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EXTENDING ECONOMIC MODELS WITH TESTABLE ASSUMPTIONS: THEORY AND APPLICATIONS

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This article studies the identification of complete economic models with testable assumptions. We start with a local average treatment effect (*LATE*) model where the "No Defiers," the independent IV assumption, and the exclusion restrictions can be jointly refuted by some data distributions. We propose two relaxed assumptions that are not refutable, with one assumption focusing on relaxing the "No Defiers" assumption while the other relaxes the independent IV assumption. The identified set of *LATE* under either of the two relaxed assumptions coincides with the classical *LATE* Wald ratio expression whenever the original assumption is not refuted by the observed data distribution. We propose an estimator for the identified *LATE* and derive the estimator's limit distribution. We then develop a general method to relax a refutable assumption A. This relaxation method requires finding a function that measures the deviation of an econometric structure from the original assumption A, and a relaxed assumption $\tilde{A}$ is constructed using this measure of deviation. We characterize a condition to ensure the identified sets under $\tilde{A}$ and A coincide whenever A is not refuted by the observed data distribution and discuss the criteria to choose among different relaxed assumptions.

1. INTRODUCTION

Empirical researchers often make convenient model assumptions which usually come from economic theories or intuitions. For example, the "No Defiers" assumption in Imbens and Angrist (1994) assumes that the instrument has a monotone effect on the decision to take treatment, the "Monotone Instrument" assumption in Manski and Pepper (2000) imposes that an instrument monotonically shifts the conditional mean of the potential outcomes, and the "Perfect Sector Selection" assumption in Roy (1951) assumes that employees perfectly observe their future earnings in two job sectors and choose the job sector that maximizes discounted lifetime earnings. Such assumptions simplify the identification and make the results easier to interpret.

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Unfortunately, assumptions in the three examples above, when combined with some other reasonable assumptions, can be rejected by some distributions of observables.[1] When the imposed assumption is refuted by data, the econometrician is faced with an empty identified set for the parameter of interest. As a result, the econometrician cannot give a useful interpretation of the economic environment.

A way to prevent the data rejection problem is to find a relaxed assumption $\tilde{A}$ so that no distributions of observables can reject $\tilde{A}$. Such a relaxed assumption is called non-refutable. One way to construct a non-refutable relaxed assumption is to consider a data-dependent assumption selection method: For each possible data distribution, we pick a particular assumption that can rationalize the data. When picking the assumption, we also do not want to deviate too much from the original assumption A, since it reflects the economic theory behind it. Specifically, given a parameter of interest θ, we want the identified set under the relaxed assumption $\tilde{A}$ to be the same as the identified set under A whenever A is not rejected by the data distribution. In other words, we want to preserve the identified set.

Constructing a non-refutable assumption $\tilde{A}$ via a data-dependent approach also allows us to view the identified set of parameters of interest as a correspondence whose further properties will be discussed. In particular, we are interested to know whether the identified set of the parameter of interest is continuous in the observed data distribution under the relaxed assumption $\tilde{A}$. If such a continuity property fails for the identified set under $\tilde{A}$, we may question the appropriateness of using $\tilde{A}$: On the one hand, it is hard to use economic intuitions to justify why the identified set changes abruptly due to a small change in the observed data distribution. On the other hand, the discontinuity of the identified set also poses challenges to estimating the identified set.[2]

We aim to provide a general method to construct a relaxed assumption from the original assumption such that the relaxed assumption preserves the identified set. To achieve this goal, we start with the local average treatment effect (*LATE*) model (Imbens and Angrist, 1994) with the classical "No Defiers" assumption, the independent IV assumption, and the exclusion restrictions, which are jointly refutable (Kitagawa, 2015; Mourifié and Wan, 2017). We use this application as a leading example to illustrate the way to construct relaxed assumptions that preserve the identified set. The application also sheds light on some subtle issues when we construct a relaxed assumption, and it serves as a running example where abstract definitions in the later section can be matched. We then generalize the insights in the *LATE* example and present a general approach to construct a relaxed assumption. The general approach allows us to construct multiple relaxed assumptions and we discuss several criteria for econometricians to decide which relaxed assumptions to use.

[1] Kitagawa (2015) and Mourifié and Wan (2017) propose tests for the assumptions of the Imbens and Angrist (1994) model. Hsu et al. (2019) propose a test for the monotone instrument assumption. Mourifié et al. (2020) study the testability of the Roy model.

[2] We discuss the continuity of identified sets in Appendix B.

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The rest of the article is organized as follows. Section 2 introduces the *LATE* example. The classical *LATE* model makes three assumptions on the distribution of potential outcomes, potential treatments, and instruments (denoted by G): the "No Defiers" assumption, the independent IV assumption, and the exclusion restrictions. We propose two relaxed assumptions that focus on relaxing the "No Defiers" assumption and the independent IV assumption, respectively. To construct the assumption that relaxes the "No Defiers" assumption, we use the probability of defiers as a measure of the deviation from the "No Defiers" assumption. We then construct a relaxed assumption by specifying what distributions G should be put into the relaxed assumption when a particular data distribution is observed. For each possible observed data distribution, we only consider the G that are consistent with the observed data distribution and have the minimal probability of defiers. We call this the minimal defiers relaxed assumption. In the construction of the minimal defiers relaxed assumption, we also emphasize the importance of imposing a weaker instrument independence condition: we require the instrument to be independent of the potential outcomes conditional on economic agents' compliance types,[3] which is weaker than the independent IV assumption. The assumption that relaxes the independent IV assumption is constructed in a similar way: we first define a function that measures the deviation of a G from the independent IV assumption; we then specify what G to put in the relaxed assumption for each possible observed data distribution, and call this the minimal marignal dependent instrument relaxed assumption.[4]

In Section 2, we also characterize the identified sets of *LATE* under the two relaxed assumptions, and the two identified sets of *LATE* coincide for all possible data distributions. Moreover, the *LATE* is point-identified. We propose an estimator of the *LATE* quantity and derive its asymptotic behavior. We then apply the identification results to study the return of college education using the dataset in Card (1993). The empirical results show that the *LATE* identified under the minimal defiers or the minimal marginal dependent instrument relaxed assumptions is more reasonable compared to the classical Wald ratio *LATE* quantity.[5]

In Section 3, we formalize a general theory to construct a non-refutable relaxed assumption out of a refutable assumption. We formalize the definitions of econometric structures, refutable assumptions, and identified sets. A further two-moment-inequality example is introduced to help illustrate the definitions. We construct a relaxed assumption similarly as we did in the *LATE* example: we consider a deviation measure m_j and define the relaxed assumption that specifies what econometric structures to be included in the relaxed assumption for each observed data distribution. Moreover, all econometric structures included in the

[3]Compliance types include always-takers, never-takers, compliers, and defiers.

