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SIMULTANEOUS DYNAMICAL DIOPHANTINE APPROXIMATION IN BETA EXPANSIONS

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Abstract

Let $\beta > 1$ be a real number and define the β-transformation on $[0, 1]$ by $T_\beta : x \mapsto \beta x \pmod 1$. Let $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ be two Lipschitz functions. The main result of the paper is the determination of the Hausdorff dimension of the set

$$W(f, g, \tau_1, \tau_2) = \{(x, y) \in [0, 1]^2 : |T_\beta^n x - f(x)| < \beta^{-n\tau_1(x)}, |T_\beta^n y - g(y)| < \beta^{-n\tau_2(y)} \text{ for infinitely many } n \in \mathbb{N}\},$$

where τ_1, τ_2 are two positive continuous functions with $\tau_1(x) \le \tau_2(y)$ for all $x, y \in [0, 1]$.

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1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with a consistent metric d. The orbit of a point $x \in X$ is the sequence $(x, T(x), T^2(x), \ldots)$ of elements of X. If T is ergodic with respect to the measure μ, Birkhoff's ergodic theorem implies that, for any $x_0 \in X$, almost surely,

$$\liminf_{n\to\infty} d(T^n x, x_0) = 0.$$

This result is qualitative in nature and hence a natural question is to investigate how fast the lim inf tends to zero. To this end, the focus is on the size of the set

$$W(T, \psi) := \{x \in X : d(T^n x, x_0) < \psi(n) \text{ for i.m. } n \in \mathbb{N}\},$$

where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a positive function such that $\psi(n) \to 0$ as $n \to \infty$. Here and throughout the paper, '*i.m.*' stands for 'infinitely many'. The set $W(T, \psi)$ is the dynamical analogue of the classical ψ-approximable set

$$W(\psi) := \{x \in \mathbb{R} : |x - p/q| < \psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N}\},$$

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which has been well studied (see [1, 2, 9]). It is easy to see that the set $W(T, \psi)$ can also be viewed as the collection of points in X whose T-orbit hits a shrinking target infinitely often. For background and more results on shrinking target problems, the reader is referred to [3, 4, 6, 7, 12–15].

We consider the beta-dynamical system $([0, 1], T_\beta)$ for general $\beta > 1$. Define the β-transformation on $[0, 1]$ by $T_\beta : x \mapsto \beta x \pmod 1$. It is well known that the beta-dynamical system is, in general, not a subshift of finite type with mixing properties. This causes difficulties in studying metrical questions related to β-expansions. The one-dimensional metrical theory associated with the beta-dynamical system is well studied, but hardly anything is known in higher dimensional settings.

Let $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ be two Lipschitz functions. Define the set

$$W(f, g, \tau_1, \tau_2) = \{(x, y) \in [0, 1]^2 : |T_\beta^n x - f(x)| < \beta^{-n\tau_1(x)},\\ |T_\beta^n y - g(y)| < \beta^{-n\tau_2(y)} \text{ for i.m. } n \in \mathbb{N}\}.$$

When the two Lipschitz functions take fixed values this is a shrinking target set and if they are identity functions then it is a recurrence set. The aim of this note is to make contributions in understanding the Diophantine properties in two-dimensional settings. In particular, we treat the dimensional theory for shrinking target problems and quantitative recurrence properties for β-expansions in a unified way, by investigating the dimensions of the two sets

$$\{(x, y) \in [0, 1]^2 : |T_\beta^n x - x_0| < \beta^{-n\tau_1(x)}, |T_\beta^n y - y_0| < \beta^{-n\tau_2(y)} \text{ for i.m. } n \in \mathbb{N}\}$$

and

$$\{(x, y) \in [0, 1]^2 : |T_\beta^n x - x| < \beta^{-n\tau_1(x)}, |T_\beta^n y - y| < \beta^{-n\tau_2(y)} \text{ for i.m. } n \in \mathbb{N}\}.$$

Throughout, we use the notation $\dim_{\mathcal{H}}$ to denote the Hausdorff dimension of a set.

