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ASYMPTOTIC PROPERTIES OF THE GAUGE AND POWER OF STEP-INDICATOR SATURATION

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Detecting multiple structural breaks at unknown dates is a central challenge in time-series econometrics. Step-indicator saturation (SIS) addresses this challenge during model selection, and we develop its asymptotic theory for tuning parameter choice. We study its frequency gauge—the false detection rate—and show it is consistent and asymptotically normal. Simulations suggest that a smaller gauge minimizes bias in post-selection regression estimates. For the small gauge situation, we develop a complementary Poisson theory. We compare the local power of SIS to detect shifts with that of Andrews' break test. We find that SIS excels when breaks are near the sample end or closely spaced. An application to U.K. labor productivity reveals a growth slowdown after the 2008 financial crisis.

1. INTRODUCTION

Step-indicator saturation (SIS), suggested by Castle et al. (2015), is a model selection algorithm designed to address location shifts in time series without restrictions on their number, date, and distance from each other or sample boundaries. In its most general form, the initial specification has a k-variate regressor x_i, which can be of the exogenous, (trend-)stationary, or random walk type, and as many step indicators as observations:

$$y_i = \beta' x_i + \sum_{j=1}^{n} \delta_j 1_{(i \le j)} + \varepsilon_i \qquad \text{for } i = 1, \dots, n, \tag{1}$$

where the parameters satisfy $\beta \in \mathbb{R}^k$ and $(\delta_1, \dots, \delta_n)' \in \mathbb{R}^n$. Thus, level shifts are deterministic. If the number of $\delta_j \neq 0$ and their location j were small and known, the model could be estimated by least squares. In practice, the nature of location

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shifts is often unknown, so the general specification (1) must be estimated through repeated use of least squares over subsets of step-indicators. For each j, there is a binary choice of whether $\delta_j = 0$ or not. Taken together, this gives 2^n relevant submodels. An exhaustive search is computationally infeasible and must be replaced by a good approximation. A block-search algorithm has been proposed by Doornik (2009) based on Hoover and Perez (1999) and Hendry and Krolzig (2005). Such algorithms depend on a tuning parameter, which can be chosen indirectly by controlling the type I error. Castle et al. (2015) measured type I errors in terms of the frequency of falsely detected shifts, which we will refer to as the gauge. We develop an asymptotic theory to understand the gauge of SIS. Research on the related impulse-indicator saturation (IIS) algorithm indicates that an analysis of the Doornik (2009) algorithm would be difficult. A better starting point would be to analyze the exhaustive algorithm or a simplified algorithm as done for IIS in Berenguer-Rico, Johansen, and Nielsen (2023) and in Johansen and Nielsen (2009, 2016b), respectively. Here, we develop an asymptotic theory to understand the gauge of simplified versions of SIS, and show that for conformable values of the gauge, the procedure maintains power to detect shifts.

Location shifts are a common phenomenon in observed time series (Perron, 1989; Andrews, 1993; Bai and Perron, 1998), and a failure to address them can affect model selection probabilities of variables (Castle and Hendry, 2014), distort parameter estimation (Hendry and Mizon, 2011), and result in forecast failure (Clements and Hendry, 1998). The growing importance of SIS in tackling location shifts is reflected in its applications in fields as varied as economics (Chuffart and Hooper, 2019; Bernstein and Martinez, 2021; Pellini, 2021), climate science (Raggad, 2018; Pretis et al., 2018b; Koch et al., 2022; O'Callaghan, Yau, and Hepburn, 2022), and public health (Doornik, Castle, and Hendry, 2022). However, despite its popularity, no study of its asymptotic properties exists. This study fills the gap using theoretical insights to shed light on four pivotal areas for practitioners: First, the tuning parameter (gauge) can be closely aligned with the investigator's preferences without detailed knowledge of the regressor type. Second, the bias in post-selection regression estimates can be addressed by choosing a small gauge, or switching to the Poisson theory for the gauge when it is vanishing. Third, SIS can detect minor shifts after a short period of upheaval and maintains power near the end of the sample. Fourth, SIS has weak regularity conditions for the regressors.

In this article, we study the split-half SIS algorithm. This is a simplified version of the SIS algorithm as implemented in tools like EViews (2020), *gets* in R (Pretis, Reade, and Sucarrat, 2018a; Sucarrat, 2020), and *Autometrics* in Oxmetrics (Doornik, 2009). Simulations by Castle et al. (2015) indicate that the general SIS has the same gauge properties as split-half SIS, but detects a broader range of shifts with greater power. Split-half SIS splits the sample into two subsets with n_1 and $n - n_1$ observations. It then applies stylized SIS to both subsamples. Stylized SIS, when applied to the second subsample, excludes the first set of step-indicators. For example, it is imposed that $\delta_j = 0$ for $j \leq n_1$ in (1). The model is then estimated

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by OLS to determine which of the coefficients δ_j for $j > n_1$ are significant. An analysis of split-half SIS can shed light on more general versions of the algorithm and provide mathematical tools for examining related algorithms.

Split-half SIS results in n decisions about the inclusion of step-indicators $1_{(i\leq j)}$. This method requires setting a tuning parameter: a common cut-off c for selecting step-indicators. Drawing inspiration from classical test theory, we aim to determine the cut-off c indirectly from a measure of type I error. Classical testing problems focus on single-decision problems in which the critical value—or the cut-off—is chosen from the size of the test, which is the probability of a type I error of falsely rejecting the hypothesis. In multiple-decision problems, there are many alternative ways of measuring type I error. We study the gauge, which is based on a count of the false rejections. The gauge is also referred to as the expected error rate (Miller, 1981) or the per-comparison error rate (Dudoit and van der Laan, 2010). A concept similar to the gauge was introduced by Hoover and Perez (1999). The term gauge originates in Hendry and Santos (2010) and Castle, Doornik, and Hendry (2011) (see also Hendry and Doornik, 2014, p. 122).

When comparing the gauge to alternative measures of type I error in the context of multiple-decision testing problems, we note that the gauge is more amenable to asymptotic analysis. These alternatives include the probability error rate (Miller, 1981; Dudoit and van der Laan, 2010), also called the family-wise error rate (Dudoit and van der Laan, 2010), and the false discovery rate (Benjamini and Hochberg, 1995). The probability error rate is the probability of at least one false detection. It requires a detailed assessment of the dependence of the individual decisions. In contrast, the gauge ignores this dependence structure. The false discovery rate is the expected value of the proportion of type I errors among the rejected hypotheses. Under our null hypothesis of no location shift in the data-generating process, the false discovery rate equals unity.

To formalize the notion of the gauge, consider two equivalent approaches to formulate stylized SIS. We introduced this algorithm by imposing $\delta_j = 0$ for $j \leq n_1$ in (1), estimating the model by least squares and then investigating the significance of the remaining step-indicators. An equivalent alternative formulation is to first regress y_i on x_i and an intercept for $i \leq n_1$. This yields least squares estimators $\hat{\beta}_1$ and $\hat{\sigma}_1^2$. These estimators will be consistent if there are no location shifts in the first subsample. We then compute the scaled residuals in the second subsample. As pointed out by Castle et al. (2015) and as shown in Section 2, we can then inspect the forward differenced residuals for outliers. That is, if there are n observations of (1), compute

$$(\nabla y_i - \hat{\beta}_1' \nabla x_i)/\sqrt{2}\hat{\sigma}_1 \qquad \text{for } i = n_1 + 1, \ldots, n-1, \tag{2}$$

with $\nabla y_i = y_i - y_{i+1}$, and where the $\sqrt{2}$-factor arises since the variance of $\nabla \varepsilon_i$ is twice the variance of ε_i. A location shift is declared if the absolute value of the forward differenced residual exceeds a cut-off, c. The frequency of declared location shifts in the stylized SIS algorithm is the frequency gauge:

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$$\hat{\gamma}_n = \frac{1}{n-n_1-1}\sum_{i=n_1+1}^{n-1} 1_{(|\nabla y_i - \hat{\beta}_1' \nabla x_i| \geq \sqrt{2}\hat{\sigma}_1 c)}. \tag{3}$$

If the data-generating process has no location shifts, then all declarations of shifts are false, so that $\hat{\gamma}_n$ is the average type I error. We show the consistency

$$\hat{\gamma}_n \xrightarrow{\mathsf{p}} \gamma = \mathsf{P}(|\nabla \varepsilon_i| \geq \sqrt{2}\sigma c), \tag{4}$$

for a wide range of time-series regressors x_i, including stationary and non-stationary regressors. We can then choose the cut-off c from the limiting gauge γ. In simulations, we confirm the consistency result and provide some further analysis. In our considerations of the gauge, we focus on the situation where the data-generating process has no location shifts although the concept could be adapted to the situation where there are shifts (Hendry and Doornik, 2014; Johansen and Nielsen, 2016c, p. 122).

The consistency of the frequency gauge for a variety of time-series regressors shows that it is possible to control the type I error of SIS without prior knowledge of the detailed time-series structure. The regressors do have a second-order effect on this consistency result, which we investigate through an asymptotic expansion of the normalized frequency gauge $n^{1/2}(\hat{\gamma}_n - \gamma)$. We find that it is asymptotically zero mean normal, but its variance depends on the correlation structure of ∇x_i and $\nabla \varepsilon_i$. Numerical approximations confirm that the asymptotic variance of the frequency gauge is strictly larger for split-half SIS than for split-half IIS. In contrast to split-half IIS, the asymptotic variance of split-half SIS depends on the temporal persistence of the time series. A small gauge substantially reduces its asymptotic variance.

A challenge to the asymptotic analysis of the frequency gauge for SIS is the temporal and cross-sectional correlation due to the forward differencing of x_i and ε_i in (2). For instance, in the autoregression $x_{i+1} = \rho x_i + \varepsilon_i$ with independent ε_i and x_i, we get that $\nabla \varepsilon_i = \varepsilon_i - \varepsilon_{i+1}$ is temporally and cross-sectionally correlated with $\nabla x_i = x_i - x_{i+1} = (1-\rho)x_i - \varepsilon_i$. In the related asymptotic analysis of IIS by Hendry, Johansen, and Santos (2008) and Johansen and Nielsen (2009, 2013, 2016a, 2016b), the use of impulse-indicators of the form $1_{(i=j)}$ avoids the temporal and cross-sectional correlation structure. Therefore, IIS can be analyzed using a version of the empirical process theory of Koul and Ossiander (1994) (see also Giraitis, Koul, and Surgailis, 2012). Our analysis of the SIS overcomes the correlation problem by combining the empirical process theory with mixingale theory of McLeish (1977).

A simulation study shows that split-half SIS can introduce a bias in the updated estimates for β in (1) that does not vanish asymptotically. The bias is largest when regressors are lagged dependent variables with an autoregressive coefficient close to unity, and when the frequency gauge is large. For split-half SIS, the empirical setting after the selection over the step-indicators resembles an unbalanced panel regression with a small temporal and large cross-sectional range. Each interval in between two consecutive retained step-indicators can be interpreted as another i in the panel that introduces a new individual fixed effect. The incidental parameter problem arises because with a nonzero frequency gauge γ, the number of breaks is approximately $n\gamma$, so that the number of observations in each interval is on average $1/\gamma$ and therefore finite even as the sample size increases. This matches the situation of a panel data model with large cross-sectional dimension and finite time-series dimension, in which biases arise for the dynamic parameter estimators. We conjecture that the bias is due to a combination of the incidental parameter problem (Lancaster, 2000, 2002) and the correlation of the retained step-indicators with the innovations (Arellano and Bond, 1991).

We suggest two different approaches to address the bias in the estimation of β under split-half SIS. First, simulations suggest investigators to use a small frequency gauge, as a smaller frequency gauge is associated with a smaller bias. In a sample of 100 observations, we would recommend a frequency gauge of 1% if one would normally conduct inference at the 5% level. Second, we develop a theory for shrinking the gauge with increasing sample sizes. For this, we consider the absolute gauge

$$\hat{\Gamma}_n = \sum_{i=n_1+1}^{n-1} 1_{(|\nabla y_i - \hat{\beta}_1' \nabla x_i| \geq \sqrt{2}\hat{\sigma}_1 c_n)}, \tag{5}$$

for increasing sequences of the cut-off c_n that satisfy, for some $\lambda > 0$,

$$\mathsf{P}(|\nabla \varepsilon_i| > \sqrt{2}\sigma c_n) = \lambda/n. \tag{6}$$

As the c_n increases with the sample size, the absolute gauge is smaller than the frequency gauge as the sample grows. By modifying the theory of Johansen and Nielsen (2016b), we show that the absolute gauge $\hat{\Gamma}_n$ is asymptotically Poisson distributed. The asymptotic result is the same whether the regressors are stationary or non-stationary. In the proof, we encounter the same dependence issue between ∇x_i and $\nabla \varepsilon_i$. We address this using the Poisson limit theorem of Chen (1975).

An alternative to SIS is the Bai and Perron (1998) procedure. It builds on the Andrews (1993) breaks test and provides estimates for timing and location of breaks. Comparing the power properties of SIS and the Bai–Perron procedure is challenging due to the inherent complexity of both methods. Instead, we compare stylized SIS with the Andrews test. We consider two types of scenarios: first, scenarios where the Andrews test is consistent, with power approaching unity as the sample size increases, while the stylized SIS has power less than unity; second, scenarios where the stylized SIS achieves power approaching unity as the parameter magnitude increases, while the Andrews test has only trivial power. On balance, we find that the Andrews test is preferable if there is one break or two well-separated breaks in the middle of the sample. SIS is preferable for a break near the end of the sample. Such a break is important to discover and address in forecasting contexts (Clements and Hendry, 1998). In turn, SIS is also preferable if two close breaks offset each other, for instance, if the growth rate moves from one level to a slightly different level through a short period of upheaval (see Castle, Hendry, and Martinez (2023) and the empirical illustration). We argue that the results carry over to a comparison with the Bai–Perron procedure.

The proof of the local power results uses convergence on the $D[0,1]$ space of discontinuous functions. We handle the one-break case by the Skorokhod (1956) J_1-metric discussed by Billingsley (1968). However, in order to establish convergence in the two-close-breaks case, we use Skorokhod's M_1 metric in line with Whitt (2002).

