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APPROXIMATION AND ESTIMATION OF SCALE FUNCTIONS FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

HARUKA IRIE,* AND
YASUTAKA SHIMIZU,* ** *Waseda University*

Abstract

The scale function plays a significant role in the fluctuation theory of Lévy processes, particularly in addressing exit problems. However, its definition is established through the Laplace transform, which generally lacks an explicit representation. This paper introduces a novel series representation for the scale function, utilizing Laguerre polynomials to construct a uniformly convergent approximation sequence. Additionally, we conduct statistical inference based on specific discrete observations and propose estimators for the scale function that are asymptotically normal.

Keywords: Spectrally negative Lévy process; scale function; Laguerre function; discrete observation; asymptotically normal estimator

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1. Introduction

On a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (right-continuous) filtration, we consider a *spectrally negative Lévy process* $X = (X_t)_{t\geq 0}$ starting at $x \in \mathbb{R}$,

$$X_t = x + ct + \sigma W_t - L_t, \quad t \geq 0,$$

where $x, c \in \mathbb{R}$ are constants, $W = (W_t)_{t\geq 0}$ is an $\mathbb{F}$-Wiener process, and $L = (L_t)_{t\geq 0}$ is an $\mathbb{F}$-Lévy subordinator (possibly of infinite activity) with the Lévy measure ν on $(0, \infty)$ satisfying

$$\int_0^1 z\, \nu(\mathrm{d}z) < \infty, \quad \nu([1, \infty)) < \infty.$$

Note that $W_0 = L_0 = 0$ a.s. The *Laplace exponent* of X is defined as

$$\psi_X(\theta) := \log \mathbb{E}[\mathrm{e}^{\theta(X_1 - x)}] = c\theta + \frac{\sigma^2}{2}\theta^2 + \int_0^\infty (\mathrm{e}^{-\theta z} - 1)\, \nu(\mathrm{d}z), \quad \theta \geq 0.$$

We are interested in the *q-scale function* of X, $W^{(q)}\colon \mathbb{R} \to \mathbb{R}_+ := [0, \infty)$ for some $q \geq 0$, defined as follows: $W^{(q)}(x) = 0$ on $(-\infty, 0)$ and otherwise $W^{(q)}$ is the unique continuous

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* Postal address: Department of Applied Mathematics, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, JAPAN
** Email address: shimizu@waseda.jp

function that is right-continuous at the origin with the Laplace transform

$$\int_0^\infty e^{-\theta z} W^{(q)}(z)\, dz = \frac{1}{\psi_X(\theta) - q} \quad \text{for } \theta > \Phi(q),$$

where $\Phi(q)$ is called the *Lundberg exponent*:

$$\Phi(q) := \sup\{\theta \geq 0 \mid \psi_X(\theta) = q\}.$$

The scale functions play essential roles in fluctuation theory of Lévy processes and have various applications in insurance and finance. For example, defining two stopping times $\tau_\alpha^+ = \inf\{t > 0 \mid X_t > \alpha\}$ and $\tau_\alpha^- = \inf\{t > 0 \mid X_t < \alpha\}$ for each $\alpha \in \mathbb{R}$, we have a fluctuation identity for $a > 0$ such that

$$\mathbb{E}\big[e^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\big] = \frac{W^{(q)}(x)}{W^{(q)}(a)} \quad \text{for } x \in [0, a],$$

which is an essential identity in the theory of two-sided exit problems and useful in analyzing credit risks and barrier options, among others. As $q = 0$, we have $\mathbb{P}(\tau_0^- < \tau_a^+) = 1 - W^{(0)}(x)/W^{(0)}(a)$, and so assuming that

$$\psi'(0+) = c - \int_0^\infty z\, \nu(dz) > 0,$$

called the *net profit condition* in ruin theory, which implies that $W^{(0)}(\infty) = 1/\psi'(0+)$, we have the well-known identity for the ruin probability in classical ruin theory:

$$\mathbb{P}(\tau_0^- < \infty) = 1 - \psi'(0+) W^{(0)}(x). \tag{1.1}$$

See Kyprianou [14] for details regarding these identities. Moreover, Biffis and Kyprianou [5], as well as Feng and Shimizu [9], demonstrated that scale functions are useful tools for representing more general ruin-related risks. They showed that certain generalized Gerber–Shiu functions – whose classical version was introduced by Gerber and Shiu [10] – can also be expressed in terms of q-scale functions.

Scale functions also play a significant role in optimal dividend problems; see Loeffen [17]. Additionally, they are involved in the study of Parisian ruin probabilities, as explored by Loeffen *et al.* [18] and Baurdoux *et al.* [1], among others.

The connection between ruin theory and q-scale functions is deeply rooted in the potential theory of spectrally negative Lévy processes and the Wiener–Hopf factorization; see e.g. Bertoin [3, 4] and Roger [20]. For further details, the reference lists in Kyprianou [14], Kyprianou and Rivero [15], Feng and Shimizu [9], and Kuznetsov *et al.* [13] provide useful historical context and insights into their applications across various fields.

In considering such an application, it is important to recognize the practical need to identify the scale function and estimate it statistically from observations of a given Lévy process. Indeed, the identification and approximation of scale functions have attracted considerable attention in recent years.

We can find an explicit representation for some simple cases, such as compound Poisson processes; see e.g. Hubalek and Kyprianou [11]. However, obtaining a general representation via the Laplace transform is generally challenging because Laplace inversion is too difficult to implement. Therefore, attempts have been made to obtain an approximate representation.

The earliest work on an approximation of scale functions is due to Egami and Yamazaki [8], who constructed an approximate sequence of q-scale functions using a compound Poisson-type Lévy process with phase-type jumps, forming a dense family in the class of spectrally negative Lévy processes. Landrault and Willmot [16] proposed an asymptotic expansion for Wiener–Poisson risk models by inverting the Laplace transform of scale functions and investigating some examples where explicit expansions are obtained. Behme *et al.* [2] extended their results to more general Lévy processes with infinite jumps. Moreover, Xie *et al.* [27] focused on a specific probabilistic representation of the q-scale function and provided an approximation formula using a Laguerre series expansion. See also Martín-González *et al.* [19] for an alternative expansion, and Surya [25] for numerical methods, among others.

Thus there are many discussions on the approximation of scale functions, but to the best of our knowledge, statistical inference based on underlying data has not yet been discussed. Our paper's novelty lies not only in providing a new series approximation of the q-scale function but also a data-based statistical estimation of it. Among these, we are particularly concerned with problems in insurance actuarial practice. In modern actuarial practice, it is standard to use the spectrally negative Lévy process $X = (X_t)_{t\geq 0}$ for the surplus or asset processes of insurance companies, and certain discrete observations of X are available as real data; see Section 3.1. However, it is usually not clear from the data which Lévy process these data follow. We therefore propose a method for estimating scale functions without specifying a model of X, by using a non-parametric method for estimating quantities associated with the Lévy measure.

We must take two steps to identify the q-scale function in practice. First we introduce a new approximation formula. We focus on a compound geometric integral representation of the q-scale function obtained by Feng and Shimizu [9]. We derive a Laguerre series expansion of the corresponding compound geometric distribution function, and the Stieltjes integral with respect to it gives the expansion of the q-scale function. Although we also use a Laguerre expansion, as in Xie *et al.* [27], our approach differs from theirs, and the formula is fundamentally different, which constitutes the primary contribution of this paper.

Second, we proceed to statistical inference. Two studies, by Zhang and Su [29] and Shimizu and Zhang [24], are instructive in this regard. The former proposes an estimator of Gerber–Shiu functions by deriving its Laguerre series expansion and estimating the coefficients for each term. They show the consistency of their proposed estimator. Shimizu and Zhang [24] applied the same approach to ruin probability and further showed that the estimator is asymptotically normal. As shown in equation (1.1), the ruin probability is represented by $W^{(0)}(x)$, so their estimator is also an asymptotically normal estimator for the 0-scale function. This paper constructs an asymptotically normal estimator of the q-scale function, which generalizes the results of [24].

The paper is organized as follows. Section 2 introduces the series representation of the q-scale function obtained by Feng and Shimizu [9]. Under the net profit condition, the q-scale function has an integral representation in the form of the expected value of the compound geometric distribution. Section 3 covers statistical inference. Assuming X to be a surplus model, we construct auxiliary statistics for each unknown parameter under a reasonable discrete observation scheme. Finally, in Section 3.2, we construct an estimator for the Laguerre expansion of the q-scale function based on these auxiliary statistics. The proposed estimators are shown to be consistent and asymptotically normal. Supplementary lemmas are summarized in the Appendix.

Notation. Throughout the paper, we use the following notation.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}_+ := [0, \infty)$.
- For a $d \times d$ matrix $A = (a_{ij})_{1\le i,j\le d}$, let $|A| := \left(\sum_{i,j=1}^d a_{ij}^2\right)^{1/2}$.
- For $\boldsymbol{a} = (a_1, \ldots, a_d)^\top$ and $\boldsymbol{b} = (b_1, \ldots, b_d)^\top$, the inner product is defined by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{k=1}^d a_k b_k$.
- Let $\mathbf{0}_d$ denote the zero vector of dimension d, and let I_d be the $d \times d$ identity matrix. Moreover, $N_d(\boldsymbol{a}, \Sigma)$ denotes the d-dim Gaussian distribution with mean vector $\boldsymbol{a}$ and covariance matrix Σ. In particular, $N := N_1$.
- The indicator function on a set $A \subset \mathbb{R}$ is given by $\mathbf{1}_A(x) = 1$ if $x \in A$; 0 otherwise.
- For functions f, g on $\mathbb{R}$, let $f(x) \lesssim g(x)$ if there exists a constant $C > 0$ such that $f(x) \le Cg(x)$ for any $x \in \mathbb{R}$.
- For a function $f(x_1, x_2, \ldots, x_d)$, let

$$\partial_{x_i} f := \frac{\partial f}{\partial x_i} \quad \text{and} \quad \partial_{(x_1,\ldots,x_d)} f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)^\top.$$

- For $s \ge 1$ and $\alpha > 0$, let

$$L^s_\alpha(\mathbb{R}_+) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} : \int_0^\infty f^s(x)\, \mathrm{e}^{-\alpha x}\, \mathrm{d}x < \infty \right\}.$$

  In particular, we let $L^s(\mathbb{R}_+) := L^s_0(\mathbb{R}_+)$, and for $f, g \in L^2(\mathbb{R}_+)$,

$$\langle f, g\rangle := \int_0^\infty f(x) g(x)\, \mathrm{d}x, \quad \|f\| := \sqrt{\langle f, f\rangle}.$$

- For functions f, g on $\mathbb{R}_+$ of finite variation, the convolution is defined as

$$f * g(x) := \int_{[0,x]} f(x-y) g(y)\, \mathrm{d}y, \quad x \ge 0.$$

  Moreover, the kth convolution is defined by $f^{*0} = \delta_0$ (Dirac's delta function concentrated on 0) and $f^{*k} = f * f^{*(k-1)}$ for $k \in \mathbb{N}$.
- For a measure ν on $\mathbb{R}_+$ and a function $\boldsymbol{f} = (f_1, \ldots, f_d)^\top : \mathbb{R}_+ \to \mathbb{R}^d$, we let

$$\nu(\boldsymbol{f}) := \int_{\mathbb{R}_+} \boldsymbol{f}(x)\, \nu(\mathrm{d}x) = \left(\int_{\mathbb{R}_+} f_1(x)\, \nu(\mathrm{d}x), \ldots, \int_{\mathbb{R}_+} f_d(x)\, \nu(\mathrm{d}x) \right)^\top.$$

2. Series representation of *q*-scale functions

2.1. A compound geometric representation of *q*-scale functions

A closed-form expression for the q-scale function of the process X was derived by Feng and Shimizu [9] as an expectation with respect to a compound geometric distribution under the

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following *net profit condition*:

NPC $c > \int_0^\infty z\,\nu(\mathrm{d}z)$.