[4]Similarly, when we decide what G to put into the relaxed assumption, we only consider the G that achieves the minimal deviation from the independent IV assumption.

[5]In Imbens and Angrist (1994), *LATE* is identified as the Wald ratio under the classical assumptions: $LATE^{Wald} \equiv (E[Y_i|Z_i=1]-E[Y_i|Z_i=0])/(E[D_i|Z_i=1]-E[D_i|Z_i=0])$.

relaxed assumption are required to have a minimal deviation from the original assumption measured by the m_j function. We call such a construction the minimal deviation relaxed assumption. We then characterize the conditions on m_j such that the identified set of parameters can be preserved under the minimal deviation relaxed assumption whenever the original assumption is not refuted by the observed data distribution. Multiple relaxed assumptions can be constructed with different features, and we recommend using empirical relevance and continuity of the identified set as criteria for selecting the appropriate relaxed assumption. We conclude the section with an extension to the incomplete models, where the minimal deviation construction of the relaxed assumption may fail to preserve the identified set.

Section 4 concludes. All proofs are collected in the Appendixes.

Related Literature

First, we contribute to the literature on model misspecification and refutation. In the macroeconomic literature, researchers use robust control to avoid the misspecification issue in their baseline model (Hansen et al., 2006; Hansen and Sargent, 2007). The robust control approach aims to accommodate local perturbations to the baseline model rather than solve the refutability of the baseline model. In econometrics, Bonhomme and Weidner (2022) use the local asymptotics framework to study the local perturbation of econometric models while Christensen and Connault (2023) consider a moment inequality model with misspecified parametric distributional assumption. Focusing on the model refutation issue, Masten and Poirier (2021) studies the refuted linear IV model. Our analysis extends the approach in Masten and Poirier (2021) to discuss several issues when using a relaxed assumption for refutable models.

Second, we also contribute to the *LATE* literature. Since Kitagawa (2015) proves the sharp testable implication of Imbens and Angrist (1994), literature relaxes the "No Defiers" condition. De Chaisemartin (2017) discusses the economic meaning of the conventional Wald ratio expression when defiers exist. He shows that, under the additional assumption that a subgroup of compliers accounts for the same population proportion as defiers and they have the same *LATE*, the Wald ratio identifies the net average treatment effect of a subgroup of compliers after deducting the average treatment effect of defiers.

Notations

Throughout this article, we use X to denote the vector of observed variables, and we use F to denote the distribution of X. We use ϵ to denote the vector of primitive variables, and we use G to denote the distribution of ϵ. We use s to denote an econometric structure. We use A to denote a refutable assumption and use $\tilde{A}$ to denote a relaxed assumption. We use $\kappa_1 \asymp \kappa_2$ to denote the same order of magnitude between κ_1 and κ_2.

2. A LEADING EXAMPLE: THE TREATMENT EFFECTS APPLICATION

We start with an application with a binary treatment and a binary instrument (Imbens and Angrist, 1994). We first set up the notations and then briefly summarize the results in Kitagawa (2015) that the assumptions in Imbens and Angrist (1994) can be rejected by some data distributions. Notations are consistently set up to connect to definitions in Section 3. We characterize the identified *LATE* under these relaxed assumptions. We conclude the section with an empirical illustration with the Card (1993) data.

2.1. The Potential Outcome Framework

An econometrician observes an outcome variable Y_i, a treatment decision D_i, and a binary instrument Z_i. The observed outcome variable Y_i and treatment decision D_i are generated through

$$
\begin{aligned}
&Y_i = Y_i(1,1)D_iZ_i + Y_i(0,1)(1-D_i)Z_i + Y_i(1,0)D_i(1-Z_i) + Y_i(0,0)(1-D_i)(1-Z_i),\\
&D_i = D_i(1)Z_i + D_i(0)(1-Z_i),
\end{aligned}
\tag{2.1}
$$

where $D_i(1), D_i(0)$ are potential treatment decisions, $Y_i(d,z)$ is the potential outcome under $(D_i = d, Z_i = z)$, and Z_i is the binary instrument.

We collect all variables on the RHS of (2.1) in a vector and call them the primitive variables: $\epsilon_i = (D_i(1), D_i(0), Y_i(0,0), Y_i(1,0), Y_i(0,1), Y_i(1,1), Z_i)$. We collect the observed variables in $X_i = (Y_i, D_i, Z_i)$. Let $\mathcal{Y}$ be the metric space of Y_i and let $\mathcal{B}$ be the Borel-sigma algebra on $\mathcal{Y}$. The space of distributions of X_i is

$$
\mathcal{F} = \{F_X(y,d,z) : \text{the support of } F_X \text{ is contained in } \mathcal{Y} \times \{0,1\}^2\}, \tag{2.2}
$$

and the space of distributions of primitive variables is

$$
\mathcal{G} = \left\{G(\epsilon) : \text{the support of } G \text{ is contained in } \{0,1\}^2 \times \mathcal{Y}^4 \times \{0,1\}\right\}. \tag{2.3}
$$

Given the potential outcome equation (2.1), we can define a unique mapping $M : \mathcal{G} \to \mathcal{F}$, such that for any $G^s \in \mathcal{G}$:

$$
\begin{aligned}
M(G^s) = \Big\{F \in \mathcal{F} : &Pr_F(Y_i \in B, D_i = d, Z_i = z) = Pr_{G^s}(Y_i(d,z) \in B, D_i(z) = d, Z_i = z),\\
&\forall B \in \mathcal{B}, \quad d,z \in \{0,1\}\Big\}.
\end{aligned}
\tag{2.4}
$$

In other words, $M(G^s)$ is the push-forward distribution of G^s under the mapping (2.1), and $M(G^s)$ contains a unique F.

In the potential outcome framework, we call G^s an econometric structure s: different econometric structures have different primitive variable distributions, and they may imply different economic interpretations through G^s through the same M, which determines the economic relationship between primitive variables and

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observed variables. We therefore can consider the following set of structures as the structural space:

$$\mathcal{S} = \{s | \; G^s \in \mathcal{G}\}. \tag{2.5}$$

A commonly used assumption for the potential outcome model is the Imbens–Angrist Monotonicity assumption (IA-M) where we assume the exogeneity, exclusion, and monotonicity of the instrument Z_i. We can write the IA-M assumption (denoted by A) as a subset of $\mathcal{S}$:

$$\begin{aligned}
A &= A^{ND} \cap A^{IV} \cap A^{ER}, \quad \text{where} \\
A^{ND} &= \{s : D_i(1) \geq D_i(0), \; G^s - \text{a.s.}\}, \\
A^{IV} &= \{s : G^s \text{ satisfies } Z_i \perp (Y_i(1,1), Y_i(0,1), Y_i(1,0), Y_i(0,0), D_i(1), D_i(0))\}, \\
A^{ER} &= \{s : Y_i(1,1) = Y_i(1,0) \quad \text{and} \quad Y_i(0,1) = Y_i(0,0), \; G^s - \text{a.s.}\}.
\end{aligned} \tag{2.6}$$