Theorem 1.1. *Let τ_1, τ_2 be two positive continuous functions on $[0, 1]$ such that $\tau_1(x) \le \tau_2(y)$ for all $x, y \in [0, 1]$ and let $f, g : [0, 1] \to [0, 1]$ be two Lipschitz functions. Then, for any $\beta > 1$,*

$$\dim_{\mathcal{H}} W(f, g, \tau_1, \tau_2) = \min\left\{\frac{2}{1 + \tau_{1,\min}}, \frac{2 + \tau_{2,\min} - \tau_{1,\min}}{1 + \tau_{2,\min}}\right\},$$

where

$$\tau_{1,\min} = \min_{x\in[0,1]} \tau_1(x), \quad \tau_{2,\min} = \min_{y\in[0,1]} \tau_2(y).$$

Remark 1.2. In case the functions $\tau_1(x), \tau_2(y), f(x), f(y)$ are constants, $\tau_1(x) = \tau_1$, $\tau_2(y) = \tau_2$, $f(x) = x_0$, $g(y) = y_0$ for any $x, y \in [0, 1]$, the Hausdorff dimension of the corresponding set has already been determined in [8, Theorem 1.2].

The paper is organised as follows. In the next section, we recall some elementary properties of the β-expansion. The main theorem is proven in the Section 3.

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2. Preliminaries

The β-expansion of real numbers was introduced by Rényi [11] by the following algorithm. For any $\beta > 1$, let

$$T_\beta(0) := 0, \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor \quad (x \in [0, 1)), \tag{2.1}$$

where $\lfloor \xi \rfloor$ is the integer part of $\xi \in \mathbb{R}$. By taking

$$\epsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1} x \rfloor \in \mathbb{N}$$

recursively for each $n \geq 1$, every $x \in [0, 1)$ can be uniquely expanded into a finite or infinite sequence

$$x = \frac{\epsilon_1(x, \beta)}{\beta} + \frac{\epsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\epsilon_n(x, \beta)}{\beta^n} + \frac{T_\beta^n x}{\beta^n},$$

which is called the β-expansion of x, and the sequence $\{\epsilon_n(x, \beta)\}_{n \geq 1}$ is called the digit sequence of x. We also write this sequence as $\epsilon(x, \beta) = (\epsilon_1(x, \beta), \ldots, \epsilon_n(x, \beta), \ldots)$. The system $([0, 1], T_\beta)$ is called the β-dynamical system or just the β-system.

Definition 2.1. A finite or infinite sequence $(w_1, w_2, \ldots)$ is said to be admissible (with respect to the base β) if there exists an $x \in [0, 1)$ such that the digit sequence of x equals $(w_1, w_2, \ldots)$.

Let Σ_β^n denote the collection of all admissible sequences of length n and Σ_β that of all infinite admissible sequences.

Let us now turn to the infinite β-expansion of 1, which plays an important role in the study of the β-expansion. Applying the algorithm (2.1) to the number $x = 1$ gives

$$1 = \frac{\epsilon_1(1, \beta)}{\beta} + \frac{\epsilon_2(1, \beta)}{\beta^2} + \cdots + \frac{\epsilon_n(1, \beta)}{\beta^n} + \cdots.$$

If the above series is finite, that is, there exists $m \geq 1$ such that $\epsilon_m(1, \beta) \neq 0$ but $\epsilon_n(1, \beta) = 0$ for $n > m$, then β is called a simple Parry number. In this case, we write

$$\epsilon^*(1, \beta) := (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots) = (\epsilon_1(1, \beta), \ldots, \epsilon_{m-1}(1, \beta), \epsilon_m(1, \beta) - 1)^\infty,$$

where $(w)^\infty$ denotes the periodic sequence $(w, w, w, \ldots)$. If β is not a simple Parry number, we write

$$\epsilon^*(1, \beta) := (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots) = (\epsilon_1(1, \beta), \epsilon_2(1, \beta), \ldots).$$

In both cases, the sequence $(\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots)$ is called the infinite β-expansion of 1 and

$$1 = \frac{\epsilon_1^*(\beta)}{\beta} + \frac{\epsilon_2^*(\beta)}{\beta^2} + \cdots + \frac{\epsilon_n^*(\beta)}{\beta^n} + \cdots.$$

The lexicographical order $\prec$ between infinite sequences is defined as follows:

$$w = (w_1, w_2, \ldots, w_n, \ldots) \prec w' = (w_1', w_2', \ldots, w_n', \ldots)$$

if there exists $k \geq 1$ such that $w_j = w_j'$ for $1 \leq j < k$, while $w_k < w_k'$. The notation $w \preceq w'$ means that $w \prec w'$ or $w = w'$. This ordering can be extended to finite blocks by identifying a finite block $(w_1, w_2, \ldots, w_n)$ with the sequence $(w_1, w_2, \ldots, w_n, 0, 0, \ldots)$.