To state the representation of $W^{(q)}$, we prepare some notation. We will fix a number $q \geq 0$ for $W^{(q)}$, and let

$$D := \frac{\sigma^2}{2}, \quad \gamma := \Phi(q), \quad \beta = \begin{cases} cD^{-1} + \gamma, & D > 0, \\ \gamma, & D = 0, \end{cases}$$

$$\widetilde{f}_q(x) := \begin{cases} D^{-1} \int_0^x \mathrm{d}y \int_y^\infty \mathrm{e}^{-\beta(x-y)} \mathrm{e}^{-\gamma(z-y)}\,\nu(\mathrm{d}z), & D > 0, \\ c^{-1} \int_x^\infty \mathrm{e}^{-\gamma(z-x)}\,\nu(\mathrm{d}z), & D = 0. \end{cases}$$

We define a distribution F_q with a probability density obtained from the function $\widetilde{f}_q$:

$$F_q(x) := \int_0^x f_q(z)\,\mathrm{d}z, \quad f_q(z) = p^{-1}\widetilde{f}_q(z), \quad p := \int_0^\infty \widetilde{f}_q(z)\,\mathrm{d}z.$$

In fact, $p \in (0, 1)$ under the NPC condition and the probability density f_q is well-defined; see Lemma C.1. Then the following representation is the immediate consequence from Feng and Shimizu [9, Proposition 4.1].

Lemma 2.1. *Suppose the NPC condition for the process X. Then the q-scale function $W^{(q)}$ of X has the following integral form:*

$$W^{(q)}(x) = \begin{cases} \frac{\mathrm{e}^{\gamma x} - \mathrm{e}^{-\beta x}}{D(1-p)(\beta+\gamma)} - \frac{1}{D(1-p)(\beta+\gamma)} \int_{[0,x)} \left[\gamma \mathrm{e}^{\gamma(x-z)} + \beta \mathrm{e}^{-\beta(x-z)}\right] \overline{G}_q(z)\,\mathrm{d}z, & D > 0, \\ \frac{\mathrm{e}^{\gamma x}}{c(1-p)} - \frac{1}{c(1-p)} \int_{[0,x)} \mathrm{e}^{\gamma(x-z)} \overline{G}_q(z)\,\mathrm{d}z, & D = 0, \end{cases}$$

where $p \in (0, 1)$ and G_q is a compound geometric distribution function

$$G_q(x) = \sum_{k=0}^\infty (1-p)p^k \int_0^x f_q^{*k}(z)\,\mathrm{d}z, \quad x \geq 0, \tag{2.1}$$

and $G_q(x) \equiv 0$ for $x < 0$.

Proof. See Section C.4.

2.2. Laguerre series expansion

Let $L_k(x)$ be the *Laguerre polynomial* of order k,

$$L_k(x) = \frac{1}{n!} \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{-x} x^n) = \sum_{j=0}^k \binom{k}{j} \frac{(-x)^j}{j!}, \quad x \in \mathbb{R}_+ := [0, \infty),$$

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by which we define the *Laguerre function* for each $\alpha > 0$ as follows:

$$\varphi_{\alpha,k}(x) := \sqrt{2\alpha} L_k(2\alpha x)\, \mathrm{e}^{-\alpha x}, \quad x \in \mathbb{R}_+.$$

Note that the system $\{\varphi_{\alpha,k}\}_{k\in\mathbb{N}_0}$ consists of an orthogonal basis of $L^2(\mathbb{R}_+)$ satisfying

$$\sup_{k\in\mathbb{N}_0, x\in\mathbb{R}_+} |\varphi_{\alpha,k}(x)| \le \sqrt{2\alpha}, \tag{2.2}$$

and that for any $k \in \mathbb{N}_0$, there exists some $\delta > 0$ such that

$$\varphi_{\alpha,k}(x) = \mathrm{O}\big(\mathrm{e}^{-(\alpha-\delta)x}\big), \quad x \to \infty,$$

since L_k is a polynomial of order k. Therefore we can control the decay order of the function $\varphi_{k,\alpha}$ by choosing $\alpha > 0$; see Remark 2.1.

Let $\mathcal{W}_\alpha^r(\mathbb{R}_+)$ be the *Sobolev–Laguerre space* (Bongioanni and Torrea [6]) for α, $r > 0$:

$$\mathcal{W}_\alpha^r(\mathbb{R}_+) := \left\{ f \in L^2(\mathbb{R}_+) \colon \sum_{k=0}^{\infty} k^r |\langle f, \varphi_{\alpha,k}\rangle|^2 < \infty \right\}.$$

There are many useful connections between Laguerre systems and the Sobolev–Laguerre space, as discussed in Comte and Genon-Catalot [7]. In particular, Proposition 7.1 and its remark provide an equivalent condition for a function to belong to $\mathcal{W}_\alpha^r(\mathbb{R}_+)$, as follows.

Lemma 2.2. (Comte and Genon-Catalot [7].) *Let $r \in \mathbb{N}$ and $\alpha > 0$. Then $f \in \mathcal{W}_\alpha^r(\mathbb{R}_+)$ if and only if f admits the derivatives of order r and that*

$$x^{m/2}\partial_x^m f(x) \in L_\alpha^2(\mathbb{R}_+), \quad 0 \le m \le r. \tag{2.3}$$

The following uniform convergence of the Laguerre series expansion is obtained by Shimizu and Zhang [24, Proposition 3].

Lemma 2.3. *For any $K \in \mathbb{N}$, the partial sum of the Laguerre series expansion of $f \in \mathcal{W}_\alpha^r(\mathbb{R}_+)$,*

$$f_K(x) := \sum_{k=0}^{K} \langle f, \varphi_{\alpha,k}\rangle \varphi_{\alpha,k}(x), \quad x \ge 0,$$

satisfies

$$\sup_{x\in\mathbb{R}_+} |f_K(x) - f(x)| = \mathrm{O}(K^{-(r-1)/2}), \quad K \to \infty.$$

2.3. Laguerre expansion of $\overline{G}_q$

According to Willmot and Lin [26], for example, we see that the tail function $\overline{G}_q := 1 - G_q$ should satisfy the following defective renewal equation (DRE):

$$\overline{G}_q(x) = p\overline{F}_q + pf_q(x) * \overline{G}_q(x), \quad x \ge 0. \tag{2.4}$$

We have the Laguerre expansion of pf_q, $p\overline{F}_q$, and $\overline{G}_q$ since these belong to $L^2(\mathbb{R}_+)$ as shown in Section C.1: for $x \ge 0$,

$$pf_q(x) = \sum_{k=0}^{\infty} a_{\alpha,k}^f \varphi_{\alpha,k}(x), \quad p\overline{F}_q(x) = \sum_{k=0}^{\infty} a_{\alpha,k}^F \varphi_{\alpha,k}(x), \quad \overline{G}_q(x) = \sum_{k=0}^{\infty} a_{\alpha,k}^G \varphi_{\alpha,k}(x),$$

where $a_{\alpha,k}^f := \langle pf_q, \varphi_{\alpha,k}\rangle$, $a_{\alpha,k}^F := \langle p\overline{F}_q, \varphi_{\alpha,k}\rangle$, and $a_{\alpha,k}^G := \langle \overline{G}_q, \varphi_{\alpha,k}\rangle$.

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For arbitrary $K \in \mathbb{N}_0$, letting

$$\boldsymbol{d}^f_{\alpha,K} := \left(d^f_{\alpha,0}, d^f_{\alpha,1}, \ldots, d^f_{\alpha,K}\right)^\top,$$

$$\boldsymbol{a}^F_{\alpha,K} := \left(a^F_{\alpha,0}, a^F_{\alpha,1}, \ldots, a^F_{\alpha,K}\right)^\top,$$

$$\boldsymbol{a}^G_{\alpha,K} := \left(a^G_{\alpha,0}, a^G_{\alpha,1}, \ldots, a^G_{\alpha,K}\right)^\top,$$

we have the following relation among these coefficients; see Zhang and Su [28, Section 2] or Shimizu and Zhang [24, Proposition 2].

Lemma 2.4. *Define a* $(K+1) \times (K+1)$*-matrix* $A^f_K = (a_{kl})_{1 \le k, l \le K+1}$*, whose components are given as follows:*

$$a_{kl} := \begin{cases} 1 - \dfrac{1}{\sqrt{2\alpha}} d^f_{\alpha,0}, & k = l, \\ -\dfrac{1}{\sqrt{2\alpha}} \left(d^f_{\alpha,k-l} - d^f_{\alpha,k-l-1}\right), & k > l, \\ 0, & k < l. \end{cases}$$

Then the matrix A^f_K *is invertible, and it holds that*

$$\boldsymbol{a}^G_{\alpha,K} = \left(A^f_K\right)^{-1} \boldsymbol{a}^F_{\alpha,K}.$$

Using this system, we can construct a Laguerre expansion of $\overline{G}_q(x)$: for a vector $\boldsymbol{\varphi}_{\alpha,K} = (\varphi_{\alpha,0}, \varphi_{\alpha,1}, \ldots, \varphi_{\alpha,K})^\top$,

$$\overline{G}_{q,K}(x) := \sum_{k=0}^{K} a^G_{\alpha,k} \varphi_{\alpha,k}(x) = \boldsymbol{a}^G_{\alpha,K} \cdot \boldsymbol{\varphi}_{\alpha,K}, \quad x \in \mathbb{R}_+. \tag{2.5}$$

Lemma 2.5. *Suppose the tail function of the Lévy measure* ν*, say* $\overline{\nu}(x) := \int_x^\infty \nu(\mathrm{d}z)$*, admits derivatives up to order* $r > 1$ *and that the distribution* G_q *given in* (2.1) *has moments of any polynomial order. Moreover, suppose there exists a constant* $\kappa > 0$ *such that*

$$\partial_x^m \overline{\nu}(x) = \mathrm{O}(1 + x^\kappa), \quad x \to \infty, \tag{2.6}$$

for any $0 \le m \le r-2$*. Then we have* $\overline{G}_q \in \mathcal{W}^r_\alpha(\mathbb{R}_+)$*, and therefore*

$$\sup_{x \in \mathbb{R}_+} \left|\overline{G}_{q,K}(x) - \overline{G}_q(x)\right| = \mathrm{O}(K^{-(r-1)/2}) \to 0, \quad K \to \infty. \tag{2.7}$$

Furthermore, if the condition (2.6) *is much milder, such as*

$$\partial_x^m \overline{\nu}(x) = \mathrm{O}(\mathrm{e}^{\kappa x}), \quad x \to \infty, \tag{2.8}$$

then the consequence (2.7) *also holds if we choose* $\alpha > 2\kappa$*.*

Proof. See Section C.5.

Remark 2.1. Lemma 2.5 explains the significance of the tuning parameter $\alpha > 0$ in the Laguerre function. When considering a model in which the Lévy measure satisfies (2.8), we can choose α such that $a > 2\kappa$. However, in most standard cases where (2.6) holds, we can choose any $\alpha > 0$; in particular, setting $\alpha = 1$ is both simple and sufficient.

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2.4. Laguerre-type expansion for q-scale functions

We can obtain a series expansion of the q-scale function by replacing $\overline{G}_q(z)$ in the expression of $W^{(q)}$ in Lemma 2.1 with the corresponding Laguerre expansion (2.5).