In (2.6), we refer to A^{ND} as the "No Defiers" assumption, A^{IV} as the independent IV assumption, and A^{ER} as the exclusion restriction. Our main parameter of interest is the *LATE* for compliers:

$$LATE(G^s) \equiv E[Y_i(1,1) - Y_i(0,0) | D_i(1) = 1, D_i(0) = 0]. \tag{2.7}$$

We focus on the *LATE* because it is analyzed frequently in empirical papers as a policy-relevant quantity and it can be identified as a simple Wald ratio (Imbens and Angrist, 1994):

$$LATE^{Wald}(F) = \frac{E_F[Y_i|Z_i = 1] - E_F[Y_i|Z_i = 0]}{E_F[D_i|Z_i = 1] - E_F[D_i|Z_i = 0]}, \quad \text{if IA-M (2.6) holds.} \tag{2.8}$$

2.2. The Sharp Testable Implication

While the IA-M assumption clearly identifies the *LATE* quantity, it can be rejected by some data distributions. For example, $E_F[D_i|Z_i = 1] \geq E_F[D_i|Z_i = 0]$ must hold under the observed data distribution F to satisfy the IA-M assumption. We summarize the results in Kitagawa (2015), who derives the sharp testable implications of the IA-M assumption (2.6). Let us define two quantities for all $B \in \mathcal{B}$ and $d \in \{0, 1\}$:

$$\begin{aligned}
P(B, d) &\equiv Pr_F(Y_i \in B, D_i = d | Z_i = 1), \\
Q(B, d) &\equiv Pr_F(Y_i \in B, D_i = d | Z_i = 0).
\end{aligned} \tag{2.9}$$

We abuse the notation and suppress the dependence of $P(\cdot, d)$ and $Q(\cdot, d)$ on F.

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LEMMA 2.1. *Let $P(\cdot,d)$ and $Q(\cdot,d)$, $d \in \{0,1\}$, be absolutely continuous with respect to some measure μ_F.*[6] *For any structure $s \in A$, $F \in M(G^s)$, and any Borel set $B \in \mathcal{B}$, the F must satisfy:*

$$P(B,1) \geq Q(B,1), \quad \text{and} \quad Q(B,0) \geq P(B,0). \tag{2.10}$$

Moreover, for any F satisfying (2.10), there is an $s \in A$ such that $F \in M(G^s)$.

Kitagawa (2015) proposes using the core determining class (Galichon and Henry, 2011) to test (2.10). As shown in Lemma 2.1, the IA-M assumption A is refutable. When A is rejected by data, then the identified set for *LATE* should be empty because no *LATE* value implied by the IA-M assumption is consistent with the data. In many empirical applications, researchers do not test this implication, nor do they specify what should be done when the A is rejected. In the next section, we use a relaxed assumption approach to find relaxed assumptions $\tilde{A}$ that are non-refutable, characterize the identified set under the relaxed assumptions, and discuss the estimation and inference on *LATE*.

2.3. Relax the IA-M Assumption

Depending on the empirical contexts, researchers may consider different aspects of the IA-M assumption to be the source of the model refutation. In this section, we study two ways to relax the IA-M assumption: the first relaxed assumption mainly targets the "No Defiers" assumption while maintaining part of the IV assumption; the second approach maintains the "No Defiers" assumption and targets the IV assumption. We maintain the exclusion restriction $Y_i(d,z) = Y_i(d,1-z)$ for the rest of the section and leave the discussion of relaxing the exclusion restriction to Section D of the Supplementary Material.

2.3.1. *The Minimal Defiers Extension.* In some contexts, the instrument may have heterogeneous effects on economic agents' decisions to take treatment and defiers may exist. In such cases, the empirical researcher may want to relax the "No Defiers" assumption. However, allowing for defiers alone cannot solve the model refutation problem (Kitagawa, 2021). For example, if $Y_i \in \{0,1\}$ is also binary, then the exclusion restriction and independent IV assumptions still imply that $P_F(Y_i = 1, D_i = 0|Z_i = 0) \geq P_F(Y_i = 1, D_i = 0|Z_i = 1) - P_F(D_i = 1|Z_i = 0)$must hold for the observed data distribution.[7]

To circumvent the refutability of the independent IV assumption, we introduce a weaker version of the IV assumption, which is called the conditional type independence assumption:

[6] Such dominating measure always exists, for example, define $\mu_F(B) = P(B,1) + Q(B,1) + P(B,0) + Q(B,0)$ for all $B \in \mathcal{B}$.

[7] To see this testable implication, we first impose the exclusion restriction and use $Y_i(d)$ to denote the potential outcomes $Y_i(d,1)$ and $Y_i(d,0)$. We want to show a failure of $(Y_i(d), D_i(1), D_i(0)) \perp Z_i$. We look at two terms Term $1 \equiv Pr_G(Y_i(0) = 1, D_i(1) = D_i(0) = 0|Z_i = 0)$ and Term $2 \equiv Pr_G(Y_i(0) = 1, D_i(1) = D_i(0) = 0|Z_i = 1)$. These two terms

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$$A^{TI} = \left\{ s \middle| G^s \text{ satisfies } Z_i \perp (Y_i(1,1), Y_i(0,1), Y_i(1,0), Y_i(0,0)) \,| D_i(1), D_i(0) \right\}.$$

We call A^{TI} the conditional type independence assumption because the conditioning variables $D_i(1), D_i(0)$ define whether the individual i is an always-taker/never-taker/complier/defier. The independent IV assumption implies the conditional type independence assumption because $A^{IV} \subseteq A^{TI}$. The conditional type independence assumption is also used in other empirical contexts to study the *LATE* (Kedagni, 2019). By using the conditional type independence assumption, we can avoid the refutation issue.

We now state an equivalent representation of the IA-M assumption using the conditional type independence representation.

LEMMA 2.2. *The IA-M assumption defined in (2.6) can be equivalently written as the intersection $A = A^{ER} \cap A^{TI} \cap A^{EM-NTAT} \cap A^{ND}$, where $A^{EM-NTAT}$ is the set of structures s such that:*

$$\begin{aligned} &E_{G^s}[\mathbb{1}(D_i(1) = D_i(0) = 1)|Z_i = 1] = E_{G^s}[\mathbb{1}(D_i(1) = D_i(0) = 1)|Z_i = 0], \\ &E_{G^s}[\mathbb{1}(D_i(1) = D_i(0) = 0)|Z_i = 1] = E_{G^s}[\mathbb{1}(D_i(1) = D_i(0) = 0)|Z_i = 0]. \end{aligned} \tag{2.11}$$

The assumption $A^{EM-NTAT}$ says that the measure of always/never-takers is independent of the instrument. It is weaker than the condition $(D_i(1), D_i(0) \perp Z_i)$ because it allows the probability of compliers or defiers to depend on the instrument. It should be noted that $A^{IV} \subseteq A^{EM-NTAT}$ so $A^{EM-NTAT}$ is also weaker than the IV assumption. The alternative representation breaks the independent IV assumption into the intersection of A^{TI} and $A^{EM-NTAT}$.