Our theory for simplified versions of SIS requires knowledge of the innovation distribution. The normal distribution is the standard choice. Just as in a standard regression, the normality assumption will be testable from the residuals once the model has been fitted. With a finite cut-off, the standard cumulant-based normality test may have to be adjusted. Indeed, this is the case when applying outlier detection with finite cut-off (Berenguer-Rico and Nielsen, 2023). In contrast, standard heteroscedasticity tests remain valid after outlier detection with finite cut-off (Berenguer-Rico and Wilms, 2021). It should be noted that other procedures, such as that of Andrews (1993) only require distributional assumptions that are sufficient to apply a Central Limit Theorem. In turn, SIS requires weaker assumptions to the regressors.

We apply our split-half SIS theory to analyze the U.K. labor productivity from 1980 to 2021. While there is a growing consensus about the decline of productivity growth in the United Kingdom (Chadha, 2022), a simple autoregressive model is not rejected by the standard diagnostic tests. This indicates that location shifts are not always obvious to the investigator. Using a 1% gauge, the split-half SIS algorithm identified multiple shifts in U.K. productivity growth: 0.56% before 2000, 0.37% up to 2008 and 0.04% up to 2020. These findings also illustrate the ability of SIS to find minor shifts around episodes of upheaval, in our case, the 2008 financial crisis and the 2020 COVID pandemic.

Section 2 outlines the model and the SIS algorithm. Sections 3 and 4 present the asymptotic results on the frequency gauge for the stylized and split-half SIS, respectively, while Section 5 presents the Poisson theory for the absolute gauge. Power analysis is found in Section 6. Simulation results are given in Section 7. An empirical illustration follows in Section 8. Section 9 concludes. Proofs are given in an online appendix.

2. MODEL AND ALGORITHMS

We begin by presenting the linear regression model to which we apply the SIS algorithm. Subsequently, we introduce two simplified versions of the SIS: stylized SIS and split-half SIS. Lastly, given that the decision rules on the retaining of step-indicators pertain to differenced innovations, we discuss their notable properties.

2.1. The Model

SIS aims to detect location shifts within the model:

$$y_i = \mu + \beta' x_i + \varepsilon_i \qquad \text{for } i = 1, \ldots, n. \tag{7}$$

By saturating with step-indicators of the type $1_{(i \le j)}$, we obtain equation (1) with $\delta_n = \mu$. In practice, one would expect that only a few of the δ_j parameters in (1) are nonzero, but their number and location are unknown. The regressor x_i is a k-vector, which does not include an intercept. It can include stationary, trend-stationary, and random walk variables, but excludes explosive regressors. The innovations ε_i are independent, identically, distributed with a continuous distribution that is known up to the scale. Further, the innovations are independent of current and past regressors x_j for $j \le i$. The coefficient of the intercept is identified when $\mathbb{E}\varepsilon_i = 0$, but the asymptotic theory does not depend on this constraint.

As a model selection algorithm, the idea of SIS is grounded in the general-to-specific approach to regressor selection of Hoover and Perez (1999). Its core mechanism revolves around iterative backward elimination: in each step, a regression is estimated, the least significant regressor is eliminated, and the smaller model is re-estimated. The iteration stops when the fit of the model deteriorates too much. While a single backward elimination has poor properties for correlated regressors, Hoover and Perez (1999) found that multiple backward eliminations with different starting points have better properties in recovering the original data-generating process. Algorithms, such as *PcGets* (Hendry and Krolzig, 2005) and *Autometrics* (Doornik, 2009) adopt this multi-path approach but search over many more paths to get closer to evaluating all possible subsets of regressors. *Autometrics* allows situations with more regressors than observations by searching over blocks of regressors. This permits saturation with indicators for each observation as in IIS and SIS. The saturation approach allows a simultaneous search over regressors x_i and indicator variables. The simultaneous search is helpful when there is a high sample correlation between regressors and indicator variables (see Hendry and Doornik, 2014). SIS is implemented in the R package *gets* (Pretis et al., 2018a; Sucarrat, 2020), in EViews (2020), and within a structural time-series model in Marczak and Proietti (2016). It is worth noting that *Autometrics* employs indicators of the form $1_{(i \le j)}$ as here, while *gets* utilizes $1_{(i \ge j)}$. A related algorithm based on sensitivity analysis was presented by Becker, Paruolo, and Saltelli (2021).

2.2. Stylized SIS Estimation and Forward Differencing

The simplest block search algorithm is *stylized* SIS. We apply it to (1). It begins by dividing the observations into two parts: the first n_1 observations and the remaining $n_2 = n - n_1$ observations. For the first half, we keep only an intercept and otherwise drop the indicator variables. This gives the model equation

$$y_i = \beta' x_i + \mu 1_{(i \le n_1)} + \sum_{j=n_1+1}^{n} \delta_j 1_{(i \le j)} + \varepsilon_i \qquad \text{for } i = 1, \ldots, n. \tag{8}$$

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Stylized SIS declares an outlier at observation ℓ for $n_1 < \ell < n$ if the t-statistic for $\delta_\ell = 0$ exceeds a cut-off c. Subsequently, we analyze the gauge, which is the count of falsely detected level shifts. We will show that when the cut-off c is chosen as the two-sided $1-\gamma$ quantile of the standard normal distribution, the frequency gauge approaches γ.

It is useful to rewrite equation (8) to get a simple expression for the decision rule. Since the second half-sample is saturated with indicators, that half will have perfect fit. The consequence of this observation is best seen through reparameterization. Multiply x_i by unity, written as a sum of indicators for the first half $(i \le n_1)$ and for the impulses $(i = \ell)$ for $n_1 < \ell \le n$. Decompose the indicator for $(i \le j)$ likewise. This gives

$$y_i = \beta' x_i 1_{(i \le n_1)} + \Big(\mu + \sum_{j=n_1+1}^{n} \delta_j\Big) 1_{(i \le n_1)} + \sum_{\ell=n_1+1}^{n} \beta' x_i 1_{(i=\ell)} + \sum_{j=n_1+1}^{n} \delta_j \sum_{\ell=n_1+1}^{j} 1_{(i=\ell)} + \varepsilon_i.$$

Interchanging the summation order in the last δ-term gives the reparameterization

$$y_i = \beta' x_i 1_{(i \le n_1)} + \nu 1_{(i \le n_1)} + \sum_{\ell=n_1+1}^{n} \eta_\ell 1_{(i=\ell)}, \tag{9}$$

where

$$\nu = \mu + \sum_{j=n_1+1}^{n} \delta_j, \qquad \eta_\ell = \beta' x_\ell + \sum_{j=\ell}^{n_1} \delta_j. \tag{10}$$

As the indicators are orthogonal in (9), the least squares estimators for β, ν, σ^2 are found by standard multiple regression on x_i and the intercept using the first sample, while η_ℓ is estimated by $\hat{\eta}_\ell = y_\ell$. Solving the expression for η_ℓ in (10) for δ_ℓ shows that

$$\hat{\delta}_\ell = \big(\hat{\eta}_\ell - \hat{\beta}' x_\ell\big) - \big(\hat{\eta}_{\ell+1} - \hat{\beta}' x_{\ell+1}\big) = \nabla y_\ell - \hat{\beta}' \nabla x_\ell \qquad \text{for } n_1 < \ell < n, \tag{11}$$

while $\hat{\delta}_n = \hat{\eta}_n - \hat{\beta}' x_n = y_n - \hat{\beta}' x_n$. The t-statistics for $\delta_\ell = 0$ are $\hat{\delta}_\ell / (\omega_\ell \hat{\sigma} \sqrt{2})$, where

$$\omega_\ell^2 = 1 + (\nabla x_\ell)' \Big\{ 2 \sum_{k \le n_1} (x_k - \bar{x}_1)(x_k - \bar{x}_1)' \Big\}^{-1} \nabla x_\ell,$$

with $\bar{x}_1 = n_1^{-1} \sum_{i=1}^{n} x_i$ and where $n_1 < \ell < n$.

Figure 1 illustrates stylized SIS. Panels (a) and (b) show data generated with and without a location shift. The data-generating processes are, respectively, $y_i = \varepsilon_i$ and $y_i = 4 \times 1_{(i \ge 75)} + \varepsilon_i$, with standard normal errors ε_i. The sample size is $n = 100$. The horizontal lines show the expected values of the observations. In panel (b), the vertical line at $i = 75$ indicates the level shift.

Panels (c) and (d) show the results of applying stylized SIS to a first-order autoregressive model. The vertical line at $i = 50$ indicates the split. The first half-sample is used to estimate β, μ, σ^2. Panels (c) and (d) show the t-statistics for

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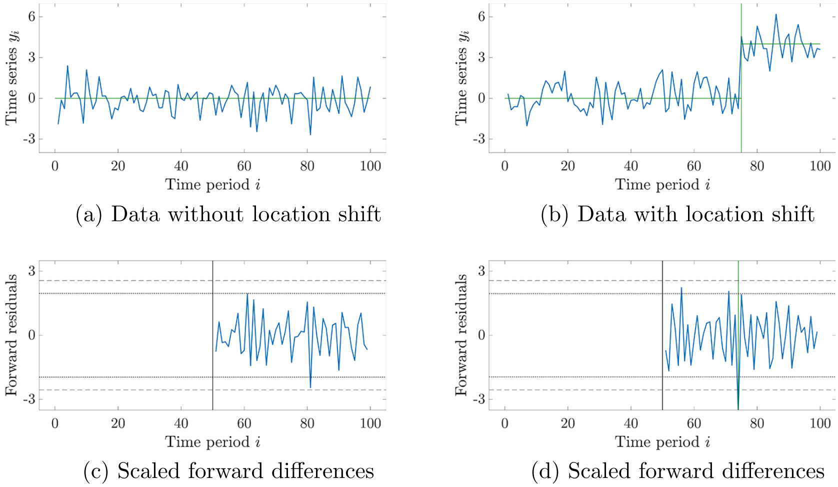

FIGURE 1. Illustration of the stylized SIS algorithm.

testing $\delta_\ell = 0$ for $50 < \ell < 100$. The horizontal lines indicate the cut-offs with frequency gauge of 5% (dotted line) and 1% (dashed line). For panel (d), we see that stylized SIS picks up the level shift and some large errors. This is the usual trade-off between errors of types I and II. The true location shift is correctly identified using both gauge values, while level shifts are only falsely detected with the higher gauge level.

2.3. SIS Algorithms

We present two simplified SIS algorithms in a more formal way. The algorithms involve splitting the sample into two consecutive parts for the n_1 first observations and the $n_2 = n - n_1$ last observations. When working with differenced variables, one observation is lost from each subsample. We define index sets

$$I_1 = (i \le n_1), \quad I_1^\circ = (i < n_1), \quad I_2 = (n_1 < i \le n), \quad I_2^\circ = (n_1 < i < n), \tag{12}$$

and counts $n_1^\circ = n_1 - 1$, $n_2^\circ = n_2 - 1$, and $n^\circ = n_1^\circ + n_2^\circ = n - 2$. For each subsample I_j, for $j = 1, 2$, we estimate the constant intercept regression model $y_i = \mu + \beta' x_i + \varepsilon_i$ by least squares regression and get the estimators

$$\bar{x}_j = n_j^{-1} \sum_{i \in I_j} x_i, \qquad \hat{\beta}_j = \left\{ \sum_{i \in I_j} (x_i - \bar{x}_j)(x_i - \bar{x}_j)' \right\}^{-1} \sum_{i \in I_j} (x_i - \bar{x}_j) y_i, \tag{13}$$

$$\bar{y}_j = n_j^{-1} \sum_{i \in I_j} y_i, \qquad \hat{\sigma}_j^2 = \frac{1}{n_j} \sum_{i \in I_j} \left\{ (y_i - \bar{y}_j) - \hat{\beta}_j'(x_i - \bar{x}_j) \right\}^2. \tag{14}$$

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We will use the estimates from the first subsample to predict location shifts in the second subsample. This corresponds to predicting outliers for the differenced series using $\nabla y_i - \hat{\beta}_1 \nabla x_i$. This gives the forecast correction factors

$$\omega_{1,i}^2 = 1 + (\nabla x_i)' \left\{ 2 \sum_{k \in I_1^\circ} (x_k - \bar{x}_1)(x_k - \bar{x}_1)' \right\}^{-1} \nabla x_i \qquad \text{for } i \in I_2, \tag{15}$$

and we define $\omega_{2,i}^2$ vice versa when replacing the index sets I_2°, I_2 by I_1°, I_1. The factors arise as follows. First, rewrite $\nabla y_i - \hat{\beta}_1' \nabla x_i = \nabla \varepsilon_i - (\hat{\beta}_1 - \beta)' \nabla x_i$ by applying equation (7). Then, assuming fixed regressors and independent normal $\mathcal{N}(0, \sigma^2)$ innovations, we get that $\nabla y_i - \hat{\beta}_1' \nabla x_i$ is normal $\mathcal{N}(0, 2\sigma^2 \omega_{1,i}^2)$. Later, we show that under mild regularity conditions $\omega_{2,i}^2$ is uniformly close to unity and it can indeed be replaced by unity for asymptotic purposes. We define the stylized SIS algorithm, which searches for location shifts in the second subsample.

Algorithm 2.1. *The stylized SIS algorithm.*

1. *Choose a cut-off value $c > 0$ to select breakpoints.*
2. *Calculate the least squares estimators $(\hat{\beta}_1, \hat{\sigma}_1^2)$ based on sample I_1.*
3. *Calculate forecast correction factors $\omega_{1,i}^2$ for $i \in I_2$.*
4. *Declare a location shift at $i+1$ if*

$$\left| \nabla y_i - \hat{\beta}_1' \nabla x_i \right| \geq \sqrt{2} \hat{\sigma}_1 \omega_{1,i} c \qquad \text{for } i \in I_2^\circ. \tag{16}$$

The frequency of location shifts declared by Algorithm 2.1 is

$$\hat{\gamma}_n^{stylized} = \frac{1}{n_2^\circ} \sum_{i \in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1' \nabla x_i| \geq \sqrt{2} \hat{\sigma}_1 \omega_{1,i} c)}. \tag{17}$$

When the data-generating process has no location shifts, so that $\mu_i = \mu$, the expression $\hat{\gamma}_n$ is the frequency gauge of the algorithm, which is the object of interest in this article.

