\bullet,1} + \frac{W}{\sqrt{n+1}}, \dots, \sqrt{\frac{n}{n+1}} Y_{\bullet,n+1} + \frac{W}{\sqrt{n+1}}$$

are mutually independent standard Gaussian random vectors on  $\mathbb{R}^d$ . Their convex hull has the same distribution as the Gaussian polytope and will be denoted by  $\mathcal{P}_{n+1,d}$  for this reason. Summarizing everything, we obtain the identity

$$\begin{aligned}\tilde{L}\mathcal{Q}_{n+1,d} &= \text{Conv}(\tilde{L}\Pi Ov_1, \dots, \tilde{L}\Pi Ov_{n+1}) \\ &= \text{Conv}(Y_{\bullet,1}, \dots, Y_{\bullet,(n+1)}) = \sqrt{\frac{n+1}{n}}\mathcal{P}_{n+1,d} - \frac{W}{\sqrt{n}},\end{aligned}\quad (5.2)$$

where  $\tilde{L}$ ,  $W$ ,  $\mathcal{Q}_{n+1,d}$  are independent. The fact that  $\mathcal{P}_{n+1,d}$  can be obtained from  $\mathcal{Q}_{n+1,d}$  by a random affine transformation stochastically independent from  $\mathcal{Q}_{n+1,d}$  was proved by Baryshnikov and Vitale [5]. We rederived it because we shall need the explicit form of the affine transformation in what follows.

*Step 3: Relating the non-absorption probabilities for  $\mathcal{P}_{n+1,d}$  and  $\mathcal{Q}_{n+1,d}$ .* Let now  $X$  be a standard Gaussian vector on  $\mathbb{R}^d$  which is independent of everything else. We know from Theorem 1.1 that for every  $\sigma^2 \geq 0$ ,

$$\mathbb{P}[\sigma X \notin \mathcal{P}_{n+1,d}] = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots).$$

It follows from (5.2) that

$$\mathbb{P}\left[\sqrt{\frac{n+1}{n}}\sigma X - \frac{W}{\sqrt{n}} \notin \tilde{L}\mathcal{Q}_{n+1,d}\right] = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots).$$

Introducing the standard normal  $d$ -dimensional vector

$$\xi := \left(\frac{n+1}{n}\sigma^2 + \frac{1}{n}\right)^{-1/2} \left(\sqrt{\frac{n+1}{n}}\sigma X - \frac{W}{\sqrt{n}}\right)$$

(which is independent of  $\tilde{L}$  and  $\mathcal{Q}_{n+1,d}$ ), we can rewrite the above as

$$\mathbb{P}\left[\sqrt{\frac{n+1}{n}}\sigma^2 + \frac{1}{n} \cdot \xi \notin \tilde{L}\mathcal{Q}_{n+1,d}\right] = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots).$$

Let  $Q \in O(d)$  be a deterministic orthogonal matrix. We claim that the random set  $\tilde{L}\mathcal{Q}_{n+1,d}$  is invariant with respect to orthogonal transformations; that is,

$$Q\tilde{L}\mathcal{Q}_{n+1,d} \stackrel{d}{=} \tilde{L}\mathcal{Q}_{n+1,d}. \quad (5.3)$$

Since  $\mathcal{Q}_{n+1,d}$  is defined as the convex hull of the columns of the matrix  $\Pi OS$ , it suffices to show that  $Q\tilde{L}\Pi OS \stackrel{d}{=} \tilde{L}\Pi OS$ , or, equivalently,  $Q\Pi GS \stackrel{d}{=} \Pi GS$ ; see (5.1). Let  $Q' \in O(n)$  be a natural extension of  $Q$  from  $\mathbb{R}^d$  to  $\mathbb{R}^n$  defined by

$$Q'e_1 = Qe_1, \dots, Q'e_d = Qe_d, \quad Q'e_{d+1} = e_{d+1}, \dots, Q'e_n = e_n.$$

Then  $Q\Pi = \Pi Q'$ , and hence it suffices to show that  $\Pi Q'GS \stackrel{d}{=} \Pi GS$ . However, since the entries of  $G$  are independent standard Gaussian and the matrix  $Q'$  is orthogonal, it is easy to check that  $Q'G \stackrel{d}{=} G$ , thus proving (5.3).