Definition 2.1. For any $K \in \mathbb{N}$ and $\alpha > 0$, the Kth-Laguerre-type expansion of $W^{(q)}$, say $W_K^{(q)}$, is given by

$$W_K^{(q)}(x) := P(x; p, \gamma, D) - \boldsymbol{Q}_{\alpha,K}(x; p, \gamma, D) \cdot \boldsymbol{a}_{\alpha,K}^G, \tag{2.9}$$

where $\boldsymbol{Q}_{\alpha,K} := (Q_{\alpha,0}, Q_{\alpha,1}, \ldots, Q_{\alpha,K})^\top$ with

$$P(x; p, \gamma, D) := \begin{cases} \dfrac{e^{\gamma x} - e^{-\beta x}}{D(1-p)(\beta+\gamma)}, & D > 0, \\ \dfrac{e^{\gamma x}}{c(1-p)}, & D = 0, \end{cases}$$

$$Q_{\alpha,k}(x; p, \gamma, D) := \begin{cases} \dfrac{\gamma \Psi_{\alpha,k}(x; \gamma) + \beta \Psi_{\alpha,k}(x; -\beta)}{D(1-p)(\beta+\gamma)}, & D > 0, \\ \dfrac{\varphi_{\alpha,k}(x) + \gamma \Psi_{\alpha,k}(x; \gamma)}{c(1-p)}, & D = 0, \end{cases}$$

and

$$\Psi_{\alpha,k}(x; b) := \int_0^x e^{b(x-z)} \varphi_{\alpha,k}(z)\, dz, \quad b \in \mathbb{R}. \tag{2.10}$$

Remark 2.2. There is an alternative version of the q-scale function $Z^{(q)}: \mathbb{R} \to [1, \infty)$, defined as

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\, dz, \quad x \in \mathbb{R},$$

where we regard that $\int_0^x = 0$ if $x < 0$. Note that $W^{(q)}(x) = q^{-1} \partial_x Z^{(q)}(x)$ as $q \neq 0$. The Laguerre-type expansion of $Z^{(q)}$ is also defined as

$$\begin{aligned} Z_K^{(q)}(x) &= 1 + q \int_0^x W_K^{(q)}(z)\, dz \\ &= 1 + q\big[P^*(x; p, \gamma, D) - \boldsymbol{Q}^*_{\alpha,K}(x; p, \gamma, D) \cdot \boldsymbol{a}_{\alpha,K}^G\big], \end{aligned} \tag{2.11}$$

where $\boldsymbol{Q}^*_{\alpha,K} := \left(Q^*_{\alpha,0}, Q^*_{\alpha,1}, \ldots, Q^*_{\alpha,K}\right)^\top$ with

$$P^*(z; p, \gamma, D) := \begin{cases} \dfrac{\gamma^{-1}(1 - e^{\gamma x}) - \beta^{-1}(1 - e^{-\beta x})}{D(1-p)(\beta+\gamma)}, & D > 0, \\ \dfrac{1 - e^{\gamma x}}{\gamma c(1-p)}, & D = 0, \end{cases}$$

$$Q^*_{\alpha,k}(x; p, \gamma, D) := \begin{cases} \dfrac{\Psi_{\alpha,k}(x; \gamma) - \Psi_{\alpha,k}(x; -\beta)}{D(1-p)(\beta+\gamma)}, & D > 0, \\ \dfrac{\Psi_{\alpha,k}(x; \gamma)}{c(1-p)}, & D = 0. \end{cases}$$

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According to Lemma 2.3, if $\overline{G}_q \in \mathcal{W}_\alpha^r(\mathbb{R}_+)$ (e.g. $f := \overline{G}_q$ satisfies (2.3)), then

$$\sup_{x\in\mathbb{R}_+} |W_K^{(q)}(x) - W^{(q)}(x)| \to 0, \quad K \to \infty;$$

see (2.7). Therefore it follows for any compact sets $V \subset \mathbb{R}_+$ that

$$\sup_{x\in V} |Z_K^{(q)}(x) - Z^{(q)}(x)| \to 0, \quad K \to \infty.$$

Remark 2.3. The approximation formulas obtained in equations (2.9) and (2.11) are fundamentally different from those in Xie *et al.* [27], while being more elementary. Their approximation focuses on the relationship between the probability density of the 'killed process' X_{e_t} (where e_t is an exponential random variable with mean 1) and its scale function, cleverly expanding the probability density into a Laguerre series. In our approximation, we utilize a Laguerre series expansion for the compound geometric distribution $\overline{G}_q$, which is essentially similar to the approach of Shimizu and Zhang [24]. The utility of this approach becomes apparent in the subsequent discussion.

3. Statistical inference: main theorems

We will proceed with the statistical estimation of scale functions. When considering the practical application of q-scale functions, we typically focus on actuarial science, where $X = (X_t)_{t\geq 0}$ represents the dynamics of an insurance surplus model. In this context, the coefficient of the linear term, c, corresponds to the premium rate. Therefore, in this paper, we specifically treat the value of c as known.

3.1. Sampling scheme

Let $n \in \mathbb{N}$. We assume that the process $X = (X_t)_{t\geq 0}$ is observed at discrete time points $t_i^n := i\Delta_n$ $(i = 0, 1, \ldots, n)$ for some number $\Delta_n > 0$:

$$\boldsymbol{X}^n := \{X_{t_i^n} \mid i = 0, 1, \ldots, n\}, \quad T_n := n\Delta_n.$$

In particular, the initial value $X_{t_0^n} = x$ is assumed to be known. Moreover, we assume that 'large' jumps of X are observed. That is, for some number $\epsilon_n > 0$, we can identify the jumps whose sizes are larger than ϵ_n, which are finitely many on $[0, T_n]$:

$$\boldsymbol{J}^n := \{\Delta X_t := X_t - X_{t-} \mid t \in [0, T_n],\ |\Delta X_t| > \epsilon_n\}.$$

Assume that we have data $\boldsymbol{X}^n \cup \boldsymbol{J}^n$, and consider the following asymptotics:

$$\Delta_n \to 0, \quad T_n \to \infty, \quad \epsilon_n \to 0, \tag{3.1}$$

as $n \to \infty$. We shall always use the limit $n \to \infty$ when considering the asymptotic symbols, and assume (3.1) for the sampling scheme $(\Delta_n, T_n, \epsilon_n)$ without further comment.

Remark 3.1. The data $\boldsymbol{X}^n$ are assumed to represent the data on the remaining reserves that insurance companies record on a regular basis. On the other hand, $\boldsymbol{J}^n$ are assumed to be 'large' claims. It may seem unnatural to consider such a model with infinitely many jumps when modeling insurance surplus, but this is a standard surplus approximation in risk theory. Arguing under asymptotics such as (3.1) with data like $\boldsymbol{J}^n$ is a common way of theoretically justifying

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that the more detailed claims data you collect, the better the estimation. In practice, it is not necessary to observe infinitely many ‘small’ jumps.

We shall make some assumptions on the scheme $(\Delta_n, T_n, \epsilon_n)$:

S1 $n\Delta_n^2 \to 0$,

S2 $\int_0^{\epsilon_n} z\, \nu(\mathrm{d}z) + \int_0^{\epsilon_n} z^2\, \nu(\mathrm{d}z) = \mathrm{o}\big(T_n^{-1/2}\big)$.

In addition, to ensure some integrability with respect to ν, we will prepare the following moment conditions on ν:

M[*k*] For some given $k > 0$, there exists some $\epsilon \in (0, 1)$ such that $\nu(|z| \vee |z|^{k+\epsilon}) < \infty$.

3.2. Main theorems

For the statistical argument, we let p_0, D_0, and γ_0 denote the true values of parameters p, D, and γ, respectively. We will provide estimators for approximations $W_K^{(q)}$ and $Z_K^{(q)}$, in which the parameters p, D, and γ are replaced by their true value.

According to the expressions (2.9) and (2.11), we construct estimators of $W_K^{(q)}$ and $Z_K^{(q)}$ as follows:

$$\widehat{W}_K^{(q)}(x) := P(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) - \boldsymbol{Q}_{\alpha,K}(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) \cdot \widehat{\boldsymbol{a}}_{\alpha,K}^G,$$

$$\widehat{Z}_K^{(q)}(x) := q\big[P^*(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) - \boldsymbol{Q}_{\alpha,K}^*(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) \cdot \widehat{\boldsymbol{a}}_{\alpha,K}^G\big].$$

The consistency and asymptotic normality for $\widehat{W}_K^{(q)}$ are obtained for each $K \in \mathbb{N}$ as follows.

Theorem 3.1. *Suppose the assumptions NPC, S1, S2, and M[2] hold. Then, for any $q > 0$, $K \in \mathbb{N}$, and $x \in \mathbb{R}_+$, we have*

$$\widehat{W}_K^{(q)}(x) \xrightarrow{p} W_K^{(q)}(x).$$

In particular, when $q = 0$, we have uniform consistency:

$$\sup_{x \in \mathbb{R}_+} \big|\widehat{W}_K^{(0)}(x) - W_K^{(0)}(x)\big| \xrightarrow{p} 0.$$

In addition, suppose M[4] holds. Then we have

$$\sqrt{T_n}\big(\widehat{W}_K^{(q)}(x) - W_K^{(q)}(x)\big) \xrightarrow{d} N(0, \sigma_K(x)),$$

where $\sigma_K(x) := [C_K(x)\Gamma_K]\Sigma_K[C_K(x)\Gamma_K]^\top$ with Σ_K and Γ_K given in (B.1) *and* (B.2)*, respectively, and C_K is the $(2K + 4)$-dim vector given by*

$$C_K(x) :=$$

$$\Big(\boldsymbol{Q}_{\alpha,K}(x; p_0, \gamma_0, D_0)^\top \big(A_K^f\big)^{-1} B_K,\ \partial_{(p,\gamma)}\big[P(x; \gamma_0, p_0, D_0) - \boldsymbol{Q}_{\alpha,K}(x; p_0, \gamma_0, D_0) \cdot \boldsymbol{a}_{\alpha,K}^G\big]\Big)^\top,$$

with the matrix B_K given in Corollary B.1.

Proof. See Section C.6.

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Theorem 3.2. *Suppose the assumptions NPC, S1, S2, and M[2] hold. Then, for any* $q > 0$, $K \in \mathbb{N}$, *and* $x \in \mathbb{R}_+$, *we have*

$$\widehat{Z}_K^{(q)}(x) \xrightarrow{p} Z_K^{(q)}(x).$$

In particular, when $q = 0$, *we have uniform consistency: for any compact set* $V \subset \mathbb{R}_+$,

$$\sup_{x \in V} \left| \widehat{Z}_K^{(0)}(x) - Z_K^{(0)}(x) \right| \xrightarrow{p} 0.$$

In addition, suppose M[4] holds. Then we have

$$\sqrt{T_n}\left(\widehat{Z}_K^{(q)}(x) - Z_K^{(q)}(x)\right) \xrightarrow{d} N\left(0, \sigma_K^*(x)\right),$$

where $\sigma_K^*(x) := q^2[C_K^*(x)\Gamma_K]\Sigma_K[C_K^*(x)\Gamma_K]^\top$ *with* Σ_K *and* Γ_K *given in* (B.1) *and* (B.2), *respectively, and* C_K^* *is the* $(2K+4)$*-dim vector given by*

$$C_K^*(x) :=$$

$$\left(\boldsymbol{Q}_{\alpha,K}^*(x; p_0, \gamma_0, D_0)^\top \left(A_K^f\right)^{-1} B_K,\ \partial_{(p,\gamma)}\left[P^*(x; \gamma_0, p_0, D_0) - \boldsymbol{Q}_{\alpha,K}^*(x; p_0, \gamma_0, D_0) \cdot \boldsymbol{a}_{\alpha,K}^G\right]\right)^\top,$$

with the matrix B_K *given in Corollary B.1.*

Proof. See Section C.7.

Corollary 3.1. *Suppose the assumptions NPC, S1, S2, and M[2] hold. Then, for any* $q > 0$, $K \in \mathbb{N}$, *and* $x \in \mathbb{R}_+$, *we have*

$$\sqrt{T_n}\begin{pmatrix} \widehat{W}_K^{(q)}(x) - W_K^{(q)}(x) \\ \widehat{Z}_K^{(q)}(x) - Z_K^{(q)}(x) \end{pmatrix} \xrightarrow{d} N_2\left(\mathbf{0}_2, \widetilde{\Sigma}_K(x)\right),$$

where

$$\widetilde{\Sigma}_K(x) := \left[\begin{pmatrix} C_K(x) \\ qC_K^*(x) \end{pmatrix} \Gamma_K\right] \Sigma_K \left[\begin{pmatrix} C_K(x) \\ qC_K^*(x) \end{pmatrix} \Gamma_K\right]^\top.$$

Proof. See Section C.8. □

Remark 3.2. Our asymptotic results for $\widehat{W}_K^{(q)}$ and $\widehat{Z}_K^{(q)}$ are all for a fixed $K \in \mathbb{N}$, and but one may also be concerned with the case where $K = K_n \to \infty$ as well as $n \to \infty$. It is not a straightforward extension, even for consistency. For example, if we could show that for each $x \in \mathbb{R}_+$,

$$\sup_{K \in \mathbb{N}} \left| \widehat{W}_K^{(q)}(x) - W_K^{(q)}(x) \right| \xrightarrow{p} 0, \quad n \to \infty,$$

then we can exchange the order of the limits $n \to \infty$ and $K \to \infty$, which concludes that $\widehat{W}_\infty^{(q)}(x) \xrightarrow{p} W_\infty^{(q)}(x)$ for each $x \in \mathbb{R}_+$. On this point, we need further study. As for the extension of asymptotic normality, a more complicated discussion is needed to extend the results to the case where K depends on n with $K = K_n \to \infty$. This will lead to a high-dimensional setting, and we will need a high-dimensional central limit theorem (CLT) for triangular arrays, even without a martingale property. A sophisticated CLT would still need to be proved.