Castle et al. (2015) refer to a split-half SIS algorithm, which is a symmetrized version of the above algorithm. For reference, we define that algorithm, including a statement on how to update the estimators for β, σ^2 in light of the identified location shifts. We allow the subsamples to be of unequal size, but retain the split-half descriptor.

Algorithm 2.2. *The split-half SIS algorithm.*

1. *Choose a cut-off value $c > 0$ to select breakpoints.*
2. *Calculate the least squares estimators $(\hat{\beta}_j, \hat{\sigma}_j^2)$ based on sample I_j for $j = 1, 2$.*
3. *Calculate forecast correction factors $\omega_{j,i}^2$ for $i \notin I_j$ and $j = 1, 2$.*
4. *Declare a location shift at $i+1$ if*

$$|\nabla y_i - \hat{\beta}_j' \nabla x_i| \geq \sqrt{2} \hat{\sigma}_j \omega_{j,i} c \qquad \text{for } i \in I_{3-j}^\circ \text{ and } j = 1, 2. \tag{18}$$

For notational simplicity, we do not consider the possibility of a location shift from $i = n_1$ to $i = n_1 + 1$. The split-half SIS algorithm of Castle et al. (2015)

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continues to re-estimate β, σ on the full sample while taking the detected location shifts into account.

The frequency of declared location shifts by Algorithm 2.2 is

$$\hat{\gamma}_n^{split} = \frac{1}{n^\circ}\Big\{\sum_{i\in I_1^\circ} 1_{(|\nabla y_i - \hat{\beta}_2'\nabla x_i| \geq \sqrt{2}\hat{\sigma}_2\omega_{2,i}c)} + \sum_{i\in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1'\nabla x_i| \geq \sqrt{2}\hat{\sigma}_1\omega_{1,i}c)}\Big\}. \quad \textbf{(19)}$$

2.4. Properties of the Differenced Innovations

The scaled innovations ε_i/σ have density f. In applications, we often assume f to be the normal density. The forward differenced innovations are denoted

$$\nabla\varepsilon_i = \varepsilon_i - \varepsilon_{i+1}, \qquad \chi_i = \nabla\varepsilon_i/(\sqrt{2}\sigma). \quad \textbf{(20)}$$

The scaled forward differenced innovations χ_i have the convolution density

$$\mathsf{h}(x) = \sqrt{2}\int_{-\infty}^{\infty} \mathsf{f}(y)\mathsf{f}(\sqrt{2}x + y)\,\mathrm{d}y, \quad \textbf{(21)}$$

and distribution function H. Following (4), let

$$\gamma = \mathsf{P}(|\chi_i| \geq c). \quad \textbf{(22)}$$

We highlight four properties of the density h.

THEOREM 2.3. *Assume ε_i/σ are i.i.d. and continuous with density* f. *The density* h *then satisfies the following properties:*

(a) *symmetry:* $\mathsf{h}(x) = \mathsf{h}(-x)$;
(b) *suppose* f *has a second moment. Then,* $\mathsf{f} = \mathsf{h}$ *if and only if* f *is standard normal;*
(c) *for* $k \in \mathbb{N}_0$: $\sup_{v\in\mathbb{R}} |v|^k \mathsf{f}(v) < \infty \Rightarrow \sup_{v\in\mathbb{R}} |v|^k \mathsf{h}(v) < \infty$;
(d) *for* $k \in \mathbb{N}_0$: $\sup_{v\in\mathbb{R}} (1+|v|^k)|\dot{\mathsf{f}}(v)| < \infty$ *and* $\mathbb{E}|\varepsilon_i^k| < \infty \Rightarrow \sup_{v\in\mathbb{R}} |v^k \dot{\mathsf{h}}(v)| < \infty$.

Theorem 2.3 implies that when the reference distribution f for ε is standard normal, so is the distribution h for χ_i. Thus, the gauge γ is associated with a cut-off c chosen as the normal $(1-\gamma/2)$ quantile.

3. THE MAIN RESULTS FOR STYLIZED SIS

We present an asymptotic theory for the frequency gauge of stylized SIS. The first-order result is consistency. This allows us to choose the cut-off c indirectly from the gauge. We obtain consistency for a wide range of stationary and non-stationary regressors. We will also develop a second-order expansion of the gauge with a view to understand how uniform the consistency result is. In this section, we give an asymptotic expansion, which is developed into an asymptotic theory for split-half SIS in the subsequent section. We then find that the asymptotic distribution is

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normal for a wide range of regressors, but with an asymptotic variance depending on the type of regressors.

We require the following time-series structure for innovations ε_i and regressors x_i.

Assumption 3.1. Let $\mathcal{F}_i$ be a filtration so that ε_{i-1} and x_i are $\mathcal{F}_{i-1}$-adapted, and ε_i/σ has unit variance and is independent of $\mathcal{F}_{i-1}$ with distribution function F and positive density f on $\mathbb{R}$ with derivative $\dot{\mathsf{f}}$.

Assumption 3.1 implies that ε_i are i.i.d. distributed. Endogeneity of the form $\mathsf{Cov}(x_i, \varepsilon_i) \neq 0$ is ruled out, but pre-determined time-series regressors are allowed. The innovations need not have zero mean, as Theorem 2.3(a) implies $\mathbb{E}\nabla\varepsilon_i = 0$ even if $\mathbb{E}\varepsilon_i \neq 0$. Jiao (2019) exploits the techniques developed here to analyze situations with endogeneity.

The theory results allow for stationary and non-stationary regressors. For this purpose, we introduce normalization matrices N_j for each subsample $j = 1, 2$. This yields normalized regressors

$$x_{in} = N_j' x_i, \quad \nabla x_{in} = N_j'(x_i - x_{i+1}) \quad \text{for } i \in I_j^\circ, \tag{23}$$

where we have suppressed the index j in the definition of the normalized regressor x_{in}. We choose the normalizations depending on the stochastic properties of x_i so that

$$\widehat{\Sigma}_{jn} = \sum_{i \in I_j} N_j'(x_i - \bar{x}_j)(x_i - \bar{x}_j)' N_j \quad \text{where} \quad \widehat{\Sigma}_{jn}^{-1} = \mathrm{O}_{\mathsf{P}}(1). \tag{24}$$

In the asymptotic theory, we will require that

$$\widehat{V}_{jn} = \sum_{i \in I_j} N_j'(x_i - \bar{x}_j)(\varepsilon_i - \mathbb{E}\varepsilon_i) = \mathrm{O}_{\mathsf{P}}(1); \qquad \mathbb{E}\sum_{i \in I_j^\circ} |\nabla x_{in}|^2 = \mathrm{O}(1). \tag{25}$$

For the practitioner, it will be possible to choose the cut-off c without precise knowledge of the type of regressors and hence the normalization. The knowledge of the type is only needed for the second-order theory.

We give some examples of normalizations. If x_i is stationary, then ∇x_i is also stationary. Thus, we let $N_j = n_j^{-1/2} I_{\dim x}$ and find that $\widehat{\Sigma}_{1n}$, $\widehat{V}_{1n}$ and $\mathbb{E}\sum_{i \in I_2^\circ} |\nabla x_{in}|^2$ converge under mild regularity conditions. If x_i is a random walk, then ∇x_i is i.i.d. and we let $N_j = n_j^{-1} I_{\dim x}$. Then, under mild regularity conditions, $\widehat{\Sigma}_{1n}$, $\widehat{V}_{1n}$ converge, while $\mathbb{E}\sum_{i \in I_2^\circ} |\nabla x_{in}|^2$ vanishes. Thus, the asymptotic expansions simplify in the latter case. As an example of cointegrated regressors, we could have

$$N_1 = \begin{pmatrix} n^{-1/2} & 0 \\ 0 & n^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{if} \quad x_i = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sum_{j=1}^{i-1} \varepsilon_j + z_i$$

for some stationary, bivariate process z_i. We note that this N_1 is non-diagonal.

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In most applications, the density of the innovations ε_i will be normal. However, the density needs neither be centered at zero nor be symmetric, as the theory results will only depend on the implied density for the differenced innovations $\nabla\varepsilon_i = \varepsilon_i - \varepsilon_{i+1}$. Our theory does require that the density f of the innovations ε_i and its derivative are bounded. The condition is satisfied for a wide range of densities, including the normal density. Moreover, the differenced innovations' conditional density, given the differenced regressors and the past, should also be bounded. If the regressors are pre-determined, this reduces to the boundedness of the density of the differenced innovations and follows from the boundedness of the density f of the innovations ε_i due to Theorem 2.3.

Assumption 3.2. Suppose that

(i) the density f satisfies $(a)\sup_{v\in\mathbb{R}}\mathsf{f}(v) < \infty$, $(b)\sup_{v\in\mathbb{R}}(1+v^2)|\dot{\mathsf{f}}(v)| < \infty$;
(ii) the conditional density $\mathsf{m}_i(y|x)$ of χ_i given ∇x_i and $\mathcal{F}_{i-1}$ exists for $i = n_1+1,\dots,n$, it is differentiable in y and satisfies $\max_{n_1+1\le i\le n}\sup_{y\in\mathbb{R},x\in\mathbb{R}^p}(1+|y|)\,|\dot{\mathsf{m}}_i(y|x)| < \infty$;
(iii) the regressors x_i satisfy, with $\widehat{\Sigma}_{1n}$, $\widehat{V}_{1n}$ defined in (24) and (25): $(a)\widehat{\Sigma}_{1n}^{-1} = \mathrm{O}_{\mathsf{P}}(1)$, $(b)\widehat{V}_{1n} = \mathrm{O}_{\mathsf{P}}(1)$, and $(c)\mathbb{E}\sum_{i\in I_2^\circ}|\nabla x_{in}|^2 = \mathrm{O}(1)$;
(iv) the subsample lengths satisfy $(n_2/n_1)^{1/2}$, $N_2^{-1}N_1 = \mathrm{o}(n_2^{1/4-\eta})$ for some $\eta > 0$.

We start by showing that the forecast correction factor $\omega_{1,i}^2$ can be replaced by unity with negligible asymptotic consequences.

THEOREM 3.3. *Consider the gauge of the stylized SIS Algorithm 2.1. Suppose Assumptions 3.1 and 3.2(ia, iii, iv) apply and that no location shifts are present, so that $\mu_i = \mu$. Then, we get for fixed $c \in \mathbb{R}$ that*

$$\hat{\gamma}_n^{stylized} = \frac{1}{n_2^\circ}\sum_{i\in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1'\nabla x_i| \ge \sqrt{2}\hat{\sigma}_1 c)} = \frac{1}{n_2^\circ}\sum_{i\in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1'\nabla x_i| \ge \sqrt{2}\hat{\sigma}_1\omega_{1,i}c)} + \mathrm{o}_{\mathsf{P}}(n_2^{-1/2}).$$

The next result presents the expansion for the frequency gauge $\hat{\gamma}_n^{stylized}$ of stylized SIS as defined in (17) around the population gauge $\gamma = \mathsf{P}(|\chi_i| \ge c) = \mathsf{P}(|\nabla\varepsilon_i| \ge \sqrt{2}\sigma c)$. The data-generating process is assumed to have no location shifts.

THEOREM 3.4. *Consider the gauge of the stylized SIS Algorithm 2.1. Suppose Assumptions 3.1 and 3.2 apply and that no location shifts are present so that $\mu_i = \mu$. Let*

$$\xi_{2n}(c) = n_2^{-1/2}\sum_{i\in I_2^\circ}\mathbb{E}_{i-1}(\nabla x_{in} \mid \chi_i = c) = \mathrm{O}_{\mathsf{P}}(1).$$

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Then, we get for fixed $c \in \mathbb{R}$ that

$$n_2^{1/2}(\hat{\gamma}_n^{stylized} - \gamma) = n_2^{-1/2}\sum_{i\in I_2^\circ}\{1_{(|\chi_i|\geq c)} - \mathbb{E}1_{(|\chi_i|\geq c)}\}$$
$$- c\mathsf{h}(c)(n_2/n_1)^{1/2}n_1^{-1/2}\sum_{i\in I_1}(\varepsilon_i^2/\sigma^2 - 1)$$
$$- \mathsf{h}(c)(\sqrt{2}\sigma)^{-1}\{\xi_{2n}(c) - \xi_{2n}(-c)\}' N_2^{-1}N_1\widehat{\Sigma}_{1n}^{-1}\widehat{V}_{1n} + \mathrm{o_P}(1). \tag{26}$$

Finally, $\hat{\gamma}_n^{stylized}$ is consistent in that $\hat{\gamma}_n^{stylized} \to \gamma$ in probability and in mean.

The consistency statement in Theorem 3.4 for the stylized SIS algorithm is nuisance parameter-free. It can be used for calibrating the SIS algorithm. The result provides the rationale for choosing c to match the desired population gauge γ: We specify our tolerance for false positives expressed by γ. Given the innovation density f, we obtain a selection quantile c. For example, if the innovations ε_i are normal, then the forward differenced innovations χ_i are standard normal by Theorem 2.3. If the sample is $n = 100$ and $\gamma = 1\%$, we choose $c = 2.58$, which is the normal 99.5% quantile.

The expansion in Theorem 3.4 has three components. The first component is a binomial term. The next two components relate to the estimation uncertainty from the initial estimation. They involve factors n_2/n_1 and $N_2^{-1}N_1$, respectively, where $N_2^{-1}N_1$ is an increasing function of n_2/n_1. These factors are allowed to diverge at an $\mathrm{o}(n^{1/4-\eta})$ rate. This means that the expansion would apply if we choose, in a stationary context, $n_1 = n^{7/8}$ and $n_2 = n - n_1$, so that $n_2/n_1 = \mathrm{O}(n^{1/8})$, which requires that $\eta < 1/8$ in Assumption 3.2(*iv*). In other words, the length of the subsample used for the initial estimation may be of a lower order of magnitude than the subsample used to search for location shifts. This feature is implicitly exploited in more complicated versions of the algorithm, which search for small subsets of observations without location shifts.