Let  $R_1 := |\xi|$ , so that  $R_1^2$  has  $\chi^2$  distribution with  $d$  degrees of freedom. By the orthogonal invariance of the random set  $\tilde{L}\mathcal{Q}_{n+1,d}$ , we can replace  $\xi$  by, say,  $R_1 e_{n-d}$ , thus obtaining

$$\mathbb{P} \left[ \sqrt{\frac{n+1}{n}\sigma^2 + \frac{1}{n}} \cdot R_1 e_{n-d} \notin \tilde{L}\mathcal{Q}_{n+1,d} \right] = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots).$$

Observe that  $R_2 := |\tilde{L}^{-1}e_{n-d}|^{-1}$  satisfies  $R_2^2 \sim \chi_{n-d+1}^2$  because of the structure of the lower-triangular matrix  $\tilde{L}$ . The random set  $\mathcal{Q}_{n+1,d}$  is also orthogonally invariant (which follows from its definition); hence

$$\mathbb{P} \left[ \sqrt{\frac{n+1}{n}\sigma^2 + \frac{1}{n}} \cdot \frac{R_1}{R_2} e_1 \notin \mathcal{Q}_{n+1,d} \right] = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots).$$

The random variables  $R_1 \sim \chi_d$  and  $R_2 \sim \chi_{n-d+1}$  are independent (because  $\xi$  and  $\tilde{L}$  are independent); hence the density of  $R_1/R_2$  is given by

$$h(u) = \frac{2u^{d-1}(1+u^2)^{-(n+1)/2}}{B\left(\frac{d}{2}, \frac{n-d+1}{2}\right)}, \quad u \geq 0.$$

We can finally rewrite the above as

$$\int_0^\infty h(u) f_{n+1,d}^* \left( \sqrt{\frac{n+1}{n}\sigma^2 + \frac{1}{n}} \cdot u \right) du = 2(b_{n+1,d-1}(\sigma^2) + b_{n+1,d-3}(\sigma^2) + \dots),$$

and Theorem 1.6 follows after replacing  $n+1$  by  $n$ .  $\square$

**Remark 5.1.** If instead of the regular simplex with  $n+1$  vertices inscribed into  $\mathbb{S}^{n-1}$  we rotate and project the regular simplex  $\text{Conv}(e_1, \dots, e_n)$ , the proof simplifies. The random vectors  $Y_{\bullet,1}, \dots, Y_{\bullet,n}$  (their number is  $n$  rather than  $n+1$ ) are independent standard Gaussian and there is no need of introducing  $W$ .

### Acknowledgement

Z. K. is grateful to Alexander Marynych for suggesting the use of partial integration in the proof of Proposition 1.5.

### References

- [1] ABRAMOWITZ, M. AND STEGUN, I. (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards Applied Mathematics Series **55**). U.S. Government Printing Office, Washington.
- [2] AFFENTRANGER, F. (1991). The convex hull of random points with spherically symmetric distributions. *Rend. Semin. Mat. Torino* **49**, 359–383.
- [3] AFFENTRANGER, F. AND SCHNEIDER, R. (1992). Random projections of regular simplices. *Discrete Comput. Geom.* **7**, 219–226.
- [4] AMELUNXEN, D. AND LOTZ, M. (2017). Intrinsic volumes of polyhedral cones: a combinatorial perspective. *Discrete Comput. Geom.* **58**, 371–409.
- [5] BARYSHNIKOV, Y. M. AND VITALE, R. A. (1994). Regular simplices and Gaussian samples. *Discrete Comput. Geom.* **11**, 141–147.
- [6] BETKE, U. AND HENK, M. (1993). Intrinsic volumes and lattice points of crosspolytopes. *Monatshefte Math.* **115**, 27–33.