The following result due to Parry is a criterion for the admissibility of a sequence.

Lemma 2.2 (Parry [10]). *Let $\beta > 1$ be a real number. A nonnegative integer sequence $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$ is admissible if and only if $(\epsilon_k, \epsilon_{k+1}, \ldots) \prec (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots)$ for any $k \geq 1$.*

Lemma 2.3 (Rényi [11]). *Let $\beta > 1$. For any $n \geq 1$,*

$$\beta^n \leq \#\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1},$$

where # denotes the cardinality of a finite set.

For any $(\epsilon_1, \ldots, \epsilon_n) \in \Sigma_\beta^n$ with $n \geq 1$, the set

$$I_n(\epsilon_1, \ldots, \epsilon_n) := \{x \in [0, 1) : \epsilon_j(x, \beta) = \epsilon_j, 1 \leq j \leq n\}$$

is called an nth-order cylinder (with respect to the base β). It is a left-closed and right-open interval with the left end point

$$\frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n}$$

and of length at most β^{-n}. The unit interval can be naturally partitioned into a disjoint union of cylinders: for any $n \geq 1$,

$$[0, 1] = \bigcup_{(\epsilon_1,\ldots,\epsilon_n)\in\Sigma_\beta^n} I_n(\epsilon_1, \ldots, \epsilon_n). \tag{2.2}$$

One difficulty in studying the metric properties of β-expansions is that the length of a cylinder is not regular. It may happen that $|I_n(\epsilon_1, \ldots, \epsilon_n)| \ll \beta^{-n}$. The following notion plays an important role in overcoming this difficulty.

Definition 2.4 (Full cylinder). A cylinder $I_n(\epsilon_1, \ldots, \epsilon_n)$ is called full if it has maximal length, that is, if

$$|I_n(\epsilon_1, \ldots, \epsilon_n)| = \frac{1}{\beta^n}.$$

Correspondingly, we also call the word $(\epsilon_1, \ldots, \epsilon_n)$ a full word.

We collect some results about the distribution of full cylinders.

Lemma 2.5 (Bugeaud and Wang [5]). *For $n \geq 1$, among every $n + 1$ consecutive cylinders of order n, there exists at least one full cylinder.*

Lemma 2.6 (Wang [17, Lemma 3.1]). *For any full word $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \Sigma_\beta^n$, there exists a point $x_n^*(w)$ in the closure of $I_n(\epsilon_1, \ldots, \epsilon_n)$ such that:*

(1) $T_\beta^n x_n^*(w) = f(x_n^*(w))$ *when* $x_n^*(w) \in I_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$;
(2) $f(x_n^*(w)) = 1$ *when*

$$x_n^*(w) = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \frac{1}{\beta^n}.$$

PROOF. For any $y_0 \in [0, 1]$, define a sequence $\{y_k\}_{k\geq 1}$ recursively by

$$y_{k+1} = f\Big(\frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \frac{y_k}{\beta^n}\Big) \quad \text{for all } k \geq 0.$$

Then

$$|y_{k+1} - y_k| \leq \frac{L}{\beta^n}|y_k - y_{k-1}| \leq \cdots \leq \Big(\frac{L}{\beta^n}\Big)^k |y_1 - y_0|.$$

Thus, $\{y_k\}_{k\geq 1}$ is a Cauchy sequence and its limit, $y_n^*(w) \in [0, 1]$, satisfies

$$y_n^*(w) = f\Big(\frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \frac{y_n^*(w)}{\beta^n}\Big).$$

Let

$$x_n^*(w) = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \frac{y_n^*(w)}{\beta^n}.$$

(1) When $0 \leq y_n^*(w) < 1$,

$$x_n^*(w) \in I_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \quad \text{and} \quad T_\beta^n x_n^*(w) = y_n^*(w) = f(x_n^*(w)).$$

(2) When $y_n^*(w) = 1$,

$$x_n^*(w) = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \frac{1}{\beta^n}$$

is the right end point of $I_n(\epsilon_1, \ldots, \epsilon_n)$ and $f(x_n^*(w)) = 1$. □

The proof of Theorem 1.1 crucially relies on the following result, called the mass transference principle for lim sup sets generated by rectangles.