The alternative presentation in Lemma 2.2 comes with the following two desirable features: first, when the "No Defiers" assumption is effective, we preserve the IA-M assumption; second, when the "No Defiers" assumption is given up, as we will later see, the rest of the assumptions $A^{ER} \cap A^{TI} \cap A^{EM-NTAT}$ are not refutable. We will use the non-refutation feature to construct a non-refutable relaxed assumption, and the preservation property will be used to show a good property of the *LATE* quantity under the relaxed assumption.

must be equal because of the independent IV assumption. By the potential outcome model (2.1), Term 1 $\leq Pr_F(Y_i = 1, D_i = 0|Z_i = 0)$. For Term 2, we can derive a sequence of relations:

$$\begin{aligned} \text{Term 2} &=_{(a)} Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - Pr_G(Y_i(0) = 1, D_i(1) = 0, D_i(0) = 1|Z_i = 1) \\ &=_{(b)} Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - Pr_G(Y_i(0) = 1, D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &\geq_{(c)} Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - Pr_G(D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &=_{(d)} Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - [Pr_F(D_i = 1|Z_i = 0) - Pr_G(D_i(1) = D_i(0) = 1|Z_i = 0) \\ &\geq Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - Pr_F(D_i = 1|Z_i = 0), \end{aligned}$$

where (a) follows by the potential outcome model (2.1), (b) follows by the independent IV assumption, (c) follows because we subtract a larger probability, and (d) follows by the potential outcome model (2.1). When the lower bound $Pr_F(Y_i = 1, D_i = 0|Z_i = 1) - Pr_F(D_i = 1|Z_i = 0)$ overtakes the upper bound $P_F(Y_i = 1, D_i = 0|Z_i = 0)$, the independent IV assumption must fail.

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We consider a relaxed assumption that uses the following measure of defiers:

$$m^d(s) = E_{G^s}[\mathbb{1}\{D_i(1) = 0, D_i(0) = 1\}]. \quad \mathbf{(2.12)}$$

This quantity serves as a lens through which we can view the deviation of a structure s from the "No Defiers" assumption. We relax the original IA-M by allowing for a certain amount of defiers and it is characterized by the following.

Assumption 2.1. Let $m^{min,d}(F) \equiv \inf\{m^d(s) : F \in M^s(G^s)$ *and* $s \in A^{ER} \cap A^{TI} \cap A^{EM-NTAT}\}$ be the minimal defier amount under F. We call

$$\tilde{A} = \cup_{F \in \mathcal{F}} \left\{ s \in A^{ER} \cap A^{TI} \cap A^{EM-NTAT} : m^d(s) = m^{min,d}(F),\ F \in M(G^s) \right\}$$

the minimal defiers relaxed assumption.

Assumption $\tilde{A}$ is constructed by a data-dependent method which specifies what we should consider in our relaxed assumption for every F in the observed data distribution space. Assumption 2.1 says that for each F, we only focus on the set of the structures that achieve the minimal deviation from the A^{ND} assumption. The construction of $\tilde{A}$ may be rationalized by an econometrician who has a belief about the possible value of defiers. Since defiers are abnormal, her prior about the amount of defiers is decreasing. The data-dependent approach for $\tilde{A}$ is to first update her belief for each possible F. The posterior of the defiers given the data distribution F will be the prior belief conditioning on the set $m^{max,d}(F) \geq m^d(s) \geq m^{min,d}(F)$.[8] Since the prior is decreasing, the minimal defier $m^d(s) = m^{min,d}(F)$ maximizes the posterior likelihood and the data-dependent construction is to focus on these posterior-likelihood-maximizing structures.

The construction of $\tilde{A}$ is similar to the methods (Masten and Poirier, 2021), and we add to the discussion by clarifying some subtle issues: first, the value of the deviation measure $m^{min,d}(F)$ can indicate the possibility of finding a non-refutable relaxed assumption. If there still exists an F_0 such that the infimum is taken over an empty set, then $m^{min,d}(F_0) = +\infty$ and there is no hope we can construct a non-refutable relaxed assumption using the $m^d(s)$ function. Second, when constructing $\tilde{A}$, the econometrician is also required to justify whether the relaxed assumption is structurally reasonable. In particular, the econometrician needs to justify the infimum quantity $m^{min,d}(F)$ is achieved by some structure s so that the set $\tilde{A}$ is well-defined. These two concerns for $\tilde{A}$ in Assumption 2.1 are addressed later in Section 2.4 and further discussed in Section 3.

2.3.2. *The Minimal Marginal Dependence Extension.* Testable implications (2.10) also arise due to the failure of the independent IV assumption, which happens in many natural experiments. In this section, we provide another relaxed assumption that uses the deviation from the independent IV assumptions but keeps the "No Defiers" assumption.

[8]The upper bound $m^{max,d}(F) \equiv \sup\{m^d(s) : F \in M(G^s)$ *and* $s \in A^{ER} \cap A^{TI} \cap A^{EM-NTAT}\}$ can be less than 1.

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The independent IV assumption is an infinite-dimensional constraint on the G^s. As a result, there are infinitely many ways to relax the independent IV assumption, and they may result in different identified sets when the IA-M assumption is rejected. Here, we present a choice of the relaxation measure that will lead to a clean identified *LATE*.