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Appendix A. Some auxiliary statistics

A.1. Estimator of D_0

For $D = \sigma^2/2$, we consider the following estimator proposed by Jacod [12] and Shimizu [22, 23]: for each fixed $T > 0$,

$$\widehat{D}_n^T := \frac{1}{2T}\left(\sum_{i=1}^{\lfloor T\Delta_n^{-1}\rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^2 - \sum_{s\le T} |\Delta L_s|^2 \mathbf{1}_{\{\Delta L_s > \epsilon_n\}}\right).$$

Lemma A.1. (Shimizu [22], Remark 3.2.) *Under the assumptions S1 and S2, the estimator* $\widehat{D}_n^T$ *is consistent with* D_0 *with the rate of convergence being faster than* $\sqrt{T_n}$ *such that for any* $t > 0$,

$$\sqrt{T_n}\big(\widehat{D}_n^T - D_0\big) \xrightarrow{p} 0.$$

Remark A.1. Since the constant $T > 0$ in the estimator $\widehat{D}_n^T$ can be arbitrary, we will fix it to be $T = 1$ without loss of generality, and set

$$\widehat{D}_n := \widehat{D}_n^1.$$

In practice, the value of T should be chosen appropriately based on the amount of data and the size of Δ_n.

A.2. Estimator of ν-functionals

First, we would like to estimate the integral-type functional $\nu(\boldsymbol{H}_\theta)$, where $\boldsymbol{H}_\theta : \mathbb{R}_+ \to \mathbb{R}^d$ is a ν-integrable function with an unknown parameter $\theta \in \Theta$:

$$\nu(\boldsymbol{H}_\theta) := \left(\int_0^\infty H_\theta^{(1)}(z)\,\nu(\mathrm{d}z), \ldots, \int_0^\infty H_\theta^{(d)}(z)\,\nu(\mathrm{d}z)\right),$$

where Θ is an open and bounded subset of $\mathbb{R}^l$ for some $l \in \mathbb{N}$. Note that the parameter θ can be a variety of parameters depending on the context. For instance, we will see later that the parameter $p := \int_0^\infty \widetilde{f}_q(z)\,\mathrm{d}z$, the coefficients of Laguerre expansion, e.g. $a_{\alpha,k}^f$ and $a_{\alpha,k}^F$, can all be expressed in terms of the integral functional of ν.

In short, we need to estimate the parameters $(D_0, \gamma_0, \nu(\boldsymbol{H}_{\theta_0}))$, where θ_0 is the true value of θ. Hereafter, we assume that there exists an open and bounded set Θ_1 and Θ_2 of $\mathbb{R}_+$ such that

$$(D_0, \gamma_0, \theta_0) \in \Theta_1 \times \Theta_2.$$

Moreover, we make the following assumptions on an integrands $\boldsymbol{H}_\theta$, which are applied to a variety of $\boldsymbol{H}_\theta$, locally in this section.

H1[δ] For each $\theta \in \Theta$, there exists a $\delta \ge 0$ such that $\nu(|\boldsymbol{H}_\theta| \vee |\boldsymbol{H}_\theta|^{2+\delta}) < \infty$.

H2 $\sup_{\theta\in\overline{\Theta}} \nu(|\boldsymbol{H}_\theta| \vee |\boldsymbol{H}_\theta|^2) < \infty$.

H3 There exists a ν-integrable function $h_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\sup_{\theta\in\overline{\Theta}} |\boldsymbol{H}_\theta(z)| \le h_1(z).$$

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H4 There exists a function $h_2\colon \mathbb{R}_+ \to \mathbb{R}$ with $\nu\big(h_2 \vee h_2^2\big) < \infty$ such that for any $\kappa \in \mathbb{R}^l$,

$$\sup_{\theta\in\overline{\Theta}} |\boldsymbol{H}_{\theta+\kappa}(z) - \boldsymbol{H}_\theta(z)| \le h_2(z)|\kappa|.$$

H5 For each $i = 1, \ldots, d$,

$$\int_0^{\epsilon_n} H_\theta^{(i)}(z)\,\nu(\mathrm{d}z) = \mathrm{o}\big(T_n^{-1/2}\big).$$

As for functionals $\nu(\boldsymbol{H}_\theta)$, we can use the following threshold-type estimator:

$$\widehat{\nu}_n(\boldsymbol{H}_\theta) := \frac{1}{T_n}\sum_{t\in(0,T_n]} \boldsymbol{H}_\theta(\Delta L_t)\mathbf{1}_{\{\Delta L_t > \epsilon_n\}}.$$

Lemma A.2. (Shimizu [22], Propositions 3.1 and 3.2.)

(1) *Under the assumption H1[0], we have*

$$\widehat{\nu}_n(\boldsymbol{H}_\theta) \xrightarrow{p} \nu(\boldsymbol{H}_\theta), \quad \theta \in \Theta.$$

In addition, assuming further H2 and H4, we have uniform consistency:

$$\sup_{\theta\in\overline{\Theta}} |\widehat{\nu}_n(\boldsymbol{H}_\theta) - \nu(\boldsymbol{H}_\theta)| \to 0.$$

(2) *Under H1[δ] for some $\delta > 0$ and H5, we have*

$$\sqrt{T_n}(\widehat{\nu}_n(\boldsymbol{H}_\theta) - \nu(\boldsymbol{H}_\theta)) \xrightarrow{d} N_d(\mathbf{0}_d, \Sigma_\theta), \quad \theta \in \Theta,$$

where $\Sigma_\theta = \Big(\nu\Big(H_\theta^{(i)}H_\theta^{(j)}\Big)\Big)_{1\le i,j\le d}$.

It will be easy to see that the following version of the continuous mapping-type theorem holds for the estimator $\widehat{\nu}_n(\boldsymbol{H}_\theta)$.

Corollary A.1. *Under the assumptions H1[0], H2–H4, it follows that*

$$\widehat{\nu}_n\Big(\boldsymbol{H}\boldsymbol{H}_{\widehat{\theta}_n}\Big) \xrightarrow{p} \nu(\boldsymbol{H}_{\theta_0}),$$

for any random sequence such that $\widehat{\theta}_n \xrightarrow{p} \theta_0 \in \Theta$.

Proof. Note that there exists a sub-subsequence $\{\widehat{\theta}_{n'}\}$ for any subsequence of $\{\widehat{\theta}_n\}$ such that $\widehat{\theta}_{n'} \to \theta_0$ a.s. Then, under the assumption H3, we can apply the Lebesgue-dominated convergence theorem to obtain $\nu\Big(\boldsymbol{H}_{\widehat{\theta}_{n'}}\Big) \to \nu(\boldsymbol{H}_{\theta_0})$ a.s. That is, the sequence $\Big\{\nu\Big(\boldsymbol{H}_{\widehat{\theta}_n}\Big)\Big\}$ has a sub-subsequence that converges to $\nu(\boldsymbol{H}_{\theta_0})$ almost surely, which implies that $\nu\Big(\boldsymbol{H}_{\widehat{\theta}_n}\Big) \xrightarrow{p} \nu(\boldsymbol{H}_{\theta_0})$. Moreover, since $\mathbb{P}(\widehat{\theta}_n \notin \Theta) \to 0$, it follows from Lemma A.2 that for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\Big(|\widehat{\nu}_n\Big(\boldsymbol{H}_{\widehat{\theta}_n}\Big) - \nu(\boldsymbol{H}_{\theta_0})| > \epsilon\Big) &\le \mathbb{P}\Big(\sup_{\theta\in\overline{\Theta}} |\widehat{\nu}_n(\boldsymbol{H}_\theta) - \nu(\boldsymbol{H}_\theta)| > \epsilon/2,\ \widehat{\theta}_n \in \Theta\Big)\\ &\quad + \mathbb{P}\Big(|\nu\Big(\boldsymbol{H}_{\widehat{\theta}_n}\Big) - \nu(\boldsymbol{H}_{\theta_0})| > \epsilon/2,\ \widehat{\theta}_n \in \Theta\Big)\\ &\quad + \mathbb{P}(\widehat{\theta}_n \notin \Theta) \to 0.\end{aligned}$$

This completes the proof. □

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A.3. Estimator of γ_0

An estimator of the Lundberg exponent $\gamma_0 = \Phi(q)$ is found in Shimizu [22] as an M-estimator given by

$$\widehat{\gamma}_n = \mathbf{1}_{\{q>0\}} \cdot \arg\inf_{r\in\Theta_2} |cr + \widehat{D}_n r^2 - \widehat{\nu}_n(k_r) - q|^2,$$

where $k_r(z) = \mathrm{e}^{-rz} - 1$. This estimator is quite natural because the contrast function is a direct estimator of the Lundberg equation '$\psi_X(r) - q = 0$', and it is useful because it satisfies consistency and asymptotic normality, as follows.

Lemma A.3 (Shimizu [22], Lemma 3.3.) *Suppose the conditions NPC, S1, S2, and M[2] hold. Then we have*

$$\sqrt{T_n}(\widehat{\gamma}_n - \gamma_0) \xrightarrow{d} N_1\big(0, v_0^2\big),$$

where

$$v_0^2 = \frac{\nu\big(k_{\gamma_0}^2\big)}{(c + 2D_0\gamma_0 + \nu(\partial_r k_{\gamma_0}))^2}.$$

In particular, we have the representation (in the proof of Lemma 3.3 of [22]) that

$$\sqrt{T_n}(\widehat{\gamma}_n - \gamma_0) = \sqrt{T_n}(\widehat{\nu}_n(\widetilde{H}_{\gamma_0}) - \nu(\widetilde{H}_{\gamma_0})) + \mathrm{o}_p(1), \tag{A.1}$$

where $\widetilde{H}_r(z) := k_r(z)/\partial_z\psi_X(z)$ *and* $k_r(z) = \mathrm{e}^{-rz} - 1$.

Appendix B. Estimators of p_0, $\boldsymbol{a}^f_{\alpha,K}$, $\boldsymbol{a}^F_{\alpha,K}$, and $\boldsymbol{a}^G_{\alpha,K}$

First, we prepare some notation to give representations of p_0, $\boldsymbol{a}^f_{\alpha,K}$, and $\boldsymbol{a}^F_{\alpha,K}$ in terms of ν-functionals.

Hereafter we set $\theta := (D, \gamma) \in \overline{\Theta} := \overline{\Theta}_1 \times \overline{\Theta}_2$, and let the true values be

$$\theta_0 := (D_0, \gamma_0) \in \Theta := \Theta_1 \times \Theta_2,$$

where Θ_1 and Θ_2 are open and bounded subsets of $\mathbb{R}_+$.