LEMMA 2.7 (Wang *et al.* [16]). *Let $\{x_n\}_{n\geq 1}$ be a sequence of points in the unit cube $[0, 1]^d$ with $d \geq 1$ and $\{r_n\}_{n\geq 1}$ a sequence of positive numbers tending to zero. Denote d-dimensional vectors by $\mathbf{a} = (a_1, \ldots, a_d)$. Define*

$$W_1 := \{x \in [0, 1]^d : x \in B(x_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\}$$

and, for any $\mathbf{a} = (a_1, \ldots, a_d)$ with $1 \leq a_1 \leq \cdots \leq a_d$, define

$$W_{\mathbf{a}} := \{x \in [0, 1]^d : x \in B^{\mathbf{a}}(x_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $B^{\mathbf{a}}(x, r)$ denotes a rectangle with centre x and side lengths $(r^{a_1}, \ldots, r^{a_d})$. If W_1 is of full Lebesgue measure, then

$$\dim_{\mathcal{H}} W_{\mathbf{a}} \geq \min_{1\leq j\leq d} \frac{d + ja_j - \sum_{i=1}^{j} a_j}{a_j}.$$

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3. Proof of Theorem 1.1

3.1. The upper bound. As is common in obtaining upper bounds for the Hausdorff dimension, we construct a natural cover for the lim sup set $W(f, g, \tau_1, \tau_2)$ and then show that the s-dimensional Hausdorff measure of this cover set is zero whenever $s > \dim_H W(f, g, \tau_1, \tau_2)$.

From (2.2), for any $n \in N$,

$$[0, 1] \times [0, 1] = \bigcup_{w,v \in \Sigma_\beta^n} I_n(w) \times I_n(v).$$

Obviously,

$$\begin{aligned} W(f, g, \tau_1, \tau_2) &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{(x, y) \in [0, 1]^2 : |T_\beta^n x - f(x)| < \beta^{-n\tau_1(x)}, |T_\beta^n y - g(y)| < \beta^{-n\tau_2(y)}\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v \in \Sigma_\beta^n} J_n(w) \times J_n(v), \end{aligned}$$

where

$$\begin{aligned} J_n(w) \times J_n(v) = \{x \in I_n(w) : |T_\beta^n x - f(x)| < \beta^{-n\tau_1(x)}\} \\ \times \{y \in I_n(v) : |T_\beta^n y - g(y)| < \beta^{-n\tau_2(y)}\}. \end{aligned}$$

For large n, the length of $J_n(w)$ satisfies

$$|J_n(w)| \le \frac{4}{\beta^{n(1+\tau_{1,\min})}}.$$

Denote by L the Lipschitz constant of f, that is,

$$|f(x) - f(y)| \le L|x - y| \quad \text{for } x, y \in [0, 1].$$

Then, for any $x, y \in J_n(w)$,

$$\begin{aligned} 2\beta^{-n\cdot\tau_{1,\min}} &\ge |(T_\beta^n x - f(x)) - (T_\beta^n y - f(y))| \\ &\ge |T_\beta^n x - T_\beta^n y| - |f(x) - f(y)| \\ &\ge \beta^n|x - y| - L|x - y| \\ &= (\beta^n - L)|x - y|. \end{aligned}$$

The same is true for $J_n(v)$, in other words,

$$|J_n(v)| \le \frac{4}{\beta^{n(1+\tau_{2,\min})}}.$$

The lim sup set $W(f, g, \tau_1, \tau_2)$ is defined by a collection of rectangles and there are two ways to cover a single rectangle $J_n(w) \times J_n(v)$.

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Case 1. Cover using balls with radius being the shorter side length $4/\beta^{n(1+\tau_{2,\min})}$. Then $J_n(w) \times J_n(v)$ can be covered by at most

$$\frac{\beta^{n(1+\tau_{2,\min})}}{\beta^{n(1+\tau_{1,\min})}} \leq 2\beta^{n(\tau_{2,\min}-\tau_{1,\min})}$$

balls of radius $4/\beta^{n(1+\tau_{2,\min})}$. Thus, for $s > (2 + \tau_{2,\min} - \tau_{1,\min})/(1 + \tau_{2,\min})$,

$$\begin{aligned}\mathcal{H}^s(W(f, g, \tau_1, \tau_2)) &\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{w,v\in\Sigma_\beta^n} 2\beta^{n(\tau_{2,\min}-\tau_{1,\min})}\Big(\frac{4}{\beta^{n(1+\tau_{2,\min})}}\Big)^s \\ &\leq 32\Big(\frac{\beta}{\beta-1}\Big)^2 \liminf_{N\to\infty} \sum_{n=N}^{\infty} \beta^{2n}\beta^{n(\tau_{2,\min}-\tau_{1,\min}-(1+\tau_{2,\min})s)} \\ &= 0.\end{aligned}$$