The third term in the Theorem 3.4 expansion involves the nuisance quantity $\xi_{2n}(c)$. It vanishes in two distinct cases. First, if the regressors are strictly exogenous, then $\mathbb{E}_{i-1}(\nabla x_i \mid \chi_i = c) = \mathbb{E}_{i-1}\nabla x_i$ does not depend on c so that $\xi_{2n}(c) - \xi_{2n}(-c) = 0$. Second, for random walk-type regressors with stationary ∇x_i, the normalization is $N_2 = n^{-1}$ so that $\xi_{2n}(c)$ vanishes. The third term simplifies if the sequence $(\nabla x_i, \chi_i)$ is stationary. In this case, we let $N_2 = n_2^{-1/2}$ and get $\xi_{2n}(c) = n_2^{-1}\sum_{i\in I_2^\circ}\mathbb{E}_{i-1}(\nabla x_i|\chi_i = c)$. Under regularity conditions, this converges in probability to $\mathbb{E}\mathbb{E}_0(\nabla x_1|\chi_1 = c) = \mathbb{E}(\nabla x_1|\chi_1 = c)$. Under a normality assumption, this can be computed explicitly. Thus, suppose that $(\nabla x_1, \chi_1)$ are normal given $\mathcal{F}_0$ with conditional mean $(v_0, 0)$, where v_0 is $\mathcal{F}_0$-measurable with expectation $\mathbb{E}v_0 = 0$. Noting that χ_1 has unit variance, we have

$$\begin{pmatrix}\nabla x_1\\ \chi_1\end{pmatrix} \mid \mathcal{F}_0 \overset{\mathsf{D}}{=} \mathcal{N}\left\{\begin{pmatrix}v_0\\ 0\end{pmatrix}, \begin{pmatrix}\sigma_{\nabla\nabla} & \sigma_{\nabla\chi}\\ \sigma_{\chi\nabla} & 1\end{pmatrix}\right\}. \tag{27}$$

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Then, $\xi_{2n}(c) \to \mathbb{E}v_0 + c\sigma_{\nabla\chi} = c\sigma_{\nabla\chi}$ in probability, while $\xi_{2n}(c) - \xi_{2n}(-c) \to 2c\sigma_{\nabla\chi}$. For example, in the autoregression $y_i = \mu + \alpha y_{i-1} + \varepsilon_i$ so that $x_i = y_{i-1}$, we find that $\sigma_{\nabla\chi} = \mathbb{E}_0(x_1 - x_2)(\varepsilon_1 - \varepsilon_2)/(\sqrt{2}\sigma) = \mathbb{E}_0(y_0 - y_1)(\varepsilon_1 - \varepsilon_2)/(\sqrt{2}\sigma) = -\sigma/\sqrt{2}$.

In the statement of Theorem 3.4, the initial least squares estimation is based on observations with indices $I_1 = (i \le n_1)$, while the search for location shifts is based on observations with indices $I_2 = (i > n_1)$. The consecutive nature of the sets I_1 and I_2 is convenient in the proof to simplify notation. However, the result extends to situations where the sets I_1 and I_2 are more complicated. Indeed, this is possible because Theorem 3.4 is derived under the hypothesis of no location shifts. It would be possible to choose I_1 as all odd and I_2 as all even indices. In that case, all observations will be involved when computing the forward differences arising from the set I_2.

4. THE MAIN RESULTS FOR SPLIT-HALF SIS

We provide an asymptotic expansion for split-half SIS and analyze the asymptotic distribution of the frequency gauge for stationary and for random walk regressors.

4.1. Expansion of the Gauge for Split-Half SIS

We expand the gauge for split-half SIS by applying Theorem 3.4 to each subsample. This requires a symmetrized version of Assumption 3.2.

Assumption 4.1. Suppose that

(i) the density f satisfies $\sup_{v\in\mathbb{R}} \mathsf{f}(v) < \infty$, $\sup_{v\in\mathbb{R}}(1+v^2)|\dot{\mathsf{f}}(v)| < \infty$;

(ii) the conditional density $\mathsf{m}_i(y|x)$ of χ_i given ∇x_i and $\mathcal{F}_{i-1}$ exists for $i = 1,\dots,n$, it is differentiable in y and satisfies $\max_{1\le i\le n} \sup_{y\in\mathbb{R},x\in\mathbb{R}^p}(1+|y|)\,|\dot{\mathsf{m}}_i(y|x)| < \infty$;

(iii) the regressors x_i satisfy for $j = 1,2$, with $\widehat{\Sigma}_{jn}$, $\widehat{V}_{jn}$ defined in (24) and (25): (a) $\widehat{\Sigma}_{1n}^{-1} = \mathrm{O}_\mathsf{P}(1)$, $(b)\widehat{V}_{jn} = \mathrm{O}_\mathsf{P}(1)$, and $(c)\mathbb{E}\sum_{i\in I_j^\circ}|\nabla x_{in}|^2 = \mathrm{O}(1)$;

(iv) the subsample lengths satisfy $(n_2/n_1)^{1/2}$, $N_2^{-1}N_1 = \mathrm{o}(n_2^{1/4-\eta})$, and $(n_1/n_2)^{1/2}$, $N_1^{-1}N_2 = \mathrm{o}(n_1^{1/4-\eta})$ for some $\eta > 0$.

THEOREM 4.2. *Consider the gauge of the split-half SIS Algorithm 2.2. Suppose Assumptions 3.1 and 4.1 apply and that no location shifts are present so that $\mu_i = \mu$. Let $\xi_{jn}(c) = n_j^{-1/2}\sum_{i\in I_j}\mathbb{E}_{i-1}(N_j'\nabla x_i \mid \chi_i = c)$ for $j = 1,2$. Then, we get for fixed $c \in \mathbb{R}$ that*

$$\sqrt{n}(\hat{\gamma}_n^{split} - \gamma) = n^{-1/2}\sum_{i=1}^{n-1}\left\{1_{(|\chi_i|\ge c)} - \mathbb{E}1_{(|\chi_i|\ge c)}\right\}$$

$$- c\mathsf{h}(c)n^{-1/2}\sum_{i=1}^{n}\left\{n_2 n_1{}^{-1}1_{(i\in I_1)} + n_1 n_2{}^{-1}1_{(i\in I_2)}\right\}(\varepsilon_i^2\sigma^{-2} - 1)$$

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$$- \mathsf{h}(c)(\sqrt{2}\sigma)^{-1}\big[(n_1/n)^{1/2}\{\xi_{1n}(c)-\xi_{1n}(-c)\}' N_1^{-1} N_2 \widehat{\Sigma}_{2n}^{-1}\widehat{V}_{2n} + (n_2/n)^{1/2}\{\xi_{2n}(c)-\xi_{2n}(-c)\}' N_2^{-1} N_1 \widehat{\Sigma}_{1n}^{-1}\widehat{V}_{1n}\big] + \mathrm{o}_{\mathsf{P}}(1).$$

Finally, $\hat{\gamma}_n^{split}$ is consistent in that $\hat{\gamma}_n^{split} \to \gamma$ in probability and in mean.

Once again, the consistency statement in Theorem 4.2 for the split-half SIS algorithm is nuisance parameter-free.

4.2. Asymptotic Distribution in the Stationary Case

We now consider the expansion of split-half SIS when the regressors x_j are stationary. We start by introducing some notations for various moments for the innovations ε_i:

$$\varkappa_1 = \mathbb{E}\varepsilon_i/\sigma, \qquad \varkappa_2 = \mathbb{E}\varepsilon_i^2/\sigma^2 = 1, \qquad \varkappa_4 = \mathbb{E}\varepsilon_i^4/\sigma^4, \tag{28}$$

$$\varsigma_0 = \mathbb{E}\{1_{(|\chi_i|\geq c)} 1_{(|\chi_{i+1}|\geq c)}\}, \tag{29}$$

$$\varsigma_2 = \mathbb{E}\{1_{(|\chi_i|\geq c)}(\varepsilon_{i+1}^2/\sigma^2 - 1)\} = \mathbb{E}\{1_{(|\chi_i|\geq c)}(\varepsilon_i^2/\sigma^2 - 1)\}. \tag{30}$$

Further, for the stationary regressor x_i, we denote

$$\mu_x = \mathbb{E}x_i, \qquad \Sigma_x = \mathsf{Var} x_i, \tag{31}$$

and finally, for a cross moment for innovations and regressors, we denote

$$\varsigma_{1x} = \mathbb{E}\{\nabla x_i(1_{(|\chi_i|\geq c)} - \gamma)(\varepsilon_i/\sigma - \varkappa_1)\}, \tag{32}$$

$$\xi_c = \mathbb{E}(\nabla x_i \mid \chi_i = c) = \mathbb{E}(\nabla x_i \mid \chi_i = -c). \tag{33}$$

Then, the vector $s_i = \{1_{(|\chi_i|\geq s)} - \gamma, \varepsilon_i^2/\sigma^2 - 1, (\varepsilon_i/\sigma - \varkappa_1)(x_i - \mu_x)'\Sigma_x^{-1}\}'$ has variance and first-order autocovariance of the form

$$\Omega_0 = \begin{pmatrix} \gamma(1-\gamma) & \varsigma_2 & 0 \\ \varsigma_2 & \varkappa_4 - 1 & 0 \\ 0 & 0 & \Sigma_x^{-1}(1-\varkappa_1^2) \end{pmatrix}, \qquad \Omega_1 = \begin{pmatrix} \varsigma_0 - \gamma^2 & \varsigma_2 & \varsigma_{1x}'\Sigma_x^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

Finally, we define long-run variances for the summands of the frequency gauge in (19). Let (j,k) be $(1,2)$ or $(2,1)$ and define with $n_j/n \to \lambda_j > 0$ for $j = 1,2$

$$d_j = \begin{pmatrix} 1 \\ -c\mathsf{h}(c)(\lambda_k/\lambda_j) \\ -\mathsf{h}(c)\xi_c(\lambda_k/\lambda_j)/\sqrt{2} \end{pmatrix}, \qquad \omega_j^2 = d_j'\Omega_0 d_j + 2e_1'\Omega_1 d_j. \tag{35}$$

The long-run variances ω_j^2 will be assumed to be positive in order to exploit the Functional Central Limit Theorem for non-stationary mixingales in McLeish (1977).

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Example 4.1. If ε_i/σ has standard normal density φ and distribution function Φ, then $\mathsf{h}(x)=\varphi(x)$, while $\varkappa_1=0$ and $\varkappa_4=3$. It is argued in Section A.9 of the Supplementary Material that

$$\varsigma_0=2\gamma-4\{T(c,1/\sqrt{3})+T(c,\sqrt{3})\}, \qquad \varsigma_2=c\varphi(c). \tag{36}$$

Here, $T(c,a)=\int_c^\infty\varphi(x)\int_0^{ax}\varphi(y)dydx$ following Owen (1980, formula 2.2; 2.8). In particular, $T(c,a)$ is positive and decreasing in c with $T(0,1/\sqrt{3})=1/12$ and $T(0,\sqrt{3})=1/6$. Finally, if $\nabla x_1,\chi_1$ are jointly normal given $\mathcal{F}_0$ as in (27) then $\xi_c=2c\sigma_{\nabla\chi}$.

Assumption 4.3. Suppose

(i) the density f satisfies $\sup_{v\in\mathbb{R}}|v|\mathsf{f}(v)<\infty$ and $\int_{\mathbb{R}}v^{4+}\mathsf{f}(v)dv<\infty$;
(ii) the pairs x_i,ε_i are stationary with $\mathbb{E}|x_i^{2+}|<\infty$;
(iii) $\omega_1^2,\omega_2^2>0$;
(iv) let z_i be either of x_i, x_ix_i' or $\nabla x_i 1_{(|\chi_i|\geq c)}(\varepsilon_i/\sigma-\varkappa_1)$ and suppose $\mathbb{E}|\mathbb{E}_{k-m}n^{-1}\sum_{i=k+1}^{k+n}(z_i-\mathbb{E}z_i)|\to 0$ as $\min(k,m,n)\to\infty$;
(v) $n^{-1}\sum_{i=1}^n x_i=\mu_x+\mathsf{o}_\mathsf{P}(1)$.

THEOREM 4.4. *Consider the gauge of the split-half SIS Algorithm 2.2 with $n_j/n\to\lambda_j>0$ for $j=1,2$, so that $\lambda_1+\lambda_2=1$. Suppose Assumptions 3.1, 4.1, and 4.3 apply and that no location shifts are present so that $\mu_i=\mu$. Then, for fixed $c\in\mathbb{R}$, we get $n^{1/2}(\hat\gamma_n^{split}-\gamma)\overset{\mathsf{D}}{\to}\mathcal{N}(0,B)$, where*

$$\begin{aligned}B&=\lambda_1\omega_1^2+\lambda_2\omega_2^2\\&=\gamma(1-\gamma)+2(\varsigma_0-\gamma^2)-4c\mathsf{h}(c)\varsigma_2-\sqrt{2}\mathsf{h}(c)\varsigma_{1x}'\Sigma_x^{-1}\xi_c\\&\quad+(\lambda_1^2/\lambda_2+\lambda_2^2/\lambda_1)\mathsf{h}^2(c)\{c^2(\varkappa_4-1)+(1-\varkappa_1^2)\xi_c'\Sigma_x^{-1}\xi_c/2\}.\end{aligned} \tag{37}$$

Example 4.2. Let $y_i=\mu+\alpha y_{i-1}+\varepsilon_i$ be stationary so that $|\alpha|<1$ and ε_i/σ is standard normal. Then, $x_i=y_{i-1}$ has mean $\mu_x=\mu/(1-\alpha)$ and variance $\Sigma_x=\sigma^2/(1-\alpha^2)$. It is argued in Section A.9 of the Supplementary Material that $\sigma_{\nabla\chi}=-\sigma/\sqrt{2}$ in (27), that $\varsigma_{1x}=-\sigma\varsigma_2$ and that condition (*iv*) of Assumption 4.3 holds.