We define the following notation. As $D > 0$, for $\alpha > 0$ and $k \in \mathbb{N}_0$,

$$H_p(z;\theta) := D^{-1}\int_0^z \mathrm{d}y \int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-x)}\,\mathrm{d}x,$$

$$H^f_{\alpha,k}(z;\theta) := D^{-1}\int_0^z \mathrm{d}y \int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-x)}\varphi_{\alpha,k}(x)\,\mathrm{d}x,$$

$$H^F_{\alpha,k}(z;\theta) := D^{-1}\int_0^z \mathrm{d}y \int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-x)}\Psi_{\alpha,k}(x;0)\,\mathrm{d}x.$$

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Recall that $\Psi_{\alpha,k}(x;b)$ is given by (2.10) in Definition 2.1. In particular, when $D=0$,

$$H_p(z;\theta) := c^{-1}\int_0^z \mathrm{e}^{-\gamma(z-x)}\,\mathrm{d}x,$$

$$H^f_{\alpha,k}(z;\theta) := c^{-1}\int_0^z \mathrm{e}^{-\gamma(z-x)}\varphi_{\alpha,k}(x)\,\mathrm{d}x,$$

$$H^F_{\alpha,k}(z;\theta) := c^{-1}\int_0^z \mathrm{e}^{-\gamma(z-x)}\Psi_{\alpha,k}(x;0)\,\mathrm{d}x.$$

In this paper we use the convention

$$\boldsymbol{H}^f_{\alpha,K} = \big(H^f_{\alpha,0},\ldots,H^f_{\alpha,K}\big)^\top.$$

Then it is straightforward to obtain the following expression by direct computation.

Lemma B.1. *It holds that*

$$p_0 = \nu(H_p(\,\cdot\,;\theta_0)),\quad a^f_{\alpha,k} = \nu\big(H^f_{\alpha,k}(\,\cdot\,;\theta_0)\big),\quad a^F_{\alpha,k} = \nu\big(H^F_{\alpha,k}(\,\cdot\,;\theta_0)\big).$$

Proof. Use Fubini's theorem. We shall compute only $H^F_{\alpha,k}(z;\theta)$:

$$\begin{aligned} a^F_{\alpha,k} &:= \langle p\overline{F}_q,\varphi_{\alpha,k}\rangle \\ &= \int_0^\infty p\overline{F}_q(x)\varphi_{\alpha,k}(x)\,\mathrm{d}x \\ &= \int_0^\infty pf_q(x)\bigg(\int_0^x \varphi_{\alpha,k}(u)\mathrm{d}u\bigg)\,\mathrm{d}x \quad \text{(by integration by parts)} \\ &= \int_0^\infty \bigg(D^{-1}\int_0^x \mathrm{d}y\int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-y)}\nu(\mathrm{d}z)\bigg)\Psi_{\alpha,k}(x;0)\,\mathrm{d}x \\ &= \int_0^\infty \bigg(D^{-1}\int_0^z \mathrm{d}y\int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-y)}\Psi_{\alpha,k}(x;0)\,\mathrm{d}x\bigg)\nu(\mathrm{d}z) \\ &= \nu\big(H^F_{\alpha,k}(\,\cdot\,;\theta)\big). \end{aligned}$$

The others are similar and omitted. □

Thanks to this lemma, we have estimators of p_0, $a^f_{\alpha,k}$, and $a^F_{\alpha,k}$ as follows:

$$\widehat{p}_n = \widehat{\nu}_n(H_p(\,\cdot\,;\widehat{\theta}_n)),\quad \widehat{a}^f_{\alpha,k} = \widehat{\nu}_n\big(H^f_{\alpha,k}(\,\cdot\,;\widehat{\theta}_n)\big),\quad \widehat{a}^F_{\alpha,k} = \widehat{\nu}_n\big(H^F_{\alpha,k}(\,\cdot\,;\widehat{\theta}_n)\big),$$

where

$$\widehat{\theta}_n = (\widehat{D}_n,\widehat{\gamma}_n)^\top.$$

Theorem B.1. *Suppose the assumptions NPC, S1, S2, and M[2] hold. Then the estimators $\widehat{p}_n$, $\widehat{a}^f_{\alpha,k}$, and $\widehat{a}^F_{\alpha,k}$ are consistent with their true values. In particular, it follows for each $\alpha>0$ and $K\in\mathbb{N}$ that*

$$\big(\widehat{\boldsymbol{a}}^f_{\alpha,K},\widehat{\boldsymbol{a}}^F_{\alpha,K},\widehat{p}_n\big)^\top \xrightarrow{p} \big(\boldsymbol{a}^f_{\alpha,K},\boldsymbol{a}^F_{\alpha,K},p_0\big)^\top,$$

where p_0 is the true value of $p=\int_0^\infty \widetilde{f}_q(z)\,\mathrm{d}z$.

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Proof. According to Lemma B.1, $\widehat{p}_n$, $\widehat{a}^f_{\alpha,k}$, and $\widehat{a}^F_{\alpha,k}$ are all represented by ν-functionals, for which we can apply Corollary A.1 since $\widehat{\theta}_n = (\widehat{D}_n, \widehat{\gamma}_n) \xrightarrow{p} \theta_0 = (D_0, \gamma_0)$ under our assumptions by Lemmas A.1 and A.3. Therefore we can check H1[0], H2–H4-type conditions for each $H_\theta := H_p, H_{\alpha,k}$, and $H^F_{\alpha,k}$, but it is straightforward under M[2] since H2 and H3 are true by Lemma C.3, and H4 is also true by Lemma C.4. □

To state the asymptotic normality result, we define the following notation:

$$\boldsymbol{H}_{\alpha,K}(z;\theta) := \left(\boldsymbol{H}^f_{\alpha,K}(z;\theta), \boldsymbol{H}^F_{\alpha,K}(z;\theta), H_p(z;\theta)\right)^\top \in \mathbb{R}^{2K+3};$$

$$\widetilde{\boldsymbol{H}}_{\alpha,K}(z;\theta) := (\boldsymbol{H}_{\alpha,K}(z;\theta), \widetilde{H}_\gamma(z))^\top \in \mathbb{R}^{2K+4},$$

where $\widetilde{H}_\gamma$ is given in (A.1).

Theorem B.2. *Suppose the same assumptions as in Theorem B.1 and M[4] hold. Then we have asymptotic normality: for each $\alpha > 0$ and $K \in \mathbb{N}$,*

$$\sqrt{T_n}\begin{pmatrix} \widehat{\boldsymbol{a}}^f_{\alpha,K} - \boldsymbol{a}^f_{\alpha,K} \\ \widehat{\boldsymbol{a}}^F_{\alpha,K} - \boldsymbol{a}^F_{\alpha,K} \\ \widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0 \end{pmatrix} \xrightarrow{d} N_{2K+4}\left(\boldsymbol{0}_{2K+4}, \Gamma_K \Sigma_K \Gamma_K^\top\right),$$

where $\Sigma_K := (\sigma_{ij})_{1\le i,j\le 2K+4}$ with

$$\sigma_{ij} := \int_0^\infty \widetilde{H}^{(i)}_{\alpha,K}(z;\theta_0)\widetilde{H}^{(j)}_{\alpha,K}(z;\theta_0)\nu(\mathrm{d}z), \tag{B.1}$$

and Γ_K is the $(2K+4)\times(2K+4)$-matrix denoted by

$$\Gamma_K := \begin{pmatrix} I_{2K+3} & \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0)) \\ \boldsymbol{0}^\top_{2K+3} & 1 \end{pmatrix}. \tag{B.2}$$

Proof. We only show the case where $D > 0$ because the proof for $D = 0$ is similar.

Firstly, thanks to Taylor's formula, it follows that

$$\begin{aligned}
&\sqrt{T_n}\left(\widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\widehat{\theta}_n)) - \nu(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0))\right) \\
&= \sqrt{T_n}\left(\widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\widehat{\theta}_n)) - \widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0))\right) + \sqrt{T_n}(\widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0)) - \nu(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0))) \\
&= \widehat{\nu}_n\left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta^n_u)\,\mathrm{d}u\right)\sqrt{T_n}(\widehat{\gamma}_n - \gamma_0) + \widehat{\nu}_n\left(\int_0^1 \partial_D \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta^n_u)\,\mathrm{d}u\right)\sqrt{T_n}(\widehat{D}_n - D_0) \\
&\quad + \sqrt{T_n}(\widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0)) - \nu(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0))) \\
&= \left(I_{2K+3}\,\widehat{\nu}_n\left(\textstyle\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta^n_u)\,\mathrm{d}u\right)\right)\sqrt{T_n}\begin{pmatrix} \widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0)) - \nu(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0)) \\ \widehat{\gamma}_n - \gamma_0 \end{pmatrix} \\
&\quad + \left[\left(\widehat{\nu}_n\left(\int_0^1 \partial_D \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta^n_u)\,\mathrm{d}u\right)\boldsymbol{1}_{\{\widehat{\theta}_n\in\Theta\}}\right) + \widehat{\nu}_n\left(\int_0^1 \partial_D \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta^n_u)\,\mathrm{d}u\right)\boldsymbol{1}_{\{\widehat{\theta}_n\notin\Theta\}}\right] \\
&\qquad \times \sqrt{T_n}(\widehat{D}_n - D_0) =: S_1 + S_2,
\end{aligned}$$

where $\theta^n_u := \theta_0 + u(\widehat{\theta}_n - \theta_0)$, $(u \in (0,1))$.

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As for S_2, it follows from Lemma C.3 that for any $k \in \mathbb{N}_0$,

$$\left| \int_0^1 \partial_D H_{\alpha,k}(z; \theta_u^n) \, \mathrm{d}u \mathbf{1}_{\{\widehat{\theta}_n \in \Theta\}} \right| \le \sup_{\theta \in \Theta} |\partial_D H_{\alpha,k}(z; \theta)| \lesssim z.$$

Therefore the condition M[4] ensures that

$$\nu\left(\left| \int_0^1 \partial_D H_{\alpha,k}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \mathbf{1}_{\{\widehat{\theta}_n \in \Theta\}} \right|^2 \right) < \infty.$$

Moreover, it follows for any $\epsilon > 0$ that

$$\left\{ \widehat{\nu}_n \left(\int_0^1 \partial_D \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) \mathbf{1}_{\{\widehat{\theta}_n \notin \Theta\}} > \epsilon \right\} \subset \left\{ \mathbf{1}_{\{\widehat{\theta}_n \notin \Theta\}} = 1 \right\} = \{\widehat{\theta}_n \notin \Theta\}.$$

Since $\widehat{\theta}_n \xrightarrow{p} \theta_0 \in \Theta$ by S1, S2, and M[2], it follows from Lemmas A.1 and A.2 that

$$S_2 = (\mathrm{O}_p(1) + \mathrm{o}_p(1)) \sqrt{T_n} (\widehat{D}_n - D_0) \xrightarrow{p} 0.$$

Secondly, we show the following convergence:

$$\widehat{\nu}_n \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) \xrightarrow{p} \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_0)). \tag{B.3}$$

Thanks to Fubini's theorem, we have

$$\begin{aligned}
&\left| \widehat{\nu}_n \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_0)) \right| \\
&\quad \le \left| \widehat{\nu}_n \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) - \nu \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) \right| \mathbf{1}_{\{\widehat{\theta}_n \in \Theta\}} \\
&\qquad + \left| \widehat{\nu}_n \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) - \nu \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) \right| \mathbf{1}_{\{\widehat{\theta}_n \notin \Theta\}} \\
&\qquad + \left| \nu \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_0)) \right| \\
&\quad \le \int_0^1 |\widehat{\nu}_n(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n)) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n))| \, \mathrm{d}u \mathbf{1}_{\{\widehat{\theta}_n \in \Theta\}} \\
&\qquad + \left| \widehat{\nu}_n \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) - \nu \left(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n) \, \mathrm{d}u \right) \right| \mathbf{1}_{\{\widehat{\theta}_n \notin \Theta\}} \\
&\qquad + \int_0^1 |\nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_u^n)) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,; \theta_0))| \, \mathrm{d}u \\
&\quad =: S_1' + S_2' + S_3'.
\end{aligned}$$

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As for S_1', since

$$S_1' \le \sup_{\theta\in\Theta} |\widehat{\nu}_n(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta)) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta))|,$$

we see from Lemmas A.2 and C.4 that $S_1' \xrightarrow{p} 0$.

As for S_2', since

$$\{|S_2'| > \epsilon'\} \subset \left\{\mathbf{1}_{\{\widehat{\theta}_n \notin \Theta\}} = 1\right\} = \{\widehat{\theta}_n \notin \Theta\},$$

we have $S_2' \xrightarrow{p} 0$ by the consistency of $\widehat{\theta}_n$.