- [7] BINGHAM, N. H. AND DONEY, R. A. (1988). On higher-dimensional analogues of the arc-sine law. *J. Appl. Prob.* **25**, 120–131.
- [8] BÖHM, J. AND HERTEL, E. (1981). *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung* (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften (LMW), Mathematische Reihe, 70). Birkhäuser, Basel, Boston.
- [9] BÖRÖCZKY, K. AND HENK, M. (1999). Random projections of regular polytopes. *Arch. Math.* **73**, 465–473.
- [10] CARNAL, H. (1970). Die konvexe Hülle von  $n$  rotationssymmetrisch verteilten Punkten. *Z. Wahrscheinlichkeitsth.* **15**, 168–176.
- [11] DONOHO, D. L. AND TANNER, J. (2009). Counting faces of randomly projected polytopes when the projection radically lowers dimension. *J. Amer. Math. Soc.* **22**, 1–53.
- [12] DONOHO, D. L. AND TANNER, J. (2010). Counting the faces of randomly-projected hypercubes and orthants, with applications. *Discrete Comput. Geom.* **43**, 522–541.
- [13] EFRON, B. (1965). The convex hull of a random set of points. *Biometrika* **52**, 331–343.
- [14] HADWIGER, H. (1979). Gitterpunktzahl im Simplex und Wills'sche Vermutung. *Math. Ann.* **239**, 271–288.
- [15] HUG, D. (2013). Random polytopes. In *Stochastic Geometry, Spatial Statistics and Random Fields: Asymptotic Methods*, Springer, Berlin, pp. 205–238.
- [16] HUG, D., MUNSONIUS, G. O. AND REITZNER, M. (2004). Asymptotic mean values of Gaussian polytopes. *Contributions to Algebra and Geometry* **45**, 531–548.
- [17] JEWELL, N. P. AND ROMANO, J. P. (1982). Coverage problems and random convex hulls. *J. Appl. Prob.* **19**, 546–561.
- [18] JEWELL, N. P. AND ROMANO, J. P. (1985). Evaluating inclusion functionals for random convex hulls. *Z. Wahrscheinlichkeitsth.* **68**, 415–424.
- [19] KABLUCHKO, Z. AND ZAPOROZHETS, D. (2019). Expected volumes of Gaussian polytopes, external angles, and multiple order statistics. *Trans. Amer. Math. Soc.* **372**, 1709–1733.
- [20] MAJUMDAR, S. N., COMTET, A. AND RANDON-FURLING, J. (2010). Random convex hulls and extreme value statistics. *J. Statist. Phys.* **138**, 955–1009.
- [21] RAYNAUD, H. (1970). Sur l'enveloppe convexe des nuages de points aléatoires dans  $R^n$ . I. *J. Appl. Prob.* **7**, 35–48.
- [22] RÉNYI, A. AND SULANKE, R. (1963). Über die konvexe Hülle von  $n$  zufällig gewählten Punkten. *Z. Wahrscheinlichkeitsth.* **2**, 75–84.
- [23] ROGERS, C. A. (1958). The packing of equal spheres. *Proc. London Math. Soc.* **8**, 609–620.
- [24] RUBEN, H. (1954). On the moments of order statistics in samples from normal populations. *Biometrika* **41**, 200–227.
- [25] RUBEN, H. (1960). On the geometrical moments of skew-regular simplices in hyperspherical space, with some applications in geometry and mathematical statistics. *Acta Math.* **103**, 1–23.
- [26] SCHLÄFLI, L. (1950). Theorie der vielfachen Kontinuität. In *Gesammelte Mathematische Abhandlungen*, Springer, pp. 167–387.
- [27] SCHNEIDER, R. (2008). Recent results on random polytopes. *Boll. Unione Mat. Ital.* **1**, 17–39.
- [28] SCHNEIDER, R. AND WEIL, W. (2008). *Stochastic and Integral Geometry* (Probability and Its Applications). Springer, Berlin.
- [29] VERSHIK, A. M. AND SPORYSHEV, P. V. (1986). An asymptotic estimate for the average number of steps in the parametric simplex method. *Zh. Vychisl. Mat. i Mat. Fiz.* **26**, 813–826, 958.
- [30] VERSHIK, A. M. AND SPORYSHEV, P. V. (1992). Asymptotic behavior of the number of faces of random polyhedra and the neighborliness problem. *Selecta Math. Soviet.* **11**, 181–201.
- [31] WENDEL, J. G. (1962). A problem in geometric probability. *Math. Scand.* **11**, 109–111.