We start with a measurement of the deviation from the independent IV assumption. By the M mapping defined in (2.4), the probability measures $Pr_{G^s}(Y_i(d_1,1)\in B_{d_1 1}, D_i(1)=d_1, D_i(0)=d_0|Z_i=1)$ and $Pr_{G^s}(Y_i(d_0,0)\in B_{d_0 1}, D_i(1)=d_1, D_i(0)=d_0|Z_i=0)$ are absolutely continuous with respect to some dominating measure μ_F.[9] We can therefore derive the Radon–Nikodym densities for $d_1,d_0\in\{0,1\}$

$$g^s_{y1}(y,d_1,d_0|Z_i=1)\equiv\frac{dPr_{G^s}(Y_i(d_1,1)\in B_{d_1 1}, D_i(1)=d_1, D_i(0)=d_0|Z_i=1)}{d\mu_F},$$
$$g^s_{y0}(y,d_1,d_0|Z_i=0)\equiv\frac{dPr_{G^s}(Y_i(d_0,0)\in B_{d_0 0}, D_i(1)=d_1, D_i(0)=d_0|Z_i=0)}{d\mu_F}.$$

The g^s_{y1} and g^s_{y0} are the marginal densities of potential outcomes under different instrument values. If the independent IV assumption holds, then the densities must be the same. We therefore consider the following deviation measure:

$$m^{MI}(s)=\sum_{d=0}^{1}\int_{\mathcal{Y}}\left[g^s_{y1}(y,d,d|Z_i=1)-g^s_{y0}(y,d,d|Z_i=0)\right]^2 d\mu_F(y).$$

The m^{MI} measures the L^2 distance of marginal distributions of the potential outcomes when the instrument Z_i takes different values. When $d=1$, $g^s_{yz}(y,1,1|Z_i=z)$ is the marginal density of $Y_i(1,1)=Y_i(1,0)$ and $D_i(1)=D_i(0)=1$ conditional on $Z_i=z$, which focuses on the always-takers. Similarly, when $d=0$, the density concerns the never-takers group. In this measurement, the defiers group is omitted because we maintain the "No Defiers" assumption. We also omit the marginal distribution of y for the compliers. This is because for $Z_i=z$, the observed Y_i only contains information about $Y_i(z,z)$ for compliers but not $Y_i(1-z,z)$, and the independent IV assumption does not imply a relationship between the marginal densities of $Y_i(1,1)$ and $Y_i(0,0)$ for compliers. In fact, as it will be shown later, we can always construct a G^s such that the conditional type independent IV condition holds for the compliers.

The measure m^{MI} also differs from the m^d in a significant way: when $m^{MI}(s)=0$, it does not mean that s must satisfy the independent IV assumption. However, the choice of $m^{MI}(s)$ is still plausible: note that for always-takers, only the $Y_i(1,1)=Y_i(1,0)$ matters because the untreated status never occurs for always-takers and we can never get information about $Y_i(0,1)=Y_i(0,0)$ for always-takers, so we omitted the potential outcomes $(Y_i(0,1),Y_i(0,0))$ in m^{MI}. A similar logic holds for never-takers.

[9] Note that we maintain the exclusion restriction $Y_i(d,z)=Y_i(d,1-z)$, so we can omit the $Y_i(d_1,0)$ and $Y_i(d_0,1)$.

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Before we proceed to the relaxed assumption, we also introduce the conditional type independence condition for the compliers:

$$A^{TI-CP} = \left\{ s \middle| G^s \text{ satisfies } Z_i \perp (Y_i(1,1), Y_i(0,1), Y_i(1,0), Y_i(0,0)) \, |D_i(1) = 1, D_i(0) = 0 \right\}.$$

We maintain this crucial assumption so that we can get an informative bound on the *LATE* quantity. This is because, for $Z_i = z$, Y_i only contains information about $Y_i(z,z)$ for compliers and we need some independence conditions across the instrument values to make sure that we can combine the outcomes under $Z_i = 1$ and $Z_i = 0$ to identify the *LATE*.

Assumption 2.2 (Minimal Marginal Dependent Instrument). Let $m^{min,MI}(F) \equiv \inf\{m^{MI}(s) : F \in M(G^s) \text{ and } s \in A^{ER} \cap A^{TI-CP} \cap A^{ND}\}$ be the minimal distance. We call

$$\tilde{A}' = \cup_{F \in \mathcal{F}} \left\{ s \in A^{ER} \cap A^{TI-CP} \cap A^{ND} : m^{MI}(s) = m^{min,MI}(F),\ F \in M(G^s) \right\}$$

the minimal marginal dependent instrument relaxed assumption.

Similar to $\tilde{A}$, the rationale behind $\tilde{A}'$ is also a data-dependent construction of assumptions. However, it should be noted that when $m^{min,MI}(F) = 0$, $\tilde{A}'$ does not equal the IA-M assumption because $m^{MI}(s) = 0$ is not an equivalent characterization of the independent instrument condition. Nonetheless, as we will see in the next section, $\tilde{A}'$ produces a clean identified *LATE* expression.

2.4. The Identified Set Under Assumption 2.1 or 2.2

Before we characterize the identified *LATE* quantity, we first show that $\tilde{A}$ in Assumption 2.1 and $\tilde{A}'$ in Assumption 2.2 are well-defined and non-refutable. In particular, we want to show that the minimal deviation functions ($m^{min,d}(F)$ and $m^{min,MI}(F)$) are finite so that the assumption is not refutable, and the minimal deviation values can be achieved by some econometric structures.

Proposition 2.1. *The $\tilde{A}$ in Assumption 2.1 and $\tilde{A}'$ in Assumption 2.2 are well-defined and non-refutable such that: (1) $m^{min,d}(F) < \infty$ and $m^{min,MI}(F) < \infty$ for all $F \in \mathcal{F}$; (2) for any F, there exist s_1 and s_2 such that $m^d(s_1) = m^{min,d}(F)$ and $m^{MI}(s_2) = m^{min,MI}(F)$.*

The second condition in Proposition 2.1 is non-trivial. If the $m^{min,d}(F)$ is finite but not achieved by some structure s, then $\left\{ s \in A^{ER} \cap A^{TI} \cap A^{EM-NTAT} : m^d(s) = m^{min,d}(F),\ F \in M(G^s) \right\} = \emptyset$ and the data-dependent construction of relaxed assumptions does not specify what to consider when F is observed, and $\tilde{A}$ becomes refutable.

We first set up some notations and regularity conditions to describe the identified *LATE*. We use the following notation to denote the identified set of *LATE* when F is the observed data distribution and A_0 is an imposed assumption:

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$$LATE^{ID}_{A_0}(F) = \left\{ E_{G^s}[Y_i(1,1) - Y_i(0,0)|D_i(1) = 1, D_i(0) = 0] \middle| M(G^s) = F, s \in A_0 \right\}.$$

In our later discussion of the relaxed assumption, it would be easier to use the Radon–Nikodym derivatives of $P(\cdot,d)$ and $Q(\cdot,d)$ with respect to the μ_F measure. Let μ_F be the common dominating measure in Lemma 2.1. Let $p(y,d)$ and $q(y,d)$ be defined as:

$$p(y,d) = \frac{dP(B,d)}{d\mu_F} \qquad q(y,d) = \frac{dQ(B,d)}{d\mu_F}. \tag{2.13}$$

The densities $p(y,d)$ and $q(y,d)$ will be used in our subsequent theorems and proofs. We first set up some notations that will simplify the notations in the identified *LATE* expression:

$$\mathcal{Y}_d = \left\{ y : (-1)^d (q(y,d) - p(y,d)) \geq 0 \right\}, \quad d \in \{0,1\},$$

which is the set of y such that when $D_i = d$, the testable implication (2.10), in the density form, is not violated.