As for S_3', note that

$$\begin{aligned} S_3' &\le \int_0^1 |\nu(\partial_\gamma \boldsymbol{H}_{\alpha,k}(\,\cdot\,;\theta_u^n)) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,k}(\,\cdot\,;\theta_0))|\mathbf{1}_{\{|\theta_u^n-\theta_0|<\delta'\}}\,\mathrm{d}u \\ &\quad + \int_0^1 |\nu(\partial_\gamma \boldsymbol{H}_{\alpha,k}(\,\cdot\,;\theta_u^n)) - \nu(\partial_\gamma \boldsymbol{H}_{\alpha,k}(\,\cdot\,;\theta_0))|\mathbf{1}_{\{|\theta_u^n-\theta_0|\ge\delta'\}}\,\mathrm{d}u. \end{aligned}$$

The first term on the right-hand side goes to zero in probability due to the continuity of the mapping $\theta \mapsto \nu(\partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta))$, which is a consequence by the Lebesgue convergence theorem with Lemma C.3 and the condition M[2]. The second term on the right-hand side also goes to zero in probability by the consistency of θ_u^n to θ_0. Hence we see that $S_3' \xrightarrow{p} 0$, which concludes the convergence (B.3).

Finally, we can show that

$$\sqrt{T_n}\big(\widehat{\nu}_n(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0)) - \nu(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0))\big)$$

is asymptotically normal by checking the conditions in Lemma A.2 with $\boldsymbol{H}_\theta := \widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta)$. In fact, it follows from Lemma C.3 and M[4] that $\nu(|\boldsymbol{H}_\theta|^4) < \infty$ for each $\theta \in \Theta$, and that H5 also holds true since, by Lemma C.3 and (S2),

$$\begin{aligned} &\int_0^{\epsilon_n} \big(H_{\alpha,k}^f(z;\theta) + H_{\alpha,k}^F(z;\theta) + H_p(z;\theta) + H_{\gamma_0}(z)\big)\nu(\mathrm{d}z) \\ &\quad \lesssim \int_0^{\epsilon_n} z\nu(\mathrm{d}z) \\ &\quad = \mathrm{o}\big(T_n^{-1/2}\big), \qquad n \to \infty. \end{aligned}$$

As a consequence, we see from Lemma A.2 that

$$\sqrt{T_n}(\widehat{\nu}_n(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0)) - \nu(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0))) \xrightarrow{d} N_{2K+4}(\mathbf{0}_{2K+4}, \Sigma_K),$$

where Σ_K is given in (B.2). Now, putting

$$\widehat{\Gamma}_K := \begin{pmatrix} I_{2K+3} & \widehat{\nu}_n\big(\int_0^1 \partial_\gamma \boldsymbol{H}_{\alpha,K}(\,\cdot\,:\theta_0)\,\mathrm{d}u\big) \\ \mathbf{0}_{2K+3} & 1 \end{pmatrix},$$

we have the expression

$$\sqrt{T_n}\begin{pmatrix}\widehat{\boldsymbol{a}}^f_{\alpha,K}-\boldsymbol{a}^f_{\alpha,K}\\ \widehat{\boldsymbol{a}}^F_{\alpha,K}-\boldsymbol{a}^F_{\alpha,K}\\ \widehat{p}_n-p_0\\ \widehat{\gamma}_n-\gamma_0\end{pmatrix}=\sqrt{T_n}\begin{pmatrix}\widehat{\nu}_n(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\widehat{\theta}_n))-\nu(\boldsymbol{H}_{\alpha,K}(\,\cdot\,;\theta_0))\\ \widehat{\gamma}_n-\gamma_0\end{pmatrix}$$

$$=\widehat{\Gamma}_K\cdot\sqrt{T_n}\big(\widehat{\nu}_n(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0))-\nu(\widetilde{\boldsymbol{H}}_{\alpha,K}(\,\cdot\,;\theta_0))\big)+o_p(1).$$

Since $\widehat{\Gamma}_K\xrightarrow{p}\Gamma_K$ by (B.3), Slutsky's lemma yields the result, and the proof is completed. □

According to Shimizu and Zhang [24], Theorems 1 and 2 and their proofs, we find the following results.

Corollary B.1. *Under the same assumptions as in Theorem B.1, it holds that*

$$\widehat{\boldsymbol{a}}^G_{\alpha,K}\xrightarrow{p}\boldsymbol{a}^G_{\alpha,K}.$$

In addition, suppose M[2] holds. Then $\widehat{\boldsymbol{a}}^G_{\alpha,K}$ *is asymptotically normal since*

$$\sqrt{T_n}\big(\widehat{\boldsymbol{a}}^G_{\alpha,K}-\boldsymbol{a}^G_{\alpha,K}\big)=-\big(A^f_K\big)^{-1}\widehat{B}_K\sqrt{T_n}\begin{pmatrix}\widehat{\boldsymbol{a}}^f_{\alpha,K}-\boldsymbol{a}^f_{\alpha,K}\\ \widehat{\boldsymbol{a}}^F_{\alpha,K}-\boldsymbol{a}^F_{\alpha,K}\end{pmatrix},\tag{B.4}$$

where $\widehat{B}_K$ *is a consistent estimator of the the* $(K+1)\times(2K+2)$ *matrix* B_K *given as follows:* $B_K:=(B^*_K,\,-I_{K+1})$*, where* $B^*_K:=(b_{kl})_{1\le k,l\le K+1}$*, whose elements are given by*

$$b_{kl}:=\begin{cases}-\dfrac{1}{\sqrt{2\alpha}}a^G_{\alpha,0}, & k=l,\\[2ex] -\dfrac{1}{\sqrt{2\alpha}}\big(a^G_{\alpha,k-l}-a^G_{\alpha,k-l-1}\big), & k>l,\\[2ex] 0, & k<l.\end{cases}$$

For example, an estimator $\widehat{B}_K$ *is obtained by replacing* $a^G_{\alpha,k}$*, each element of* B_K*, by its consistent estimator* $\widehat{a}^G_{\alpha,k}$*.*

Appendix C. Auxiliary lemmas and some proofs

C.1. On functions f_q, F_q, and G_q

Lemma C.1 *Under the NPC condition, the probability density* f_q *is well-defined and uniformly bounded.*

Proof. We prove only the case where $D>0$, and the one for $D=0$ is similarly proved. First, we see from Lemma 3.2 in Feng and Shimizu [9] that

$$p:=\int_0^\infty\widetilde{f}_q(z)\,\mathrm{d}z\in(0,\,1),$$

under the net profit condition NPC. Therefore $f_q:=p^{-1}\widetilde{f}_q$ is well-defined.

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Moreover, since $\mathrm{e}^{-\gamma(z-y)} < 1$ for $y < z < \infty$ and $\mathrm{e}^{-\beta(x-y)} \leq 1$ for $0 < x \leq y$, it follows that f_q is uniformly bounded as follows: for any $x \in \mathbb{R}$,

$$\begin{aligned}\widetilde{f}_q(x) &= D^{-1} \int_0^x \mathrm{d}y \int_y^\infty \mathrm{e}^{-\beta(x-y)} \mathrm{e}^{-\gamma(z-y)} \nu(\mathrm{d}z) \\ &\leq D^{-1} \int_0^x \mathrm{d}y \int_y^\infty \nu(\mathrm{d}z) \\ &\leq D^{-1} \int_0^\infty z\nu(\mathrm{d}z) \\ &< \infty. \end{aligned}$$

□

Lemma C.2. *For F_q and G_q, their tail functions satisfy that $\overline{F}_q, \overline{G}_q \in L^s(\mathbb{R}_+)$ for any $s \geq 1$.*

Proof. Since $0 \leq \overline{F}_q(x), \overline{G}_q(x) \leq 1$ for any $x \in \mathbb{R}$, it suffices to show that $\overline{F}_q, \overline{G}_q \in L^1(\mathbb{R}_+)$. The Fubini theorem yields that

$$\begin{aligned}\int_0^\infty \overline{F}_q(x)\,\mathrm{d}x &= \int_0^\infty \left(\int_u^\infty f_q(x)\mathrm{d}x \right) \mathrm{d}u \\ &= \int_0^\infty x f_q(x)\,\mathrm{d}x \\ &= \int_0^\infty \left(\int_0^z \mathrm{d}y \int_y^\infty x \mathrm{e}^{-\beta(x-y)} \mathrm{e}^{-\gamma(z-y)}\,\mathrm{d}x \right) \nu(\mathrm{d}z) \\ &= \frac{1}{\beta\gamma} \left(\frac{1}{\beta} - \frac{1}{\gamma} \right) \int_0^\infty (1 - \mathrm{e}^{-\gamma z})\, \nu(\mathrm{d}z) + \frac{1}{\beta\gamma} \int_0^\infty z\, \nu(\mathrm{d}z) \\ &\leq \beta^{-2} \int_0^\infty z\, \nu(\mathrm{d}z) \\ &< \infty. \end{aligned}$$

Moreover, taking the Laplace transform of the DRE (2.4) for G_q, we have

$$\int_0^\infty \mathrm{e}^{-sx} \overline{G}_q(x)\,\mathrm{d}x = \frac{p \int_0^\infty \mathrm{e}^{-sx} f_q(x)\,\mathrm{d}x}{1 - p\int_0^\infty \mathrm{e}^{-sx} \overline{F}_q(x)\,\mathrm{d}x}.$$

Substituting with $s = 0$, we have $\overline{G}_q \in L^1(\mathbb{R}_+)$ since $f_q, \overline{F}_q \in L^1(\mathbb{R}_+)$. □

C.2. On functions H_p, $H^f_{\alpha,k}$, and $H^F_{\alpha,k}$

Lemma C.3. *For each $z \in \mathbb{R}_+$ and $m = 0, 1$, the following inequalities hold true:*

$$\sup_{\theta\in\overline{\Theta}} |\partial_D^m H_p(z;\theta)| \lesssim z, \quad \sup_{\theta\in\overline{\Theta}} |\partial_D^m H^f_{\alpha,k}(z;\theta)| \lesssim z, \quad \sup_{\theta\in\overline{\Theta}} |\partial_D^m H^F_{\alpha,k}(z;\theta)| \lesssim z.$$

Moreover, it follows that

$$\sup_{\theta\in\overline{\Theta}} |\partial_\gamma H_p(z;\theta)| \lesssim z, \quad \sup_{\theta\in\overline{\Theta}} |\partial_\gamma H^f_{\alpha,k}(z;\theta)| \lesssim z, \quad \sup_{\theta\in\overline{\Theta}} |\partial_\gamma H^F_{\alpha,k}(z;\theta)| \lesssim z + z^2.$$

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Proof. We shall show the case where $D>0$. The case where $D=0$ is similarly proved. Since $\Theta=\Theta_1\times\Theta_2$, each Θ_i is open and bounded, we can assume that

$$\Theta=\left(\eta_1,\,\eta_1^{-1}\right)\times\left(\eta_2,\,\eta_2^{-1}\right),$$

for $\eta_1,\eta_2>0$ small enough without loss of generality. Then we have

$$\begin{aligned}|H_p(z;\theta)|&=D^{-1}\int_0^z\,\mathrm{d}y\int_y^\infty \mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-y)}\,\mathrm{d}x\\&=\frac{1}{\beta D\gamma}(1-\mathrm{e}^{-\gamma z})\\&\le\frac{1}{(c+\eta_1\eta_2)\eta_1}\left(1-\mathrm{e}^{-\eta_1^{-1}z}\right)\\&\lesssim z,\end{aligned}$$

uniformly in $\theta\in\overline{\Theta}$. Then, due to (2.2), we see that $|H^f_{\alpha,k}(z;\theta)|\le\sqrt{2\alpha}|H_p(z;\theta)|$, which yields $\sup_{\theta\in\overline{\Theta}}|H^f_{\alpha,k}(z;\theta)|\lesssim z$. Moreover, since it also holds that $\sup_{k\in\mathbb{N}_0}|\Phi_{\alpha,k}(x;0)|\le\sqrt{2\alpha}x$, we have

$$\begin{aligned}\left|H^F_{\alpha,k}(z;\theta)\right|&\le\sqrt{2\alpha}D^{-1}\int_0^z\,\mathrm{d}y\int_y^\infty x\mathrm{e}^{-\beta(x-y)}\mathrm{e}^{-\gamma(z-y)}\,\mathrm{d}x\\&=\frac{\sqrt{2\alpha}}{\beta D\gamma}\left(z-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(1-\mathrm{e}^{-\gamma z})\right)\\&\le\frac{\sqrt{2\alpha}}{(c+\eta_1\eta_2)\eta_1}z\\&\lesssim z.\end{aligned}$$