Example 4.3. We consider the asymptotic distribution of the gauge for standard normally distributed error terms ε_i/σ, so that $\varkappa_1=0$ and $\varkappa_4=3$. Further, assume that the sample size in the two subsamples is equal and that the regressors x_i are stationary. The asymptotic variance (37) in Theorem 4.4 then reduces to

$$\begin{aligned}B=&\gamma(1-\gamma)+2(\varsigma_0-\gamma^2)-4c\mathsf{h}(c)\varsigma_2-\sqrt{2}\mathsf{h}(c)\varsigma_{1x}'\Sigma_x^{-1}\xi_c\\&+\mathsf{h}^2(c)(2c^2+\xi_c'\Sigma_x^{-1}\xi_c/2).\end{aligned} \tag{38}$$

Recall that if, in addition, $\nabla x_1,\chi_1$ are normal given F_0 as in (27) then $\xi_c=2c\sigma_{\nabla\chi}$. Further, in a first-order autoregression $y_i=\mu+\alpha y_{i-1}+\varepsilon_i$, the conditional covariance $\sigma_{\nabla\chi}$ equals $-\sigma/\sqrt{2}$, while $\varsigma_{1x}=-\varsigma_2$ and $\Sigma_x=\mathsf{Var}x_i=\sigma^2/(1-\alpha^2)$.

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4.3. Distribution of Split-Half SIS When ξ_{nj} Vanishes

The Theorem 4.2 expansion for the split-half SIS's frequency gauge simplifies when the term ξ_{nj} vanishes so that the third term in the expansion falls away. As remarked after Theorem 3.4, this happens for strictly exogenous or random walk regressors. The limiting long-run variance simplifies so that

$$\tilde{\omega}_j^2 = \gamma(1-\gamma) + 2(\varsigma_0 - \gamma^2) - 4c\mathsf{h}(c)(\lambda_k/\lambda_j)\varsigma_2 + c^2\mathsf{h}^2(c)(\lambda_k/\lambda_j)^2(\varkappa_4 - 1). \quad \textbf{(39)}$$

We will require that $\tilde{\omega}_j^2 > 0$.

THEOREM 4.5. *Consider the gauge of the split-half SIS Algorithm 2.2 with $n_j/n \to \lambda_j > 0$ for $j = 1,2$, so that $\lambda_1 + \lambda_2 = 1$. Let $\xi_{jn} = \mathrm{o}_{\mathsf{P}}(1)$. Suppose Assumptions 3.1, 4.1, and 4.3(i) apply, $\tilde{\omega}_j^2 > 0$ for $j = 1,2$ and that no location shifts are present so that $\mu_i = \mu$. Then, for fixed $c \in \mathbb{R}$, we get $n^{1/2}(\hat{\gamma}_n^{split} - \gamma) \xrightarrow{\mathsf{D}} \mathcal{N}(0, \tilde{B})$, where*

$$\begin{aligned}\tilde{B} &= \lambda_1\tilde{\omega}_1^2 + \lambda_2\tilde{\omega}_2^2 \\ &= \gamma(1-\gamma) + 2(\varsigma_0 - \gamma^2) - 4c\mathsf{h}(c)\varsigma_2 + c^2\mathsf{h}^2(c)(\lambda_1^2/\lambda_2 + \lambda_2^2/\lambda_1)(\varkappa_4 - 1). \quad \textbf{(40)}\end{aligned}$$

5. POISSON APPROXIMATION

We present a theory for a vanishing frequency gauge. We set the cut-off so as to control the absolute gauge, the number of falsely discovered outliers. This gives a Poisson exceedance theory. For stylized and split-half SIS, the absolute gauges are

$$\hat{\Gamma}_n^{stylized} = \sum_{i \in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1' \nabla x_i| \geq \sqrt{2}\hat{\sigma}_1 \omega_{1,i} c_n)}, \quad \textbf{(41)}$$

$$\hat{\Gamma}_n^{split} = \sum_{i \in I_2^\circ} 1_{(|\nabla y_i - \hat{\beta}_1' \nabla x_i| \geq \sqrt{2}\hat{\sigma}_1 \omega_{1,i} c_n)} + \sum_{i \in I_1^\circ} 1_{(|\nabla y_i - \hat{\beta}_2' \nabla x_i| \geq \sqrt{2}\hat{\sigma}_2 \omega_{2,i} c_n)}. \quad \textbf{(42)}$$

Here, we choose the cut-off c_n so that, for some $\lambda > 0$,

$$\mathsf{P}(|\nabla\varepsilon_i| > \sqrt{2}\sigma c_n) = \mathsf{P}(|\chi_i| > c_n) = \lambda/n. \quad \textbf{(43)}$$

The analysis builds on the Poisson exceedance theory for IIS (Johansen and Nielsen, 2016b). The analysis has two parts. The first part is a Poisson limit theorem for the case without estimation errors. For IIS, the standard Poisson limit theorem could be used. For SIS, we have that the forward differenced innovations are 1-dependent. We can then apply the Chen (1975) Poisson limit theorem. The second part is an argument that the estimation errors do not matter for the asymptotic theory. This argument is similar to that of the IIS analysis. For the analysis, we need the following high-level assumptions.

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Assumption 5.1. Suppose that

(i) the innovations ε_i are i.i.d., so that $\chi_i = \nabla\varepsilon_i/(\sqrt{2}\sigma)$ has continuous distribution function H with density h satisfying
  (a) $\mathbb{E}|\chi|^r < \infty$ for some $r > 4$;
  (b) $\mathsf{h}(c_n)/[c_n\{1-\mathsf{H}(c_n)\}] = \mathrm{O}(1)$;
  (c) $\mathsf{h}(c_n - n^{-1/4}A)/\mathsf{h}(c_n) = \mathrm{O}(1)$ for all $A > 0$;
  (d) given $\lambda > 0$ choose c_n so that for all i then $\mathsf{P}(|\chi_i| > c_n) = \lambda/n$ and suppose $n\{\mathbb{E}1_{(|\chi_i|>c_n)}1_{(|\chi_{i+1}|>c_n)}\} \to 0$;

(ii) the regressors x_i satisfy, with $j = 1,2$ and $\widehat{\Sigma}_{jn}$, $\widehat{V}_{jn}$ defined in (24) and (25):
  (a) $\widehat{\Sigma}_{jn}^{-1} = \mathrm{O}_\mathsf{P}(1)$, $(b)\,\widehat{V}_{jn} = \mathrm{O}_\mathsf{P}(1)$, and (c) $\mathbb{E}\sum_{i\in I_j^\circ}|\nabla x_{in}|^4 = \mathrm{O}(n^{-1})$;

(iii) the subsample lengths satisfy $N_2^{-1}N_1, N_1^{-1}N_2 = \mathrm{O}_\mathsf{P}(1)$.

Remark 5.1. Assumption 5.1(i) is satisfied when the innovations ε_i are normal. For parts (a)–(c), this follows from Lemma A.14 in Section A.10 of the Supplementary Material. For part (d), this follows from Lemma A.13.

THEOREM 5.2. *Suppose Assumption 5.1, that $n_2/n \to \psi$ for $0 < \psi < 1$ and that the cut-off is chosen through $\mathsf{P}(|\chi_i| > c_n) = \lambda/n$ for all i. Then,*

(a) $\hat{\Gamma}_n^{stylized} = \sum_{i\in I_2^\circ} 1_{(|\chi_i|>\sigma c_n)} + \mathrm{o}_\mathsf{P}(1) \xrightarrow{\mathsf{D}} \mathsf{Poisson}(\lambda\psi)$;

(b) $\hat{\Gamma}_n^{split} = \sum_{i=1}^{n-1} 1_{(|\chi_i|>\sigma c_n)} + \mathrm{o}_\mathsf{P}(1) \xrightarrow{\mathsf{D}} \mathsf{Poisson}(\lambda)$.

The Poisson result shows that the absolute gauge is not consistent for the target λ. Rather, it has a Poisson variation around the target. The asymptotic Poisson variation depends neither on the regressors nor on the estimation error.

6. POWER

We consider local power for stylized SIS and for the Andrews (1993) test and argue that the results carry over to the Bai and Perron (1998) procedure. Proofs are given in Section A.11 of the Supplementary Material.

6.1. Power of Stylized SIS

The power properties of the SIS algorithm are discussed by Castle et al. (2015). We give further discussion for the stylized SIS algorithm. For simplicity, we focus on the case without regressors, so the model in (8) reduces to

$$y_i = \sigma\mu 1_{(i\le n_1)} + \sum_{j=n_1+1}^{n} \sigma\delta_j 1_{(i\le j)} + \varepsilon_i \qquad \text{for } i = 1,\ldots,n, \tag{44}$$

with independent, normal $\mathcal{N}(0,\sigma^2)$ innovations and where μ and δ_j are reparameterized using the scale σ. This data-generating process allows up to $n - n_1 - 1$ breaks.

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The stylized SIS Algorithm 2.1 estimates the error variance from the first sample-half and uses forward differences throughout the second sample-half to detect location shifts (see Section 2.2). Thus, stylized SIS declares step-shifts for any observation in the second sample half, $n_1 < i < n$, if

$$\left|\nabla y_i\right| \geq \sqrt{2}\hat{\sigma}_1 c. \tag{45}$$

Theorem 3.4 analyzes the gauge of the procedure. Under normality, we choose the cut-off from the equation $\gamma = 2\{1 - \Phi(c)\}$ (see (22)); for example, $\gamma = 1\%$ corresponds to $c = 2.58$.

By the temporal independence, then y_i for $i > n_1$ is independent of the variance estimator $\hat{\sigma}_1^2$, which is asymptotically $\sigma^2\chi^2_{n_1-1}/(n_1-1)$-distributed. Assuming also normality, then the t-statistics defined from (45) are non-central t-distributed (Johnson, Kotz, and Balakrishnan, 1993). We note that for an index i in the second sample-half, then (44) can be written as $y_i = \sum_{j=i}^{n}\sigma\delta_j + \varepsilon_i$. Thus, we find with $\chi_i = \nabla\varepsilon_i/(\sqrt{2}\sigma)$ that

$$2z_i = \frac{\nabla y_i}{\sqrt{2}\hat{\sigma}_1} = \frac{\chi_i + \delta_i/\sqrt{2}}{\hat{\sigma}_1/\sigma} \overset{\mathsf{D}}{=} \mathsf{t}_{n_1-1}\left(\frac{\delta_i}{\sqrt{2}}\right) \qquad \text{for } n_1 < i < n. \tag{46}$$

A single step-shift at time $\tau+1$ of size δ comes about in model (44) if $\mu = \delta_\tau = -\delta$, with δ_n indicating the post-break level, while all other δ_i are zero. If χ represents a standard normal variable then the power to detect such a shift is

$$\begin{aligned}\mathsf{P}\left\{|z_\tau| > c\right\} &= \mathsf{P}\left\{\mathsf{t}_{n_1-1}\left(-\delta/\sqrt{2}\right)\right\} \\ &\to \mathsf{P}\left(\left|\chi - \delta/\sqrt{2}\right| > c\right) = \Phi\left(-c + \delta/\sqrt{2}\right) + \Phi\left(-c - \delta/\sqrt{2}\right).\end{aligned} \tag{47}$$

We learn a number of properties from this result. First, the power does not depend on the sign of the shift. Second, the power of the difference decision rule (45) is invariant to time τ. The power stays the same even in the boundaries of the sample. Third, the t-tests have power approaching unity when $|\delta|$ is increasing. Fourth, two decisions are dependent if they concern consecutive time periods. Otherwise, they are independent. Thus, the power is invariant to the number, magnitude, and timing of other shifts as long as they are at least two periods away. SIS can detect shifts, even if their number is large. Fifth, a slight location shift can be detected with high probability if the two episodes are separated by a short period of upheaval. For analytic simplicity, this short period is at least two periods long. Thus, suppose there is one level until τ, a location shift of size δ at $\tau+1$ followed by an opposite location shift of size $\nu - \delta$ at $\tau+3$, to a new level that is ν larger than the first level and where ν may be small. In terms of the model (44), this comes about through $\mu = \delta_\tau = -\delta$ and $\delta_{\tau+2} = \delta - \nu$, while δ_n gives the post-break level. The joint probability of correct detection is

$$\mathsf{P}\{|z_\tau| > c, |z_{\tau+2}| > c\} \to \{\Phi(-c+\delta/\sqrt{2}) + \Phi(-c-\delta/\sqrt{2})\} \times [\Phi\{-c+(\delta-\nu)/\sqrt{2}\} + \Phi\{-c-(\delta-\nu)/\sqrt{2}\}]. \quad \textbf{(48)}$$

Thus, for large n and small ν, we find

$$\mathsf{P}\{|z_\tau| > c, |z_{\tau+2}| > c\} = \{\Phi(-c+\delta/\sqrt{2}) + \Phi(-c-\delta/\sqrt{2})\}^2 + \mathrm{O}(\nu).$$

As a consequence, a small location shift can be discovered with large probability, if the upheaval δ is large. Once it has been established that there is, for instance, a shift of this type and no other shifts, it can be tested whether $\nu = 0$. This test will be consistent if τ/n and $1-\tau/n$ have nonzero limits as no search is involved anymore.

While the fifth case may seem contrived, it occurs empirically. Castle et al. (2023) find that the U.K. annual real wage growth rate increases from 0.8% prior to World War II to 1.7% after the war, with a large impulse during the war. Similarly, the U.K. annual productivity per worker increases from 1.2% prior to World War I to 1.7% after a huge deflation episode in the wake of the war. Such changes have profound implications for the economy, even if they are small relative to the residual standard error.