Assumption 2.3. There exists a constant $c \geq 0$ such that: (i) $Pr_F(Z_i = 1) \in (c, 1-c)$; (ii) $Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0) > c$ and $P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1) > c$.

Assumption 2.3 is a regularity assumption. For the identification result, we only need Assumption 2.3 to hold with $c = 0$ so that *LATE* is well-defined. For inference purposes, we will further require $c > 0$ to avoid the weak instrument issue.

THEOREM 1. *Suppose Assumption 2.3 holds. The minimal defiers probability is identified as:*

$$m^{min,d}(F) = (Q(\mathcal{Y}_1^c, 1) - P(\mathcal{Y}_1^c, 1))P_F(Z_i = 0) + (P(\mathcal{Y}_0^c, 0) - Q(\mathcal{Y}_0^c, 0))P_F(Z_i = 1),$$

and the minimal marginal dependence distance is identified as:

$$m^{min,MI}(F) = \int \max\{q(y,1) - p(y,1), 0\}^2 + \max\{p(y,0) - q(y,0), 0\}^2 d\mu_F(y).$$

The identified $LATE^{ID}_{\tilde{A}}$ and $LATE^{ID}_{\tilde{A}'}$ satisfy

$$\begin{aligned} LATE^{ID}_{\tilde{A}}(F) &= LATE^{ID}_{\tilde{A}'}(F) \\ &= \frac{\int_{\mathcal{Y}_1} y(p(y,1) - q(y,1)) d\mu_F(y)}{P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1)} - \frac{\int_{\mathcal{Y}_0} y(q(y,0) - p(y,0)) d\mu_F(y)}{Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)}. \end{aligned} \tag{2.14}$$

There are several notable implications from Theorem 1: first, whether we use $\tilde{A}$ or $\tilde{A}'$ as the relaxed assumption, we have the same identified *LATE* quantity for all F; second, the identified *LATE* coincide with the Wald ratio expression (2.8) whenever the testable implication (2.10) holds. When (2.10) is violated, the expression (2.14) is the *LATE* for compliers under the model that selects the minimal amount of defiers or minimal marginal dependence; and third, the *LATE* quantity is always point-identified for all F.

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The equality of $LATE^{ID}_{\tilde{A}}(F)$ and $LATE^{ID}_{\tilde{A}'}(F)$ is not a coincidence. For each F, the s_1 and s_2 that achieve the minimal deviations in Proposition 2.1 have the same conditional distribution of $Y_i(z,z)|Z_i = z$ for compliers. Also, both assumptions require the conditional type independence condition for compliers. As a result, the identified *LATE* is the same.

The second implication is due to the feature of our construction of $m^d(s)$. When F satisfies the testable implication (2.10), the minimal defier amount $m^{min,d}(F) = 0$. Note that $m^d(s) = 0$ is an equivalent characterization of the A^{ND} assumption, and Lemma 2.2 implies that $m^d(s) = 0$ will deliver the original IA-M assumption. As a result, $LATE^{ID}_{\tilde{A}}(F)$ and the Wald ratio coincide for F that satisfies (2.10).

The point identification result comes from the relationship of the minimal defiers amount and the marginal distribution of $Y_i(z,z)|Z_i = z$ for compliers. Whenever F violates (2.10), we must add defiers to the model. Since we want to minimize the probability of defiers, the priority is to deal with the values of y that $(-1)^d(q(y,d) - p(y,d)) < 0$, i.e., the values of y where (2.10) is violated locally. Moreover, there is a unique way to make such an adjustment. Since the densities of Y for compliers and defiers are related through the potential outcome equation (2.1),[10] adjusting the density of Y for defiers in a minimal way leads to the unique adjustment in the density of Y for compliers. Since the *LATE* is a function of the conditional densities of Y for compliers, *LATE* is point-identified. Similar logic holds for the $\tilde{A}'$ assumption.

The identified quantity $LATE^{ID}_{\tilde{A}}(F)$ coincides with the *LATE* quantity in Dahl et al. (2023). However, our rationale leading to the $LATE^{ID}_{\tilde{A}}(F)$ is quite different from that in Dahl et al. (2023). In our setting, the econometrician wants to minimize the deviation from the IA-M assumption because she believes defiers are irregular. In contrast, Dahl et al. (2023) directly assume that given a value of potential outcomes, compliers and defiers cannot coexist, which is called the local monotonicity assumption. This assumption lacks a clear rationale to justify why the compliers and defiers cannot co-exist.

We also want to stress the importance of the conditional type independence assumption A^{TI}, while Dahl et al. (2023) maintain the A^{IV} assumption. Our identification theorem also illustrates why conditional type independence assumption is crucial. In (2.14), the denominator of the first term, $P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1)$, is the probability of compliers given $Z_i = 1$, and $Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)$ is the probability of compliers given $Z_i = 0$. These two quantities are not guaranteed to coincide, and the independent IV assumption fails. The conditional type independence assumption echoes the result in Kitagawa (2021), where he shows that the independent instrument assumption is testable.

[10] Suppose we set a density for defiers $h_d(y) \equiv g(Y_i(1,0) = y, D_i(1) = 0, D_i(0) = 1|Z_i = 0)$, then the potential outcome equation implies that the always-takers will have a density $g(Y_i(1,0) = y, D_i(1) = D_i(0) = 1|Z_i = 0) = q(y,0) - h_d(y)$. Since we impose $A^{TI} \cap A^{EM-NTAT}$, we can change the instrument value for always-takers, which leads to $g(Y_i(1,0) = y, D_i(1) = D_i(0) = 1|Z_i = 1) = q(y,0) - h_d(y)$. Again, we use the potential outcome to derive that the compliers should have a density of $p(y,1) - (q(y,1) - h_d(y))$.

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2.5. Estimation and Inference

The estimator in Dahl et al. (2023) is based on numerical integration of estimated density, which can be hard to implement. In this section, we propose an estimator of $LATE^{ID}_{\tilde{A}}(F)$ that only requires a sample average with appropriate trimming and the computation is easier.

The identification results in Theorem 1 rely on the sets $\mathcal{Y}_1, \mathcal{Y}_0$. Throughout this section, we focus on the estimation and inference problem when Y_i is continuously distributed on $\mathbb{R}$, and μ_F is the Lebesgue measure.

Assumption 2.4. Y_i is continuously distributed with unbounded support and the measure $P(B,d)$, $Q(B,d)$ is absolutely continuous with respect to the Lebesgue measure.