Other proofs can be done similarly, so we omit them and end here. □

Lemma C.4. *For each $z\in\mathbb{R}_+$, $m=0,1$, and $\kappa>0$, the following inequalities hold true:*

$$\begin{aligned}\sup_{\theta\in\overline{\Theta}}\left|\partial_\gamma^m H_p(z;\theta+\kappa)-\partial_\gamma^m H_p(z;\theta)\right|&\lesssim|\kappa|z,\\ \sup_{\theta\in\overline{\Theta}}\left|\partial_\gamma^m H^f_{\alpha,k}(z;\theta+\kappa)-\partial_\gamma^m H^f_{\alpha,k}(z;\theta)\right|&\lesssim|\kappa|z,\\ \sup_{\theta\in\overline{\Theta}}\left|H^F_{\alpha,k}(z;\theta+\kappa)-H^F_{\alpha,k}(z;\theta)\right|&\lesssim|\kappa|z,\\ \sup_{\theta\in\overline{\Theta}}\left|\partial_\gamma H^F_{\alpha,k}(z;\theta+\kappa)-\partial_\gamma H^F_{\alpha,k}(z;\theta)\right|&\lesssim|\kappa|(z^2+z).\end{aligned}$$

Proof. We shall show the case where $D>0$. The case where $D=0$ is similarly proved. As in the previous proof, we assume that $\Theta=\left(\eta_1,\,\eta_1^{-1}\right)\times\left(\eta_2,\,\eta_2^{-1}\right)$ for $\eta_1,\eta_2>0$ small enough without loss of generality, and put $\kappa:=(\kappa_1,\kappa_2)^\top$.

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Let

$$h_\theta(x, y, z) := \mathrm{e}^{-\gamma(x-y)} \exp\left(-\frac{c}{D}(x-y)\right) \mathrm{e}^{-\gamma(z-y)}.$$

Note that it follows for $y < x < \infty$, $0 < y \le z$ that

$$\left|\exp\left(-\frac{c}{D+\kappa_2}(x-y)\right) - \exp\left(-\frac{c}{D}(x-y)\right)\right| \mathrm{e}^{-\gamma(z-y)} \ge |h_{\theta+\kappa}(x, y, z) - h_\theta(x, y, z)|. \quad \text{(C.1)}$$

Using this inequality, we have

$$\begin{aligned}
&|H_p(z; \theta+\kappa) - H_p(z; \theta)| \\
&\quad = \left|\int_0^z \mathrm{d}y \int_y^\infty \left(\frac{1}{D+\kappa_2} h_{\theta+\kappa}(x, y, z) - \frac{1}{D} h_\theta(x, y, z)\right) \mathrm{d}x\right| \\
&\quad \le \left|\int_0^z \mathrm{d}y \int_y^\infty \left(\frac{1}{D+\kappa_2} - \frac{1}{D}\right) h_{\theta+\kappa}(x, y, z) \,\mathrm{d}x\right| \\
&\qquad + \left|\int_0^z \mathrm{d}y \int_y^\infty \frac{1}{D}(h_{\theta+\kappa}(x, y, z) - h_\theta(x, y, z)) \,\mathrm{d}x\right| \\
&\quad \le \int_0^z \mathrm{d}y \int_y^\infty \left(\frac{1}{D} - \frac{1}{D+\kappa_2}\right) \exp\left(-\frac{c}{D+\kappa_2}(x-y)\right) \mathrm{e}^{-\gamma(z-y)} \,\mathrm{d}x \\
&\qquad + \int_0^z \mathrm{d}y \int_y^\infty \frac{1}{D}\left(\exp\left(-\frac{c}{D+\kappa_2}(x-y)\right) - \exp\left(-\frac{c}{D}(x-y)\right)\right) \mathrm{e}^{-\gamma(z-y)} \,\mathrm{d}x \\
&\quad = \frac{2}{c\gamma D}(1 - \mathrm{e}^{-\gamma z})\kappa_2 \\
&\quad \le \frac{2}{c\eta_1\eta_2}\left(1 - \mathrm{e}^{-\eta_1^{-1} z}\right)|\kappa| \lesssim |\kappa| z,
\end{aligned}$$

uniformly in $\theta \in \overline{\Theta}$.

As for $H^f_{\alpha,k}$, since

$$\left|H^f_{\alpha,k}(z; \theta+\kappa) - H^f_{\alpha,k}(z; \theta)\right| \le \sqrt{2\alpha}\,|H_p(z; \theta+\kappa) - H_p(z; \theta)|$$

by (2.2), we obtain that

$$\sup_{\theta \in \overline{\Theta}} \left|H^f_{\alpha,k}(z; \theta+\kappa) - H^f_{\alpha,k}(z; \theta)\right| \lesssim z.$$

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Similarly, from (2.2) and (C.1), we have

$$
\begin{aligned}
&\left|H^F_{\alpha,k}(z;\theta+\kappa)-H^F_{\alpha,k}(z;\theta)\right| \\
&\le \sqrt{2\alpha}\left|\int_0^z \mathrm{d}y\int_y^\infty x\left(\frac{1}{D+\kappa_2}h_{\theta+\kappa}(x,y,z)-\frac{1}{D}h_\theta(x,y,z)\right)\mathrm{d}x\right| \\
&\le \sqrt{2\alpha}\left|\int_0^z \mathrm{d}y\int_y^\infty \left(\frac{1}{D+\kappa_2}-\frac{1}{D}\right)xh_{\theta+\kappa}(x,y,z)\,\mathrm{d}x\right| \\
&\quad+\sqrt{2\alpha}\left|\int_0^z \mathrm{d}y\int_y^\infty \frac{1}{D}(xh_{\theta+\kappa}(x,y,z)-xh_\theta(x,y,z))\,\mathrm{d}x\right| \\
&\le \sqrt{2\alpha}\int_0^z \mathrm{d}y\int_y^\infty \left(\frac{1}{D}-\frac{1}{D+\kappa_2}\right)x\exp\left(\frac{c}{D+\kappa_2}(x-y)\right)\mathrm{e}^{-\gamma(z-y)}\,\mathrm{d}x \\
&\quad+\frac{\sqrt{2\alpha}}{c}\int_0^z \mathrm{d}y\int_y^\infty \frac{1}{D}\left(x\exp\left(\frac{c}{D+\kappa_2}(x-y)\right)-x\exp\left(\frac{c}{D}(x-y)\right)\right)\mathrm{e}^{-\gamma(z-y)}\,\mathrm{d}x \\
&=\frac{\sqrt{2\alpha}}{c\gamma D}\left(2z+\left(\frac{3D+2\kappa_2}{c}-\frac{2}{\gamma}\right)(1-\mathrm{e}^{-\gamma z})\right)\kappa_2 \\
&\le \frac{\sqrt{2\alpha}}{c\eta_2}\left(\frac{2}{\eta_1}z+\left(\frac{5}{c\eta_2}-2\eta_1\right)\frac{1}{\eta_1}(1-\mathrm{e}^{-\eta_1^{-1}z})\right)|\kappa| \\
&\lesssim |\kappa|z,
\end{aligned}
$$

uniformly in $\theta\in\overline{\Theta}$.

Other proofs can be done similarly, so we omit them and end here. □

C.3. On functions P, $Q_{\alpha,k}$, P^*, and $Q^*_{\alpha,k}$

Lemma C.5. *The following statements are true.*

(a) *For each $x\in\mathbb{R}$, the mappings*

$$(p,\gamma,D)\mapsto P(x;p,\gamma,D)\quad \textit{and}\quad (p,\gamma,D)\mapsto P^*(x;p,\gamma,D)$$

are of $C^1((0,1)\times\Theta)$.

(b) *For each $x\in\mathbb{R}$, the mappings*

$$(p,\gamma,D)\mapsto Q_{\alpha,K}(x;p,\gamma,D)\quad \textit{and}\quad (p,\gamma,D)\mapsto Q^*_{\alpha,K}(x;p,\gamma,D)$$

are of $C^1((0,1)\times\Theta)$.

Proof. Since the statement (a) is clear from the concrete form of P and P^*. For the proof of (b), it suffices to show that the function $b\mapsto\Psi_{\alpha,k}(x;b)$ is of $C^1(\mathbb{R})$. For $M>0$ large enough, it

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follows for $z < x$ that

$$\begin{aligned}\sup_{|b|<M}\left|\partial_b\left(b\mathrm{e}^{b(x-z)}\varphi_{\alpha,k}(z)\right)\right| &= \sup_{|b|<M}\left|\mathrm{e}^{b(x-z)}\varphi_{\alpha,k}(z) + b(x-z)\,\mathrm{e}^{b(x-z)}\varphi_{\alpha,k}(z)\right| \\ &\le \sqrt{2\alpha}\sup_{|b|<M}\,(1+|b|(x-z))\,\mathrm{e}^{b(x-z)} \\ &< \sqrt{2\alpha}(1+Mx)\,\mathrm{e}^{Mx},\end{aligned}$$

which is integrable on $(0, x]$ with respect to dz. Hence the Lebesgue convergence theorem yields that $b \mapsto \Psi_{\alpha,k}(x; b)$ is of $C^1(\mathbb{R})$. □

Lemma C.6. *Under the assumptions S1, S2, and M[2], the following hold true:*

(a) $\sup_{x\in\mathbb{R}_+} |P(x; \widehat{p}_n, 0, \widehat{D}_n) - P(x; p_0, 0, D_0)| \xrightarrow{p} 0,$

(b) $\sup_{x\in\mathbb{R}_+} |Q_{\alpha,k}(x; \widehat{p}_n, 0, \widehat{D}_n) - Q_{\alpha,k}(x; p_0, 0, D_0)| \xrightarrow{p} 0,$

(c) $\sup_{x\in\mathbb{R}_+} |P^*(x; \widehat{p}_n, 0, \widehat{D}_n) - P^*(x; p_0, 0, D_0)| \xrightarrow{p} 0,$

(d) $\sup_{x\in\mathbb{R}_+} |Q^*_{\alpha,k}(x; \widehat{p}_n, 0, \widehat{D}_n) - Q^*_{\alpha,k}(x; p_0, 0, D_0)| \xrightarrow{p} 0.$

Proof. As for (a) with $D > 0$, it follows from the mean value theorem that

$$\begin{aligned}&\sup_{x\in\mathbb{R}_+} |P(x; \widehat{p}_n, 0, \widehat{D}_n) - P(x; p_0, 0, D_0)| \\ &\le \left|\frac{1}{c(1-\widehat{p}_n)} - \frac{1}{c(1-p_0)}\right| + \frac{1}{c(1-p_0)}\sup_{x\in\mathbb{R}_+}\left|\mathrm{e}^{-c\widehat{D}_n^{-1}x} - \mathrm{e}^{-cD_0^{-1}x}\right| \\ &\lesssim \left|\frac{1}{c(1-\widehat{p}_n)} - \frac{1}{c(1-p_0)}\right| + \frac{1}{c(1-p_0)}|\widehat{D}_n - D_0| \xrightarrow{p} 0.\end{aligned}$$

The proofs for (b)–(d) are similar by using the fact (2.2). The case of $D = 0$ is also similar. □

C.4. Proof of Lemma 2.1

Note the explicit expression in Feng and Shimizu [9, Proposition 4.1]:

$$W^{(q)}(x) = \begin{cases} \dfrac{1}{D(1-p)(\beta+\gamma)}\displaystyle\int_{[0,x)}[\mathrm{e}^{\gamma(x-z)} - \mathrm{e}^{-\beta(x-z)}]\,G_q(\mathrm{d}z), & D > 0, \\ \dfrac{1}{c(1-p)}\displaystyle\int_{[0,x)}\mathrm{e}^{\gamma(x-z)}\,G_q(\mathrm{d}z), & D = 0.\end{cases}$$

By the integration by parts of this expression by noticing the jump at the origin, we have the result.