6.2. Local Power for Andrews Test

We consider the Andrews (1993) test for a single break at an unknown time in the central part of the sample. This test is consistent for a shift of fixed magnitude that is not at the ends of the sample. We consider local power for various alternatives. The test is based on the simple one-shift model

$$y_i = \sigma\mu + \sigma\delta 1_{(i\le\tau)} + \varepsilon_i \qquad \text{for } i = 1, \ldots, n,$$

with independent, normal $\mathcal{N}(0, \sigma^2)$ innovations. If the break point is known, we can form the t-statistic, Z_τ say, for the hypothesis $\delta = 0$ (see (A.28) in the Supplementary Material for a detailed expression). For the case of an unknown break point, τ, we may suppose $\underline{n} \le \tau \le \overline{n}$ for some user-chosen bounds satisfying $0 < \underline{n} \le \overline{n} < n$. The likelihood ratio test is then formed by maximizing the squared t-statistic over location. This gives the test statistic

$$LR_{\max} = \max_{\underline{n}\le t\le\overline{n}} Z_t^2. \quad \textbf{(49)}$$

Distribution under hypothesis: Critical values are found from the distribution of the test statistic under the hypothesis of no break. There are two relevant limits. We note two differences from stylized SIS. On the one hand, the Andrews test asymptotics applies for unknown error distributions while SIS requires a known error distribution. On the other hand, the Andrews test generalizes to the case of stationary regressors, but in contrast to SIS, it does not generalize to the case of non-stationary regressors.

First, when there are no restrictions on the search range, so that $1 = \underline{n}$ and $\overline{n} = n-1$, then the likelihood ratio statistic diverges at a rate of $2\log\log n$ due to the behavior of a Brownian motion near the origin as described by the law of iterated logarithms. With an appropriate logarithmic normalization, the statistic converges to an extreme value distribution (Yao and Davis, 1986; Hidalgo and Seo, 2013). This test is not so common. Perhaps because it is felt that too much power is lost by the additional normalization.

Second, when the search range is trimmed, the likelihood ratio statistic converges to a supremum of a standardized Brownian bridge (Andrews, 1993). That is, if $\mathbb{B}_u$ is a standard Brownian bridge for $0 \leq u \leq 1$, which has variance $u(1-u)$, then for large n and with $\underline{n}/n \to \underline{\lambda} > 0$ and $\overline{n}/n \to \overline{\lambda} < 1$, we get

$$LR_{\max} = \max_{\underline{n} \leq t \leq \overline{n}} Z_t^2 \overset{\mathsf{D}}{\to} \sup_{\underline{\lambda} \leq u \leq \overline{\lambda}} \frac{\mathbb{B}_u^2}{u(1-u)}.$$

The critical values increase with decreasing trimming, reaching the extreme value asymptotics when there is no trimming. Andrews provided simulated critical values. A 15% trimming is commonly used with critical value 12.35 for a 1% sized test. Bai and Perron (1998) preferred 5% trimming. The test is known to be consistent for a central break of finite magnitude δ. This contrasts with SIS. We investigate local power in various cases.

A Single Break: We consider the power against an alternative with a shift of vanishing magnitude at time $\tau = \lambda n$. We allow $0 < \lambda < 1$, while noting that the Andrews test is aimed at the trimmed interval $0 < \underline{\lambda} \leq \lambda \leq \overline{\lambda} < 1$. Local power is found when the magnitude of the break vanishes as $\delta = \phi/\sqrt{n}$ for fixed ϕ. We find in Section A.11 of the Supplementary Material that, for fixed $0 < \lambda < 1$,

$$LR_{\max} \overset{\mathsf{D}}{\to} \sup_{\underline{\lambda} \leq u \leq \overline{\lambda}} \frac{(\mathbb{B}_u + \phi s_u^\lambda)^2}{u(1-u)}, \tag{50}$$

where the function s_u^λ increases linearly from 0 at $u = 0$ to $\lambda(1-\lambda)$ at $u = \lambda$ after which it decreases linearly to 0 at $u = 1$ as given by

$$s_u^\lambda = (1-\lambda)u 1_{(u \leq \lambda)} + \lambda(1-u)1_{(u>\lambda)}. \tag{51}$$

The non-centrality term is largest for $u = \lambda$, taking the value $\phi\{\lambda(1-\lambda)\}^{1/2}$. Thus, the Andrews test has local power for this alternative, whereas asymptotically, stylized SIS has trivial power. For a finite sample, we compare the maximal pointwise non-centrality for the Andrews test of $\phi\{\lambda(1-\lambda)\}^{1/2} = \delta\{n\lambda(1-\lambda)\}^{1/2}$ with the SIS non-centrality of $\delta/\sqrt{2}$ arising from (46). Notably, the magnitude of the break δ will give neither method an advantage in the power comparison. Instead, the positioning λ and the sample size n determine the comparative performance. We compare the two non-centralities, while ignoring the simultaneity of decisions within the two procedures. The Andrews test with 15% trimming and 1% size has critical value $12.35 = (3.51)^2$, while stylized SIS has 1% critical value $2.57 = (6.63)^{1/2}$. Dividing the non-centralities with 3.51 and 2.57,

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respectively, equating and solving gives $n = (12.35/6.63)/\{2\lambda(1-\lambda)\}$, with SIS being advantageous for n smaller than those values. The implied n-values for central values $\lambda = (0.5, 0.75, 0.85)$ are $n = (4,5,7)$ so that the Andrews test is favorable. However, this changes when the break occurs in the trimmed period. The largest u considered by the test statistic is $\overline{\lambda}$, so that the Andrews test has maximal pointwise non-centrality of $\delta\{n\overline{\lambda}/(1-\overline{\lambda})\}^{1/2}(1-\lambda)$. Proceeding as before, we find $n = (12.35/6.63)(1/2)\{(1-\overline{\lambda})/\overline{\lambda}\}/(1-\lambda)^2$. Thus, for $\overline{\lambda} = 0.85$, the implied n-values for $\lambda = (0.9, 0.95, 0.99)$ are $n = (16, 66, 1650)$. The comparison indicates that stylized SIS may be competitive in small samples with a late break.

Next, consider the consequence of a break close to the sample boundaries. The above derivation can be modified to the case where $\delta(1-\tau/n) = \psi/\sqrt{n}$ while $\tau/n \to 1$ and fixed ψ. These constraints imply $|\delta|/\sqrt{n} \le |\psi|$ with equality when $\tau = n-1$. Thus, we let $\delta/\sqrt{n} \to \eta$, where $0 \le |\eta| \le |\psi|$ while $\eta\psi \ge 0$. For large n, we get

$$LR_{\max} \overset{\mathsf{D}}{\to} \sup_{\underline{\lambda} \le u \le \overline{\lambda}} \frac{(\mathbb{B}_u + \psi u)^2}{u(1-u)(1+\eta\psi)}. \tag{52}$$

To see that (52) conforms with (50), note that $u \le \overline{\lambda} < 1$ and $\tau/n \to 1$ imply that $u < \tau/n$ so that $s_u^{\tau/n} = (1-\tau/n)u$ for large n, while a small δ corresponds to $\eta = 0$. The result (52) shows that when δ diverges, then the Andrews test has local power, while the stylized SIS has power approaching unity (see (46)). In particular, we can let δ diverge at a slow rate with τ sufficiently close to n to achieve $\psi = 0$, so that the Andrews test has trivial power, while stylized SIS has power approaching unity.

Two Breaks: Let $y_i = \sigma\mu + \sigma\delta_1 1_{(i\le\tau_1)} + \sigma\delta_2 1_{(i\le\tau_2)} + \varepsilon_i$, where ε_i is i.i.d. $\mathcal{N}(0,\sigma^2)$ so that the level is changed twice at $\tau_1 < \tau_2$. Again, this alternative is outside those the Andrews test is optimized against, but relevant in practice. We consider the situation where two large location shifts are close and nearly offset each other so that $\tau_2 - \tau_1$ and $\delta_1 + \delta_2$ are close to zero. This is an empirically relevant situation where SIS performs well. Thus, we investigate local power when $\delta_1 + \delta_2 = \xi/\sqrt{n}$ and $\delta_2(\tau_2 - \tau_1) = \psi\sqrt{n}$ while $\tau_1/n = \lambda$ and $(\tau_2-\tau_1)/n \to 0$ for fixed ξ, ψ, λ. These constraints imply $|\delta_2|/\sqrt{n} \le |\psi|$ with equality when $\tau_2 = \tau_1 + 1$. Thus, we let $\delta_2/\sqrt{n} \to \eta$, where $0 \le |\eta| \le |\psi|$, while $\eta\psi \ge 0$. We find in Section A.11 of the Supplementary Material using the Skorokhod (1956) M_1-metric that

$$LR_{\max} \overset{\mathsf{D}}{\to} \sup_{\underline{\lambda} \le u \le \overline{\lambda}} \frac{\left[\mathbb{B}_u + \xi s_u^{\lambda} + \psi\{1_{(u\ge\lambda)} - u\}\right]^2}{u(1-u)(1+\eta\psi)}. \tag{53}$$

Again, when δ_2 and hence δ_1 diverge, then the Andrews test has local power while split-half SIS has power approaching unity. In particular, when δ_2 diverges slowly while τ_2 and τ_1 are so close that $\psi = 0$, then the Andrews test has trivial power while stylized SIS has power approaching unity.

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6.3. Discussion of Bai and Perron Procedure

We first summarize the findings for the Andrews test. This test is consistent for a fixed-sized central break in contrast with stylized SIS, which only has local power in that situation. Otherwise, SIS can be competitive. We found that SIS has power approaching unity, while the Andrews test has trivial power in two situations. The first case has a break near the end point of the sample. Detecting such a break is highly relevant when forecasting (Clements and Hendry, 1998). The second case is when two breaks are close and nearly offsetting. This can reveal small but important changes in, for instance, growth series (Castle et al., 2023). Thus, the Andrews test is preferable if one is content that there is only one central break or perhaps two well-separated central breaks. With more complicated series, SIS will be competitive.

The Bai and Perron (1998) (BP) procedure is developed for the situation where there is an unknown, but bounded, number of multiple well-separated breaks. This procedure provides estimates of the number of breaks and their timing. This requires trimming between breaks and at the end points of the sample and a maximal number of breaks. The usual 15% trimming eliminates too much of the sample and a 5% trimming is recommended. The above analysis suggests that the BP procedure will consistently detect fixed-sized breaks that are not too close. But, with many breaks or with close breaks, the BP procedure may have near trivial power, while SIS could have high power.

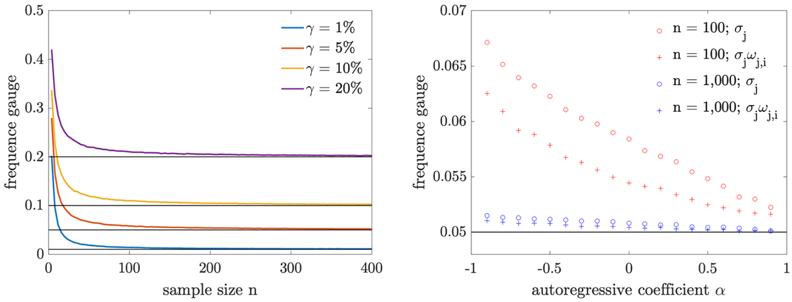
As a further point of comparison, we note that the BP procedure allows an unknown error distribution and it generalizes to stationary, but not non-stationary regressors. The SIS procedure requires a known error distribution, but allows both stationary and non-stationary regressors. We note that for many macroeconomic time series, normality is not unreasonable, but assuming stationarity of the regressors may not be appropriate.

Finally, the general SIS algorithm is designed to work jointly with regression selection, whereas the BP procedure requires a fixed set of regressors.

7. SIMULATIONS AND NUMERICAL APPROXIMATIONS

We complement the asymptotic analysis of split-half SIS with simulations and numerical approximations. These results confirm the validity of the asymptotic theory, allow comparisons to other algorithms, and inform us about the small sample properties of SIS. First, we confirm the consistency of the frequency gauge and characterize its small sample bias. Second, we use numerical approximations to decompose the components of the asymptotic variance. Third, we confirm with simulations the distributional convergence of the frequency gauge. Fourth, we consider the bias of an updated regression estimator. Fifth, we compare the power of split-half SIS with that of Andrews (1993).

All simulations have 10^4 repetitions. Each time we increase the sample size, we redraw all n observations. The simulations have been coded in MATLAB using



(a) DGP1 with varying γ and n (b) DGP2 with varying α and n

FIGURE 2. Finite sample properties of the frequency gauge.

the MFE toolbox (Sheppard, 2018). When we do not explicitly mention otherwise, we set $\hat{\omega}_{j,i}^2 = 1$ for simplification, as we are mainly concerned with evaluating the asymptotic distributions. Given a target frequency gauge γ, we choose the cut-off c in the SIS algorithm as the normal $(1-\gamma/2)$ quantile.

7.1. Analysis of Consistency of Frequency Gauge

We validate the consistency of the frequency gauge of split-half SIS as analyzed in Theorem 4.2. We consider two data-generating processes. In both cases, the algorithm is based on the model (7) with one univariate regressor x_i and $n_1 = n_2$.

DGP1 includes an exogenous regressor $y_i = \beta x_i + \varepsilon_i$, so that x_i and ε_i are independent standard normal. DGP1 is white noise if $\beta = 0$, in which case y_i is also independent standard normal. As the regressor x_i is strictly exogenous, Theorem 4.5 applies.

DGP2 is a first-order auto-regression $y_i = \alpha y_{i-1} + \varepsilon_i$, where $|\alpha| < 1$, ε_i is independent standard normal and $y_0 = 0$. Thus, $\delta_j = 0$ for all j and $\beta = \alpha$ in (1) while $x_i = y_{i-1}$.

Figure 2(a) uses DGP1 with exogenous regressor and coefficient $\beta = 0$. It shows the frequency gauge γ for an increasing sample size and different gauges γ. We use the white noise version of DGP1. We find that the small sample bias of the gauge is positive. The bias vanishes quickly with growing samples and it is modest for $n = 100$.