This shows that

$$\dim_{\mathcal{H}} W(f, g, \tau_1, \tau_2) \leq \frac{2 + \tau_{2,\min} - \tau_{1,\min}}{1 + \tau_{2,\min}}.$$

Case 2. Cover using balls with radius being the longer side length $4/\beta^{n(1+\tau_{1,\min})}$. Then $J_n(w) \times J_n(v)$ can be covered by one ball. Thus, for $s > 2/(1 + \tau_{1,\min})$,

$$\begin{aligned}\mathcal{H}^s(W(f, g, \tau_1, \tau_2)) &\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{w,v\in\Sigma_\beta^n} \Big(\frac{4}{\beta^{n(1+\tau_{1,\min})}}\Big)^s \\ &\leq 16\Big(\frac{\beta}{\beta-1}\Big)^2 \liminf_{N\to\infty} \sum_{n=N}^{\infty} \beta^{2n}\beta^{-n(1+\tau_{1,\min})s} \\ &= 0.\end{aligned}$$

This shows that

$$\dim_{\mathcal{H}} W(f, g, \tau_1, \tau_2) \leq \frac{2}{1 + \tau_{1,\min}}.$$

3.2. The lower bound. Fix $\epsilon > 0$. Let n_0 be an integer such that for all $n \geq n_0$,

$$(n + 1)\beta^{-n} \leq \beta^{-n(1-\epsilon)}, \quad \tfrac{1}{4} \geq \beta^{-n\epsilon/2}.$$

For any $w = (w_1, \ldots, w_n), v = (v_1, \ldots, v_n) \in \Sigma_\beta^n$, define

$$\begin{aligned}J_n(f, \tau_1, w) &= \{x \in I_n(w) : |T_\beta^n(x) - f(x)| < \beta^{-n\tau_1(x)}\}, \\ J_n(g, \tau_2, v) &= \{y \in I_n(v) : |T_\beta^n(y) - g(y)| < \beta^{-n\tau_2(x)}\}.\end{aligned}$$

Fix a $w = (w_1, \ldots, w_n) \in \Sigma_\beta^n$ for which $I_n(w)$ is full. Then

$$I_n(w) = \Big[\frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n}, \frac{w_1}{\beta} + \cdots + \frac{w_n + 1}{\beta^n}\Big).$$

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By Lemma 2.6, there exists a point $x_n^*(w)$ in the closure of $I_n(\epsilon_1, \dots, \epsilon_n)$ such that

$$T_\beta^n x_n^*(w) = f(x_n^*(w)).$$

For any $x \in I_n(w)$ and for large n,

$$\begin{aligned} |T_\beta^n x - f(x)| &= |T_\beta^n x - f(x) - (T_\beta^n x_n^*(w) - f(x_n^*(w))| \\ &\le (\beta^n + L)|x - x_n^*(w)| \\ &\le 2\beta^n |x - x_n^*(w)|. \end{aligned}$$

So,

$$J_n(f, \tau_1, w) \supset I_n(w) \cap B\Big(x_n^*(w), \frac{1}{2\beta^{n(1+\tau_1(x))}}\Big).$$

Thus, $J_n(f, \tau_1, w)$ contains an interval of length at least $\frac{1}{2}\beta^{-n(1+\tau_1(x))}$. We denote this interval by

$$B(x_n^{*\prime}(w), \tfrac{1}{4}\beta^{-n(1+\tau_1(x))}) \quad \text{for some } x_n^{*\prime}(w) \in I_n(w).$$

The same is true for $J_n(g, \tau_2, v)$, in other words, again for large n,

$$J_n(g, \tau_2, v) \supset B(y_n^{*\prime}(v), \tfrac{1}{4}\beta^{-n(1+\tau_2(y))}) \quad \text{for some } y_n^{*\prime}(v) \in I_n(v).$$