C.5. Proof of Lemma 2.5

We will show only the case where $D > 0$. The same calculation applies for $D = 0$. According to Lemmas 2.2 and 2.3, we should show that $f := \overline{G}_q$ satisfies (2.3).

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Let $\epsilon, M > 0$ be any constants with $\epsilon < M$. We firstly show that the infinite sum

$$\overline{G}_q(x) = \sum_{k=0}^{\infty} (1-p)p^k \int_x^{\infty} f_q^{*k}(z)\,\mathrm{d}z$$

can be differentiable under the summation sign on (ϵ, M).

We shall put the partial sum

$$g_N(x) := \sum_{k=0}^{N} (1-p)p^k \int_x^{\infty} f_q^{*k}(z)\,\mathrm{d}z$$

for an integer $N \in \mathbb{N}$, and note that $g_N(x) \to \overline{G}_q(x)$ $(N \to \infty)$ for all $x > 0$. Next, note the differential formula for the convolution: for functions $f, g \in C^1(\mathbb{R}_+)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f * g)(x) = (f * \partial_x g)(x) + f(x)g(0).$$

In particular, if $f(0) = 0$, then it follows for $k \in \mathbb{N}$ that

$$\partial_x f^{*k} = f * \partial_x f^{*(k-1)} = \cdots = f^{*(k-1)} * \partial_x f,$$

by induction. Using this formula, we have for $m = 1, 2, \ldots$ that

$$\partial_x^m g_N(x) = p(1-p^{m-2})\partial_x^{m-1} f_q(x) - p^{m-1} \sum_{k=0}^{N-m+1} (1-p)p^k f_q^{*k} * \partial_x^{m-1} f_q(x),$$

where we regard $\partial_x^0 f_q \equiv 0$ as a convention. Now, $f_q(x)$ is bounded by Lemma C.1, so f_q^{*k} $(k = 1, 2, \ldots)$ is also bounded. Moreover, it also follows that $\partial_x^m f_q$ is bounded on (ϵ, M). Indeed,

$$\partial_x f_q(x) = \frac{1}{pD}\left[-\beta f_q(x) + \overline{\nu}(x) + \int_x^{\infty} \overline{\nu}(z)\,\mathrm{d}z\right],$$

and we see by induction that for any $m \geq 2$,

$$|\partial_x^{m-1} f_q(x)| \lesssim 1 + \sum_{k=1}^{m-1} |\partial_x^{k-1} \overline{\nu}(x)| + \left|\int_x^{\infty} \overline{\nu}(z)\,\mathrm{d}z\right|.$$

Then it follows for any $\epsilon > 0$ that

$$\sup_{x\in(\epsilon,M)} |\partial_x^{m-1} f_q(x)| \lesssim 1 + \sum_{k=1}^{m-1} \sup_{x\in(\epsilon,M)} |\partial_x^{k-1} \overline{\nu}(x)| + \int_M^{\infty} \overline{\nu}(z)\,\mathrm{d}z \lesssim 1 + M^C, \qquad \text{(C.2)}$$

by the assumption (2.6). Therefore we can confirm the sequence $\{\partial_x^m g_N(x)\}_{N\in\mathbb{N}}$ is a uniform Cauchy sequence on (ϵ, ∞):

$$\lim_{N,N'\to\infty} \sup_{x\in(\epsilon,M)} |\partial_x^m g_N(x) - \partial_x^m g_{N'}(x)| = 0.$$

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Hence the term 'differential theorem' says that for any $\epsilon, M > 0$,

$$\lim_{N\to\infty} \partial_x^m g_N(x) = \partial_x^m \overline{G}_q(x), \qquad x \in (\epsilon, M).$$

Therefore it follows for any $x > 0$ that

$$\begin{aligned}\partial_x^m \overline{G}_q(x) &= p(1-p^{m-2})\partial_x^{m-1} f_q(x) - p^{m-1}\left(\sum_{k=0}^{\infty}(1-p)p^k f_q^{*k}\right) * \partial_x^{m-1} f_q \\ &= p(1-p^{m-2})\partial_x^{m-1} f_q(x) - p^{m-1}\,\mathbb{E}\big[\partial_x^{m-1} f_q(x-Z)\big],\end{aligned} \tag{C.3}$$

where Z is a random variable with the distribution G_q whose probability density is given by $\sum_{k=0}^{\infty}(1-p)p^k f_q^{*k}(x)$.

Now, as for the second term in the last right-hand side of (C.3), we see that

$$\mathbb{E}\big[\partial_x^{m-1} f_q(x-Z)\big] = \mathrm{O}(1+x^{\kappa}), \qquad x \to \infty.$$

Indeed, it follows from (C.2) that

$$\sup_{x>\epsilon} \frac{|\partial_x^{m-1} f_q(x-Z)|}{1+x^C} \lesssim \sup_{x>\epsilon} \frac{1+x^C+Z^C}{1+x^C} \le 1+|Z|^C,$$

and the last term is integrable by the assumption. Then the Lebesgue convergence theorem and the equality (C.3) yield

$$\partial_x^m \overline{G}_q(x) = \mathrm{O}(1+x^{\kappa}), \qquad x \to \infty,$$

which implies that $x^{m/2}\partial_x^m \overline{G}_q(x) \in L_\alpha^2(\mathbb{R}_+)$ for any $\alpha > 0$ and $m \ge 2$. Similarly, as in (2.8), we have

$$|x^{m/2}\partial_x^m \overline{G}_q(x)|^2 \mathrm{e}^{-\alpha x} = \mathrm{O}(x^m \mathrm{e}^{-(\alpha-2k)x}), \qquad x \to \infty,$$

and therefore $x^{m/2}\partial_x^m \overline{G}_q(x) \in L_\alpha^2(\mathbb{R}_+)$ for any $\alpha > 2\kappa$. This completes the proof.

C.6. Proof of Theorem 3.1

As for the consistency, thanks to Corollary B.1 and Lemma C.5, the continuous mapping theorem yields that

$$\widehat{W}_K^{(q)}(x) \xrightarrow{p} W_K^{(q)}(x),$$

for each $x \in \mathbb{R}_+$ and $q \ge 0$. In particular, as $q = 0$, noticing that $\widehat{\gamma}_n = \gamma_0 = 0$, we see from Lemma C.6 that

$$\begin{aligned}\sup_{x\in\mathbb{R}_+} |\widehat{W}_K^{(0)}(x) - W_K^{(0)}(x)| \le \sup_{x\in\mathbb{R}_+} &|P(x;\widehat{p}_n, 0, \widehat{D}_n) - P(x; p_0, 0, D_0)| \\ &+ \sum_{k=0}^{K} \widehat{a}_{\alpha,K}^G \sup_{x\in\mathbb{R}_+} |Q_{\alpha,k}(x;\widehat{p}_n, 0, \widehat{D}_n) - Q_{\alpha,k}(x; p_0, 0, D_0)| \\ &+ \sum_{k=0}^{K} |\widehat{a}_{\alpha,k}^G - a_{\alpha,k}^G| \sup_{x\in\mathbb{R}_+} |Q_{\alpha,k}(x; p_0, 0, D_0)| \xrightarrow{p} 0.\end{aligned}$$

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As for the asymptotic normality, we only show the case where $D > 0$ since the proof for $D = 0$ is done similarly.

Note that

$$\begin{aligned}\sqrt{T_n}\big(\widehat{W}_K^{(q)}(x) - W_K^{(q)}(x)\big) &= \sqrt{T_n}(P(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) - P(x; p_0, \gamma_0, D_0)) \\ &\quad - \sum_{k=0}^{K} \widehat{a}^G_{\alpha,k}\sqrt{T_n}(Q_{\alpha,k}(x; \widehat{p}_n, \widehat{\gamma}_n, \widehat{D}_n) - Q_{\alpha,k}(x; p_0, \gamma_0, D_0)) \\ &\quad - \sum_{k=0}^{K} Q_{\alpha,k}(x; p_0, \gamma_0, D_0)\sqrt{T_n}\big(\widehat{a}^G_{\alpha,k} - a^G_{\alpha,K}\big) \\ &=: U_1 + U_2 + U_3.\end{aligned}$$

On U_1, applying the mean value theorem, there exists some p_n^*, γ_n^*, D_n^* such that

$$\begin{aligned}U_1 &= \sum_{k=0}^{K} \partial_{(p,\gamma)}P(x; p_n^*, \gamma_n^*, D_n^*)\sqrt{T_n}\begin{pmatrix}\widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0\end{pmatrix} \\ &\quad + \sum_{k=0}^{K} \partial_D P(x; p_n^*, \gamma_n^*, D_n^*)\sqrt{T_n}(\widehat{D}_n - D_0) \\ &= \sum_{k=0}^{K} \partial_{(p,\gamma)}P(x; p_0, \gamma_0, D_0)\sqrt{T_n}\begin{pmatrix}\widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0\end{pmatrix} + o_p(1),\end{aligned}$$

by Lemma A.1 and the continuous mapping theorem. By an argument similar to that above, we have

$$U_2 = -\sum_{k=0}^{K} \widehat{a}^G_{\alpha,k}\partial_{(p,\gamma)}Q_{\alpha,k}(x; p_0, \gamma_0, D_0)\sqrt{T_n}\begin{pmatrix}\widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0\end{pmatrix} + o_p(1).$$

Moreover, on U_3, we obtain by (B.4) in Corollary B.1 that

$$U_3 = \boldsymbol{Q}_{\alpha,K}(x; p_0, \gamma_0, D_0)^\top A_K^{-1}\widehat{B}_K\sqrt{T_n}\begin{pmatrix}\widehat{\boldsymbol{a}}^f_{\alpha,K} - \boldsymbol{a}^f_{\alpha,K} \\ \widehat{\boldsymbol{a}}^F_{\alpha,K} - \boldsymbol{a}^F_{\alpha,K}\end{pmatrix}.$$

As a consequence, we have

$$U_1 + U_2 + U_3 = C_K(x)\sqrt{T_n}\begin{pmatrix}\widehat{\boldsymbol{a}}^f_{\alpha,K} - \boldsymbol{a}^f_{\alpha,K} \\ \widehat{\boldsymbol{a}}^F_{\alpha,K} - \boldsymbol{a}^F_{\alpha,K} \\ \widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0\end{pmatrix} + o_p(1), \tag{C.4}$$

and Theorem B.2 yields the result.

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C.7. Proof of Theorem 3.2

By the same argument as in the proof of Theorem 3.1 in Section C.6, we have

$$\sqrt{T_n}\big(\widehat{Z}_K^{(q)}(x) - Z_K^{(q)}(x)\big) = qC_K^*(x)\sqrt{T_n}\begin{pmatrix} \widehat{\boldsymbol{a}}^f_{\alpha,K} - \boldsymbol{a}^f_{\alpha,K} \\ \widehat{\boldsymbol{a}}^F_{\alpha,K} - \boldsymbol{a}^F_{\alpha,K} \\ \widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0 \end{pmatrix} + \mathrm{o}_p(1), \tag{C.5}$$

and Theorem B.2 yields the result.

C.8. Proof of Theorem 3.1

Noticing the expressions (C.4) and (C.5), we have

$$\sqrt{T_n}\begin{pmatrix} \widehat{W}_K^{(q)}(x) - W_K^{(q)}(x) \\ \widehat{Z}_K^{(q)}(x) - Z_K^{(q)}(x) \end{pmatrix} = \sqrt{T_n}\begin{pmatrix} C_K(x) \\ qC_K^*(x) \end{pmatrix}\begin{pmatrix} \widehat{\boldsymbol{a}}^f_{\alpha,K} - \boldsymbol{a}^f_{\alpha,K} \\ \widehat{\boldsymbol{a}}^F_{\alpha,K} - \boldsymbol{a}^F_{\alpha,K} \\ \widehat{p}_n - p_0 \\ \widehat{\gamma}_n - \gamma_0 \end{pmatrix} + \mathrm{o}_p(1).$$

Then Theorem B.2 yields the result.

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