Figure 2(b) uses the autoregressive DGP2. It considers different values of the first-order autoregressive coefficient α for two sample sizes $n = 100$ and $1,000$. We also consider the effect of including the weights $\omega_{j,i}^2$. For constant n, the small sample bias appears to decrease for increasing α. This could reflect that as the autoregressive coefficient α increases, the sample correlation between the retained step-indicators and the autoregressive process increases. Consistent with theory, the small sample bias vanishes asymptotically. The rescaling of the estimated

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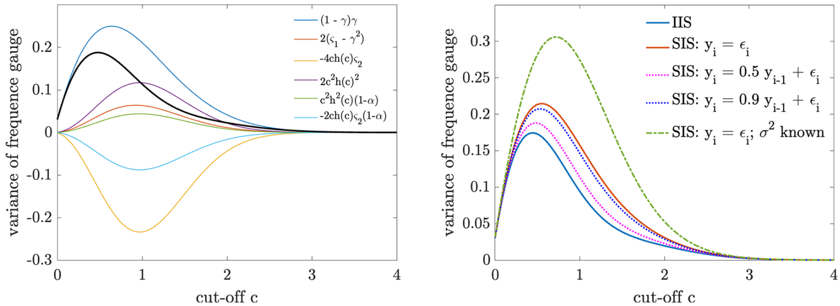

(a) Variance decomposition for DGP2 (b) Variance comparison for IIS and SIS

FIGURE 3. Analysis of the asymptotic variance of the frequency gauge for varying c.

variance, using the forward correction factors $\omega^2_{j,i}$, reduces the small sample bias by about one-third.

7.2. Analysis of Asymptotic Distribution of Frequency Gauge

We decompose the asymptotic variance of the frequency gauge of split-half SIS as a function of the cut-off c to understand the contributions of the various terms, and compare the variance to IIS. We continue to use DGP1 and DGP2.

Figure 3(a) presents a decomposition of the individual terms of the asymptotic variance of the gauge as functions of the cut-off c for the autoregressive DGP2 with $\alpha = 0.5$ as given by Theorem 4.4 and Example 4.3. The terms that do not depend on estimation errors are $(1-\gamma)\gamma$ and $2(\varsigma_1 - \psi^2)$; the terms that depend on the scale estimation error are $-4c\mathsf{h}(c)\varsigma_2$ and $2c^2\mathsf{h}(c)^2$ and the terms that depend on the location estimation error are $c^2\mathsf{h}^2(c)(1-\alpha)$ and $-2c\mathsf{h}(c)\varsigma_2(1-\alpha)$. Some terms increase the asymptotic variance, while one of the location terms and one of the scale terms decrease it.

Figure 3(b) compares the asymptotic variance of the gauge of the split-half IIS to split-half SIS.

Johansen and Nielsen (2016b, Cor. 5) gives the asymptotic distribution of the IIS gauge as

$$n^{1/2}\{\hat{\gamma}^{\mathrm{IIS}}_n(c) - \gamma\} \overset{\mathsf{D}}{\to} \mathcal{N}\{0, \gamma(1-\gamma) + 2c\mathsf{h}(c)\tilde{\varkappa}_2 + 2c^2\mathsf{h}^2(c)\}, \tag{54}$$

where $\tilde{\varkappa}_2 = \int_{-c}^{c}(u^2-1)\mathsf{f}(u)du$ is a truncated moment. Figure 3(b) displays the different asymptotic variance curves of the gauge as functions of the cut-off c for IIS and SIS for different DGPs. For IIS, we consider white noise DGP1. For SIS, we first consider the same DGP1, and second consider the autoregressive DGP2 with $\alpha = 0.5$ and $\alpha = 0.9$. Finally, we reconsider DGP1, but assume the error variance is known, so that $\hat{\omega}^2_{j,i} = \sigma^2 = 1$ and two components of the asymptotic variance become zero.

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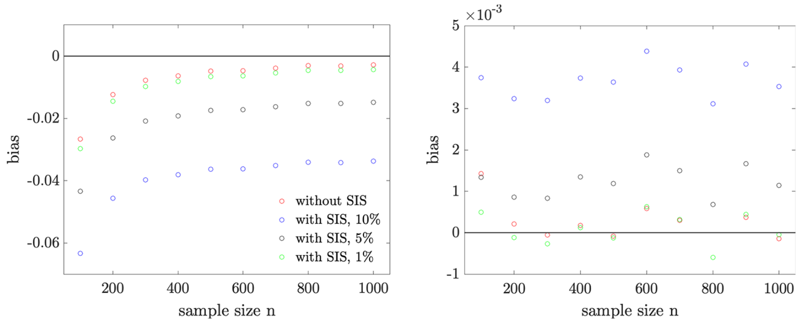
Table 1. Simulated and asymptotic variance of the frequency gauge of split-half SIS

	γ vs. n	100	400	1,600	∞
DGP1	5%	0.0516	0.0399	0.0363	0.0347
$\beta = 0$	1%	0.0160	0.0104	0.0094	0.0089
$\hat{\omega}^2_{j,i} = 1$	0.5%	0.0093	0.0057	0.0051	0.0047
	0.1%	0.0025	0.0013	0.0011	0.0010
DGP2	5%	0.0411	0.0284	0.0261	0.0249
$\alpha = 0.5$	1%	0.0149	0.0094	0.0085	0.0079
$\hat{\omega}^2_{j,i} = 1$	0.5%	0.0089	0.0056	0.0044	0.0044
	0.1%	0.0024	0.0013	0.0011	0.0010
DGP2	5%	0.0425	0.0348	0.0331	0.0323
$\alpha = 0.9$	1%	0.0134	0.0097	0.0087	0.0086
$\hat{\omega}^2_{j,i} = 1$	0.5%	0.0075	0.0052	0.0049	0.0046
	0.1%	0.0019	0.0012	0.0010	0.0010
DGP1	5%	0.0649	0.0627	0.0606	0.0610
$\beta = 0$	1%	0.0132	0.0124	0.0118	0.0117
$\hat{\sigma}^2_j = \sigma^2$	0.5%	0.0063	0.0060	0.0057	0.0057
	0.1%	0.0013	0.0011	0.0011	0.0010

We make the following observations. First, for all c, the asymptotic variance of the gauge in IIS is lower than for all four competing SIS models. Second, running SIS knowing the variance σ^2 results in a higher asymptotic variance of the frequency gauge. Third, in the autoregressive model, the α coefficient changes the asymptotic variance. The asymptotic variance is larger with $\alpha = 0.9$ than $\alpha = 0.5$. This is different from IIS, where the asymptotic variance does not include regressor-dependent terms. Finally, we observe that the asymptotic variance of the gauge falls rapidly for growing c. This motivates the choice of a large c in empirical applications, corresponding to a gauge of 1% or lower, as recommended by Castle et al. (2015).

7.3. Analysis of Distribution Convergence of Frequency Gauge

We now verify the asymptotic distribution results of the frequency gauge of split-half SIS and evaluate small sample properties. Table 1 tabulates the simulated variance and computed asymptotic variance of the frequency gauge of split-half SIS for the target gauges $\gamma = 5\%$, 1%, 0.5%, and 0.1% and sample sizes $n = 100$, 400, and 1,600. We consider the same models for split-half SIS as in Figure 3(b). Overall, the finite sample variance is quite close to the asymptotic variance when $n = 400$ and not too bad when $n = 100$. Our findings are consistent with the results in Figure 3.



(a) Autoregressive model (b) Exogenous regressor model

FIGURE 4. Bias of updated regression estimator as a function of sample size.

7.4. Updating Estimation of Regression Coefficients

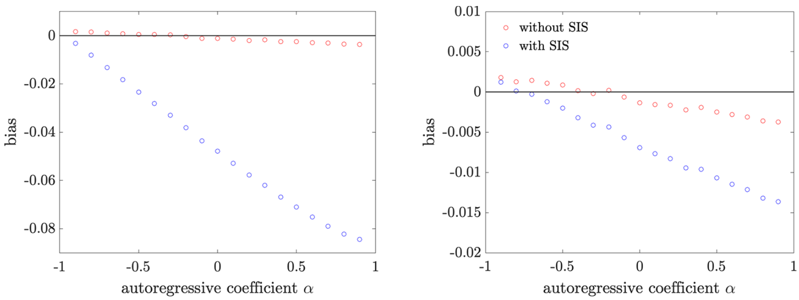
In this section, we use simulation to show that split-half SIS can introduce a bias when updating the estimates for β in (1). We conjecture that this bias can persist asymptotically with a fixed frequency gauge.

Suppose the split-half SIS Algorithm 2.1 is applied to data generated from an autoregressive model $y_i = \alpha y_{i-1} + \varepsilon_i$. This may result in $m-1$ level shifts at locations $\tau_0 = 0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m = n$. We update the α estimate by the regression

$$y_i = \mu_i + \alpha y_{i-1} + u_i \qquad \text{for } \tau_{j-1} < i \leq \tau_j \text{ and } j = 1, \ldots, m. \tag{55}$$

With a frequency gauge of γ, we will have approximately $m \approx \gamma n$ breaks so that the subsample lengths are approximately $n/m \approx 1/\gamma$. Thus, estimation of (55) corresponds to the estimation of an unbalanced dynamic panel model, with a (random) increasing cross-sectional dimension and a (random) finite time dimension. It seems like we are faced with the same issues as in the panel data of an incidental parameter problem (Lancaster, 2000, 2002) and a correlation of the retained (random) step-indicators with the dynamic regressors (Arellano and Bond, 1991). As with panel data, we would expect the bias to disappear asymptotically in a model with strictly exogenous regressors.

Figure 4 shows simulated biases of the updated estimator of the regression coefficients as a function of sample length n for different frequency gauges. Panel (a) uses the autoregressive DGP2 with $\alpha = 0.5$. As a baseline, we estimate the AR(1) model without split-half SIS. This shows the well-known negative finite sample bias that disappears asymptotically (Marriott and Pope, 1954). Then, we use split-half SIS with the frequency gauge at 1% (green), 5% (black), and 10% (blue). We find that a larger frequency gauge is associated with a larger bias that does not appear to vanish asymptotically. When we repeat this exercise in panel



(a) Frequence gauge of $\gamma = 10\%$ (b) Frequence gauge of $\gamma = 1\%$

FIGURE 5. Bias of updated regression estimator as a function of autoregressive coefficient.

(b) for exogenous regressors, we find that the bias is an order of magnitude smaller than before.

Figure 5 uses the autoregressive DGP2 and shows simulated biases as a function of the autoregressive coefficient α when the sample size is $n = 1,000$. Both panels use a standard autoregressive estimation without SIS as a benchmark along with split-half SIS estimation results. The frequency gauge is 10% in panel (a) and 1% in panel (b). We find a bias across all values of α, and it grows together with the value of α. The bias is much larger with the frequency gauge at 10% than at 1%.

Overall, the simulations provide evidence toward the presence of an incidental parameter bias when applying SIS with dynamic regressors and calibrated through the frequency gauge. The bias increases with the gauge and with the autoregressive coefficients.

7.5. Analysis of Power

We compare the power of split-half SIS and the Andrews (1993) test. We consider a one-shift data-generating process in order to validate the asymptotic theory in (46) for SIS and (50) and (52) for the Andrews test.

DGP3 has one location shift and is given by

$$y_i = \alpha y_{i-1} + \delta 1_{(i \geq \lambda n)} + \varepsilon_i \qquad \text{for } i = 1, \ldots, n, \tag{56}$$

with independent standard normal innovations. We will vary α, δ, λ, and n.

We subject the model (56) to split-half SIS and the Andrews test. For SIS, we use a 1% gauge and compute the retention frequency for the indicator at λn. The Andrews F test for detecting a single location shift with 15% trimming has a 1% critical value of 12.35. We report the power for the (maximum) test.

Table 2 shows the simulation results. The magnitude δ of the location shift is explored along columns. The location λ is explored along rows. Panels (1) and (2) consider a non-dynamic process $\alpha = 0$ for $n = 100$ and 66. Panel (3) considers a

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TABLE 2. Simulated power for the Andrews (A) test and split-half SIS

	λ	$\delta=0$ A	$\delta=0$ SIS	$\delta=2$ A	$\delta=2$ SIS	$\delta=4$ A	$\delta=4$ SIS	$\delta=8$ A	$\delta=8$ SIS
$n=100$	0.90	1.3%	1.0%	88.6%	12.0%	100.0%	57.0%	100.0%	99.9%
$\alpha=0$	0.95	1.1%	1.1%	19.2%	11.9%	51.7%	58.3%	39.5%	99.8%
	0.99	1.1%	1.2%	2.1%	11.3%	3.4%	58.1%	0.6%	99.9%
$n=66$	0.90	1.4%	1.1%	76.5%	12.8%	99.9%	58.4%	100.0%	99.8%
$\alpha=0$	0.95	1.2%	1.1%	11.0%	12.9%	24.9%	57.1%	13.0%	99.7%
	0.99	1.3%	1.2%	3.0%	12.3%	3.7%	58.2%	0.6%	99.7%
$n=66$	0.90	2.3%	0.4%	19.6%	8.5%	74.0%	55.6%	99.8%	99.9%
$\alpha=0.5$	0.95	2.4%	0.4%	4.3%	8.3%	7.4%	56.0%	6.4%	99.9%
	0.99	2.3%	0.3%	2.7%	8.8%	2.6%	56.4%	0.8%	99.9%

dynamic process $\alpha = 0.5$ for $n = 66$. The value 66 is chosen to find the δ where Andrews and SIS have equal power for $\lambda = 0.95$ as discussed in theory Section 6.

The columns marked $\delta = 0$ show the finite sample size and frequency gauge. We notice that the Andrews size is always larger than the SIS gauge. We note that the distortion is larger for $\alpha = 0.5$ than for $\alpha = 0$. The power simulations are not size corrected and are therefore favorable to the Andrews test.

The table confirms three predictions of our power theory. First, our theory predicts that the power increases with δ: We see that the SIS power is always increasing in δ. The Andrews power is also increasing in $\delta = 0$, 2, and 4, but it declines at $\delta = 8$ for $\lambda = 0.95$ and 0.99. For $\lambda = 0.99$, it even dips below the size. This may be a finite sample effect. Second, our theory predicts that the power of split-half SIS is invariant with respect to the location of λ, whereas the power of the Andrew test declines as λ approaches unity. This is confirmed in the simulations. Third, our theory predicts that the Andrews test has higher power than split-half SIS when λ is away from 1 while SIS is more powerful for λ that are close to 1. Indeed, our simulations are in favor of the Andrews test for $\lambda = 0.9$ and in favor of SIS when $\lambda = 0.99$. For the in-between case $\lambda = 0.95$, the results are mixed, with SIS being more powerful except in the first panel with $n = 100$ for $\delta = 2$.