So,

$$\begin{aligned} &W(f, \tau_1, \tau_2) \\ &\quad = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v \in \Sigma_\beta^n} J_n(f, \tau_1, w) \times J_n(g, \tau_2, v) \\ &\quad \supset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v \in \Sigma_\beta^n, I_{n,\beta}(w), I_{n,\beta}(v) \text{ full}} B(x_n^{*\prime}(w), \tfrac{1}{4}\beta^{-n(1+\tau_1(x))}) \times B(y_n^{*\prime}(v), \tfrac{1}{4}\beta^{-n(1+\tau_2(y))}) \\ &\quad \supset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v \in \Sigma_\beta^n, I_{n,\beta}(w), I_{n,\beta}(v) \text{ full}} B(x_n^{*\prime}(w), \beta^{-n(1+\tau_1(x)+\epsilon/2)}) \times B(y_n^{*\prime}(v), \beta^{-n(1+\tau_2(y)+\epsilon/2)}) \\ &\quad := W_{\mathbf{a}}. \end{aligned}$$

Since τ_1, τ_2 are two positive continuous functions on $[0, 1]$, there exists a ball $B \subset [0, 1]^2$ such that

$$\tau_1(x_1) \le \tau_{1,\min} + \frac{\epsilon}{2}, \quad \tau_1(y_1) \le \tau_{2,\min} + \frac{\epsilon}{2}$$

for any $(x_1, y_1) \in B$. Therefore,

$$B \cap W_{\mathbf{a}} \supset B \cap W_{\mathbf{a}_{\min}},$$

where

$$W_{\mathbf{a}_{\min}} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v \in \Sigma_\beta^n, I_{n,\beta}(w), I_{n,\beta}(v) \text{ full}} B(x_n^{*\prime}(w), \beta^{-n(1+\tau_{1,\min}+\epsilon)}) \times B(y_n^{*\prime}(v), \beta^{-n(1+\tau_{2,\min}+\epsilon)})$$

and

$$\mathbf{a}_{\min} = \Big(\frac{1 + \tau_{1,\min} + \epsilon}{1 - \epsilon}, \frac{1 + \tau_{2,\min} + \epsilon}{1 - \epsilon}\Big).$$

On the other hand, the interval $[0, 1]$ can be partitioned disjointly by (2.2) and, from Lemma 2.5, for any $x \in [0, 1]$, among $n + 1$ consecutive cylinders of order n, there is at least one full cylinder of order n around x. So, there exists $w \in \Sigma_\beta^n$ for which $I_n(w)$ is full such that

$$|x - x_n^{*\prime}(w)| \le (n + 1)\beta^{-n} \le \beta^{-n(1-\epsilon)}$$

for any $x_n^{*\prime}(w) \in I_n(w)$. Thus,

$$[0, 1] = \bigcup_{w\in\Sigma_\beta^n,\, I_n(w)\text{ full}} B(x_n^{*\prime}(w), \beta^{-n(1-\epsilon)}).$$

Clearly, the set

$$W_1 := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n, I_n(w), I_n(v)\text{ full}} B(x_n^{*\prime}(w), \beta^{-n(1-\epsilon)}) \times B(y_n^{*\prime}(v), \beta^{-n(1-\epsilon)})$$

equals $[0, 1]^2$, so it is of full Lebesgue measure.

Now we use the mass transference principle generated by rectangles (Lemma 2.7) to conclude that

$$\dim_{\mathcal{H}}(B \cap W_{\mathbf{a}_{\min}}) \ge \min\Big\{\frac{2}{1 + \tau_{1,\min} + \epsilon}, \frac{2 + \tau_{2,\min} - \tau_{1,\min}}{1 + \tau_{2,\min} + \epsilon}\Big\}.$$

By letting $\epsilon \to 0$,

$$\dim_{\mathcal{H}}(B \cap W_{\mathbf{a}_{\min}}) \ge \min\Big\{\frac{2}{1 + \tau_{1,\min}}, \frac{2 + \tau_{2,\min} - \tau_{1,\min}}{1 + \tau_{2,\min}}\Big\}.$$

It is clear that

$$W(f, g, \tau_1, \tau_2) \supset W_{\mathbf{a}} \supset B \cap W_{\mathbf{a}} \supset B \cap W_{\mathbf{a}_{\min}}.$$

So,

$$\dim_{\mathcal{H}} W(f, g, \tau_1, \tau_2) \ge \min\Big\{\frac{2}{1 + \tau_{1,\min}}, \frac{2 + \tau_{2,\min} - \tau_{1,\min}}{1 + \tau_{2,\min}}\Big\}.$$

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