To estimate $\mathcal{Y}_0, \mathcal{Y}_1$, we first estimate the density $p(y,d)$ and $q(y,d)$ using kernel density estimators with a bandwidth sequence h_n:

$$f_h(y,1) = \frac{\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{Y_i - y}{h_n}\right)\mathbb{1}(D_j = 1, Z_j = 1)}{\sum_{i=1}^n \mathbb{1}(Z_j = 1)} - \frac{\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{Y_i - y}{h_n}\right)\mathbb{1}(D_j = 1, Z_j = 0)}{\sum_{i=1}^n \mathbb{1}(Z_j = 0)},$$
$$f_h(y,0) = \frac{\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{Y_i - y}{h_n}\right)\mathbb{1}(D_j = 0, Z_j = 0)}{\sum_{i=1}^n \mathbb{1}(Z_j = 0)} - \frac{\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{Y_i - y}{h_n}\right)\mathbb{1}(D_j = 0, Z_j = 1)}{\sum_{i=1}^n \mathbb{1}(Z_j = 1)}. \quad \textbf{(2.15)}$$

Assumption 2.5. There exist constants M_l and M_u such that for $d = 0, 1$ such that $\mathcal{Y}_d \cap [M_u, \infty) \in \{\varnothing, [M_u, \infty)\}$ and $\mathcal{Y}_d \cap (-\infty, M_l] \in \{\varnothing, (-\infty, M_l]\}$. Moreover, we know $\mathcal{Y}_d \cap [M_u, \infty)$ and $\mathcal{Y}_d \cap (-\infty, M_l]$.

Assumption 2.5 requires that the sign of $p(y,d) - q(y,d)$ is known and fixed for large values of y. As a result, we only need to estimate the set $\mathcal{Y}_d \cap [M_l, M_u]$.[11] By controlling the tail behavior of the densities, we can avoid the ill behaviors in the density estimation and maintain a good property for the *LATE* estimator. Define the upper tail set $\mathcal{Y}_d^{ut} = \mathcal{Y}_d \cap [M_u, \infty)$, the lower tail set $\mathcal{Y}_d^{lt} = \mathcal{Y}_d \cap (-\infty, M_l]$, and we estimate $\mathcal{Y}_1, \mathcal{Y}_0$ by

$$\hat{\mathcal{Y}}_d(b_n) = \{y \in (M_l, M_u) : f_h(y,d) \geq b_n\} \cup \mathcal{Y}_d^{ut} \cup \mathcal{Y}_d^{lt},$$

where b_n is a sequence of positive constants that converge to zero. The estimated set above only uses density $f_h(y,d)$ to distinguish whether $y \in \mathcal{Y}_d$ in the range (M_l, M_u) and uses the known tail sign for large value of Y_i in Assumption 2.5 directly. For the $\tilde{A}$ or the $\tilde{A}'$ relaxed assumptions in Theorem 1, we use the sample analog to construct an estimator of (2.14) as:

[11] For example, suppose $p(y,d)$ and $q(y,d)$ have Gaussian tails: $p(y,d) = C_p e^{-y^2/\sigma_p(d)^2}$ and $q(y,d) = C_q e^{-y^2/\sigma_q(d)^2}$ for $|y| > C^{tail} > 0$, where C^{tail} defines the tail regions. If $\sigma_p(1) > \sigma_q(1)$, then $\mathcal{Y}_1 \cap [C^{tail}, \infty) = [C^{tail}, \infty)$ and $\mathcal{Y}_1 \cap (-\infty, -C^{tail}] = (-\infty, -C^{tail}]$.

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$$\widehat{LATE} = \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i \left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} \right] \mathbb{1}(Y_i \in \hat{\mathcal{Y}}_1(b_n))}{\frac{1}{n}\sum_{i=1}^{n} \left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} \right] \mathbb{1}(Y_i \in \hat{\mathcal{Y}}_1(b_n))} - \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i \left[\frac{\mathbb{1}(D_i=0,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} - \frac{\mathbb{1}(D_i=0,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} \right] \mathbb{1}(Y_i \in \hat{\mathcal{Y}}_0(b_n))}{\frac{1}{n}\sum_{i=1}^{n} \left[\frac{\mathbb{1}(D_i=0,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} - \frac{\mathbb{1}(D_i=0,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} \right] \mathbb{1}(Y_i \in \hat{\mathcal{Y}}_0(b_n))}. \tag{2.16}$$

Limit Distribution of $\widehat{LATE}$. We present the limit distribution of $\widehat{LATE}$ defined in (2.16). The following assumptions are sufficient to guarantee the $\widehat{LATE}$ in (2.16) will converge to a normal distribution.

Assumption 2.6. The kernel function K satisfies: (i) $K(u)$ is continuous and supported on $[-C_k, C_k]$ and $\int_u K(u)du = 1$; (ii) $\int_u uK(u)du = 0$; and (iii) $\int u^2 K(u)du < \infty$.

Assumption 2.7. The conditional distributions $F(y|D_i = d, Z_i = z)$ have densities $f(y|d,z)$ for all $d,z \in \{0,1\}$, and $f''(y|d,z)$ exist and are uniformly bounded above by a constant c_f; (iii) $E(Y_i^{2+\delta}) < \infty$ for some $\delta > 0$.

The above two assumptions are standard in the nonparametric density estimation literature and guarantee that the density difference estimators $f_h(y,d)$ will converge uniformly in probability to its limit $(-1)^{1-d}(p(y,d) - q(y,d))$ at a polynomial rate.

Assumption 2.8. Let $f(y,1) = p(y,1) - q(y,1)$ and $f(y,0) = q(y,0) - p(y,0)$. The following condition holds for any sequence $b_n \to 0_+$: $\int_{M_l}^{M_u} |f(y,d)|\mathbb{1}(-b_n \leq f(y,d) \leq b_n)dy = O(b_n^2)$.

Assumption 2.8 controls the bias from trimming $\{y \in [M_l, M_u] : 0 < f(y,d) < b_n\}$ and removes the asymptotic bias and sampling variation in the kernel estimator $f_h(y,d)$. Assumption 2.8 can be replaced by a sufficient primitive condition.

Assumption 2.9. Let $M_0 < \infty$ be a positive integer. For $d = 0,1$, the set $\mathcal{C}_d = \{y : f(y,d) = 0,\ y \in [M_l, M_u]\}$ has at most M_0 points. Let $B(\mathcal{C}_d, \delta) = \cup_{y \in \mathcal{C}_d} B(y,\delta)$ be the δ-neighborhood of $\mathcal{C}_d$ for $d = 0,1$. For both $d = 0,1$, we have $\sup_{y \in B(\mathcal{C}_d,\delta)} |\frac{d(f(y,d))}{dy}| > 1/C$ for some $C, \delta > 0$.

LEMMA 2.3. *Assumption 2.9 implies Assumption 2.8.*

Essentially, in Assumption 2.9, we rule out all data distributions whose $\{y : f(y,d) = 0\}$ have a positive measure, or whose density difference slops near the zero points of $f(y,d) = 0$ are too flat.