Finally, we see that the power declines with increasing temporal persistence by looking at panels (2) and (3) where $n = 66$, but the autoregressive coefficient is $\alpha = 0$ and $\alpha = 0.5$, respectively. There is an indication that the decline in performance is larger for the Andrews test than for SIS.

8. EMPIRICAL ILLUSTRATION

As an empirical example of the use of stylized SIS, consider the log U.K. labor productivity, y_i, from the first quarter of 1980 to the third quarter of 2021. This gives a sample of length $n = 167$ plus initial values. The labor productivity is

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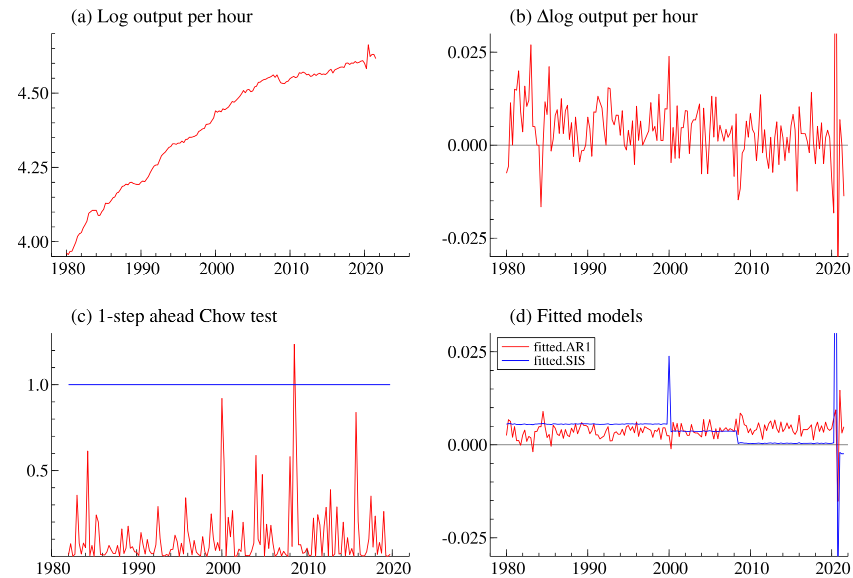

FIGURE 6. U.K. labor productivity.

measured by the U.K.'s Office of National Statistics as a chain volume measure of gross value added at basic prices divided by the number of hours worked. We used PcGive in OxMetrics 8 for the analysis (Doornik and Hendry, 2013).

Figure 6(a) shows the log labor productivity y with a marked decline in its growth rate after the 2008 financial crisis. There is considerable movement through the COVID-19 pandemic from 2020. The post-2008 decline has been of concern in the political debate for some years, see, for example, Chadha (2022), and the submission to the Treasury Committee in October 2021 by the Bank of England's Chief Economist, Huw Pill:

> "Before the global financial crisis, U.K. productivity growth averaged over two per cent per year. Since then, labor productivity (growth) has fallen considerably."

Panel (b) shows the labor productivity growth rate measured as the log difference $\Delta y_i = y_i - y_{i-1}$. Note that the y-axis has been truncated to better visualize the pre-COVID periods. We make the following observations. The series is very noisy, and one can just about visually discern a gradual decline over time. We will model the growth rate as a first-order autoregression, thus imposing that the series in levels has a unit root. We will show how SIS can help in capturing the declining level of the growth rate.

We start by fitting a first-order autoregression to the growth rate for the whole period. While not reported here, the results point to a very misspecified model, and diagnostics point to difficulties matching movements through the COVID period.

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An investigator may, therefore, drop that period and focus on the period until 2019:4. We then find the model:

$$\underset{(se)}{\widehat{\Delta y_i}} = \underset{(0.079)}{0.104}\Delta y_{i-1} + \underset{(0.0007)}{0.0035}, \tag{57}$$

$$\widehat{\sigma} = 0.0069, \qquad n = 160, \qquad RSS = 0.0074, \tag{58}$$

$$\chi^2_{norm}[2] = 5.00\,(p = 0.082), \qquad \mathsf{F}_{ar(1-5)}[5, 153] = 1.96\,(p = 0.088), \tag{59}$$

$$\max C^2 = 8.49\,(p = 0.482)\,\{\arg\max = 2008:3\}, \tag{60}$$

$$\max F = 3.52\,(p = 0.01)\,\{\arg\max = 2004:1\}. \tag{61}$$

The fitted model is reported in (57) and in Figure 6(d). The fit indicates an overall constant level for the quarterly growth rate of $0.0035/(1-0.104) = 0.39\%$.

We subjected the model (to 2019:4) to various misspecification tests. These do not tend to reject the model. A normality test based on cumulants (Kilian and Demiroglu, 2000; Doornik and Hansen, 2008) and a test for residual autocorrelation (Godfrey, 1978; Nielsen, 2006) are reported in (59). Figure 6(c) shows a one-step recursive Chow test with pointwise 1% critical values. This indicates a slight rejection in 2008:3, but the practitioner may not wish to give too much attention to this, given that about 144 tests were conducted (Hendry and Nielsen, 2007). Indeed, a joint test as shown in (60) does not reject the model (Nielsen and Whitby, 2015). The Andrews test reported in (61), used for detecting a single location shift, gives a marginal decision indicating a possible break in 2004:1. It appears that minor location-shifts are not reliably detected by conventional misspecification tests. Yet, Figure 6(a) does show a marked decline in the log labor productivity y_i since 2008.

We now apply the stylized SIS algorithm to the full sample until 2021:3. First, we fit the first-order autoregression to the first sample-half until 1999:4. This is the same as fitting the autoregression to the full sample combined with step-indicators for each observation from 2000:1 to 2021:3. We get

$$\underset{(se)}{\widehat{\Delta y_i}} = \underset{(0.111)}{0.201}\Delta y_{i-1} + \underset{(0.0010)}{0.0045} + \sum_{j=81}^{167}\widehat{\delta}_j 1_{(i\geq j)}, \tag{62}$$

$$\widehat{\sigma} = 0.0068, \qquad n = 167, \qquad RSS = 0.0036, \tag{63}$$

$$\chi^2_{norm}[2] = 3.78\,(p = 0.151), \qquad \mathsf{F}_{ar(1-5)}[5, 73] = 1.39\,(p = 0.237). \tag{64}$$

This fit indicates a constant quarterly growth rate of $0.0045/(1-0.201) = 0.56\%$ prior to 2000. Test for normality and residual autocorrelation do not reject (see (64)).

There are 87 estimated coefficients for the step-indicators. Computing the t-statistics for these 87 estimates, we find that the most extreme t-statistics are: 10.4 for 2020:3, −9.57 for 2020:4, 4.28 for 2021:1, −3.21 for 2000:2, −2.69 for

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2008:3, 2.65 for 2016:1, 2.53 for 2000:1, 2.17 for 2004:2, and 2.00 for 2008:2. Using the 1% cut-off for the normal distribution of 2.576, we keep the six most significant step-indicators. Rerunning the model gives

$$\begin{aligned}\underset{(se)}{\widehat{\Delta y}_i} &= -\underset{(0.074)}{0.008}\Delta y_{i-1} + \underset{(0.0009)}{0.0056} \\ &\quad + \underset{(0.0068)}{0.0183} I_{(i\geq 00:1)} - \underset{(0.0069)}{0.0202} I_{(i\geq 00:2)} - \underset{(0.0015)}{0.0033} I_{(i\geq 08:3)} \\ &\quad + \underset{(0.007)}{0.080} I_{(i\geq 20:3)} - \underset{(0.012)}{0.119} I_{(i\geq 20:4)} + \underset{(0.010)}{0.036} I_{(i\geq 21:1)},\end{aligned} \quad \textbf{(65)}$$

$$\widehat{\sigma} = 0.0068, \qquad n = 167, \qquad RSS = 0.0072, \quad \textbf{(66)}$$

$$\chi^2_{norm}[2] = 5.39\,(p = 0.068), \qquad \mathsf{F}_{ar(1-5)}[5, 154] = 2.39\,(p = 0.041). \quad \textbf{(67)}$$

The autoregressive coefficient is now insignificant. Adding up the constant terms and correcting for the modest autoregressive coefficient gives long-run means of 0.56% prior to 2000, then 0.37% until 2008, then 0.043% until 2020.

We identify two significant drops in productivity in 2000 and 2008, corresponding to the burst of the dot-com bubble and the financial crises, respectively. Both are characterized by pairs of offsetting step-indicators. However, the split half-SIS retains only one of the two step-indicators from 2008, which results in less accurate tracking of the series during the financial crisis.

The more comprehensive SIS algorithm in OxMetrics yields a similar model to split-half SIS, but it manages to retain two offsetting step-indicators for 2008 instead of just one. In the updated OxMetrics model, these indicators have larger t-statistics than the single 2008 indicator found in (65). It appears that the split-half SIS is too simple to track the somewhat protracted upheaval during the financial crisis.

9. CONCLUSION

In this article, we investigated the properties of the SIS algorithm that addresses location shifts in time series in the context of model selection. The growing importance of SIS in tackling location shifts is reflected in its applications in fields as varied as economics (Chuffart and Hooper, 2019; Bernstein and Martinez, 2021; Pellini, 2021), climate science (Raggad, 2018; Pretis et al., 2018b; Koch et al., 2022; O'Callaghan et al., 2022), and public health (Doornik et al., 2022). In this section, we summarize the insights gained through a study of SIS with asymptotic analysis, simulations, and numerical approximations.

The first insight is that the frequency gauge is consistent for a wide range of both stationary and non-stationary regressors. This means that even without detailed knowledge of the regressor types, an investigator can choose the cut-off of SIS from the limiting gauge. To address the sensitivity of this result, we demonstrated that the variation of the frequency gauge around its limit asymptotically follows a normal distribution. However, its variance depends on the type of regressors.

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Simulations revealed that this variation remains limited, even in small samples. As a result, the sole tuning parameter of the SIS algorithm can be finely adjusted to align with the investigator's preferences.

The second insight concerns the link between the frequency gauge and the bias in the updated regression estimator after selecting over step-indicators. This bias appears to emerge in the presence of dynamic regressors when searching for location shifts. This contrasts with the theory of IIS, where there is no such bias (Johansen and Nielsen, 2016b). The bias diminishes as the gauge decreases, suggesting that the gauge should be chosen small and possibly vanishing with sample size. For that purpose, we developed a Poisson theory for the absolute gauge. For a sample size of $n = 100$ observations, we recommend setting the absolute gauge to 1, which is equal to the frequency gauge of 1%, in line with Castle et al. (2015). In larger samples, we advise targeting the absolute gauge rather than the frequency gauge, so that the cut-off drifts slowly to infinity.

The third insight pertains to the circumstances in which stylized SIS demonstrates higher statistical power compared to the Andrews (1993) test. We developed a local power theory for stylized SIS and the Andrews test. Our findings suggest that the Andrews test maintains consistency when faced with one or two well-separated, central location shifts, whereas the SIS shows trivial power. Conversely, for location shifts near the end of the sample or for two offsetting location shifts close to each other, the SIS maintains power, while the power of the Andrews test goes down to its size. In time series observed over extended periods, major upheavals like the 2008 financial crisis and the 2020 COVID-19 pandemic might recur. Consequently, we anticipate multiple breaks in the data. These breaks may occur closely together or toward the end of the sample. In such scenarios, SIS appears to be preferable to the Andrews test. The same conclusions hold for the Bai and Perron (1998) procedure that allows more breaks but inherits the power trade-offs from the Andrews test.

The fourth insight relates to the regularity conditions of SIS compared to the Andrews test. SIS assumes a known error distribution, while the Andrews test does not. The assumption is testable and contributes to the power of SIS to detect breaks that occur closely together. The first-order theory for SIS applies to a variety of stationary and non-stationary regressors. In order to do this, the present theory is formulated in terms of normalization matrices. This implies that the theory works regardless of the choice of the normalization matrix. In contrast, the asymptotic theory for the Andrews test requires stationary regressors, introducing an additional risk of mistakes, as the investigator must carefully determine the appropriate normalization of the regressors. Furthermore, SIS is designed to be implemented along with regressor selection, which is useful when there is uncertainty about the choice of regressors.

The theory for SIS is complicated because SIS operates on the differenced residuals which are temporally dependent even for well-behaved errors. We found various technical solutions that may be useful elsewhere. The empirical process theory was developed using ideas from the McLeish (1977) mixingale theory.

The Poisson theory requires the Chen (1975) Poisson limit theorem for dependent binary variables. In addition, to allow two close breaks in the power theory, we relied on the Skorokhod (1956) M_1-metric favored by Whitt (2002) rather than the J_1-metric favored by Billingsley (1968).

A potential further development is to develop a test for the presence of location shifts along the lines of the IIS test for outliers of Jiao and Pretis (2022). The techniques for dealing with correlation between (differenced) errors and regressors turn out to be useful for analyzing instrumental variable estimation (Jiao, 2019).

Overall, our results provide theoretical underpinnings that recast SIS as an interpretable and tunable break-detection tool: its gauge can be chosen ex ante and justified ex post, its post-selection bias controlled through gauge calibration, and its power advantages are located precisely where applied work often struggles—near sample endpoints and during short, offsetting upheavals. In settings with non-stationary regressors, SIS should be the default search device; when a few well-separated breaks are suspected, the Andrews/Bai–Perron procedures remain preferable. The empirical-process, Poisson, and M_1 techniques we develop both underwrite these claims and open a path to new analysis in the model selection theory.

SUPPLEMENTARY MATERIAL

Proofs of all results are provided in Nielsen and Qian (2025): Supplement to "Asymptotic Properties of the Gauge and Power of Step-Indicator Saturation," Econometric Theory Supplementary Material. To view, please visit: https://doi.org/10.1017/S0266466625100145.

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