THEOREM 2. *Let $\widehat{LATE}$ be defined in (2.16) and $LATE_{\tilde{A}}^{ID}(F)$ be defined in (2.14). Suppose Assumption 2.3 holds for $c > 0$ and Assumptions 2.4–2.8 hold. Let $b_n \asymp$*

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$n^{-1/4}/\log n$ and $h_n \asymp n^{-1/5}$, then $\sqrt{n}(\widehat{LATE} - LATE^{ID}_{\tilde{A}}(F)) \to_d N(0, \Pi'\Gamma\Sigma\Gamma'\Pi)$, where

$$\Sigma = Var\begin{pmatrix} \mathbb{1}(Z_i=0) \\ \mathbb{1}(Z_i=1) \\ Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1) \\ Y_i\mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1) \\ Y_i\mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0) \\ Y_i\mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0) \\ \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1) \\ \mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1) \\ \mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0) \\ \mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0) \end{pmatrix},$$

and matrices Π and Γ are specified as by

$$\Pi = \begin{pmatrix} \frac{1}{\pi_3} \\ -\frac{1}{\pi_4} \\ -\frac{\pi_1}{\pi_3^2} \\ \frac{\pi_2}{\pi_4^2} \end{pmatrix}, \quad \pi \equiv \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} \int_{\mathcal{Y}_1} y(p(y,1)-q(y,1))dy \\ \int_{\mathcal{Y}_0} y(q(y,0)-p(y,0))dy \\ \int_{\mathcal{Y}_1} (p(y,1)-q(y,1))dy \\ \int_{\mathcal{Y}_0} (q(y,0)-p(y,0))dy \end{pmatrix},$$

$$\Gamma = \frac{1}{Pr(Z_i=1)Pr(Z_i=0)}\begin{pmatrix} \Gamma_1 & \Gamma_3 & \mathbf{0}_{2\times 4} \\ \Gamma_2 & \mathbf{0}_{2\times 4} & \Gamma_3 \end{pmatrix},$$

where

$$\Gamma_1 = \begin{pmatrix} E[Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1)] & -E[Y_i\mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1)] \\ -E[Y_i\mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0)] & E[Y_i\mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0)] \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} E[\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1)] & -E[\mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1)] \\ -E[\mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0)] & E[\mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0)] \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} Pr(Z_i=0) & -Pr(Z_i=1) & 0 & 0 \\ 0 & 0 & Pr(Z_i=1) & -Pr(Z_i=0) \end{pmatrix}.$$

COROLLARY 2.1. *Let $(\hat{\Gamma}, \hat{\Pi}, \hat{\Sigma}) \to_p (\Gamma, \Pi, \Sigma)$, and let $\hat{\sigma} = \sqrt{\hat{\Pi}'\hat{\Gamma}\hat{\Sigma}\hat{\Gamma}'\hat{\Pi}}$. Then the set*

$$\left[\widehat{LATE} - \frac{\hat{\sigma}}{\sqrt{n}}\Phi(\alpha/2), \widehat{LATE} + \frac{\hat{\sigma}}{\sqrt{n}}\Phi(1-\alpha/2)\right] \tag{2.17}$$

is a valid α-confidence interval for $LATE^{ID}_{\tilde{A}}(F)$, where Φ is the standard normal CDF function.

Theorem 2 shows that the $\widehat{LATE}$ in (2.14) is $\sqrt{n}$ consistent. Once the matrices Π, Γ, and Σ are estimated by consistent estimators, we can test hypotheses such as $H_0 : LATE^{ID}_{\tilde{A}}(F) = 0$. However, Assumption 2.5 requires the econometrician to know the sign of tail behavior of $p(y,1) - q(y,1)$ and $q(y,0) - p(y,0)$. In some

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empirical applications, we may want to be agnostic about tail signs or only impose less restrictive conditions on tail signs. In this case, we can calculate the confidence interval for each possible tail condition, and then take the union, but this confidence interval will be conservative.

2.6. An Empirical Illustration

In this section, we apply our results in Theorems 1 and 2 to Card (1993), who studies the causal effect of college attendance on earnings. In this application, the outcome variable Y_i is an individual i's log wage in 1976, $D_i = 1$ means individual i attended a four-year college, and $Z_i = 1$ means the individual was born near a four-year college. This data set has been used by both Kitagawa (2015) and Mourifié and Wan (2017) to test the IA-M assumption, and they both reject the IA-M assumption. If a child grows up near a college, he or she may hear more stories of heavy tuition burdens, which may discourage him or her from attending college. On the other hand, if this child grew up far away from a college, he or she may instead choose to attend college. Therefore, we would expect defiers to exist in this empirical setting. Moreover, it is unclear why this instrument is fully independent of the potential income, since the choice of residence may depend on parents' potential income, which may be correlated with their children's income.

We conditioned (Y_i, D_i, Z_i) on three characteristics: living in the south (S/NS), living in a metropolitan area (M/NM), and being an African-American ethnic group (B/NB). We follow Mourifié and Wan (2017) to exclude the subgroup NS/NM/B due to the small sample size. We also exclude subgroup NS/M/B due to the high frequency of $Z = 1$. We conduct estimation and inference on each of the remaining six subgroups and the pooled African-American group. The choices of trimming sequence b_n, kernel bandwidth h_n, upper and lower bands M_u, M_l, and tail set $\mathcal{Y}_d^{ut}, \mathcal{Y}_d^{lt}$ are available on request.

Estimation results are reported in Table 1. We also report the *LATE* estimates when we directly use the IA-M assumption as the Wald statistics. The estimated measure of compliers under $\tilde{A}$ or $\tilde{A}'$ conditioned on $Z_i = 1$ and $Z_i = 0$ are, respectively, reported as $P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1)$ and $Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)$, while the estimated measure of compliers under IA-M assumption is $E[D_i|Z_i = 1] - E[D_i|Z_i = 0]$. The estimates of $LATE_{\tilde{A}}^{ID}$ and $LATE^{Wald}$ differ the most for three groups: S/NM/NB, S/M/NB, and S/M/B. It should be noted that for all these three groups, estimated $E[D_i|Z_i = 1] - E[D_i|Z_i = 0]$ differs from $P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1)$ and $Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)$. If we blindly use the identification result under the IA-M assumption and use the standard $LATE^{Wald}$ estimator, the "identified" *LATE* can be negative (subgroups S/NM/NB and S/M/NB), or be unrealistically large (subgroup S/M/B). Once we use $\tilde{A}$ or $\tilde{A}'$, the estimated *LATE* for each of the six subgroups is positive, and the values of *LATE* are all between zero and one. When we look at the African-American group only, while $LATE^{Wald}$ is large, it fails to reject the hypothesis that education is harmful to their earnings, while our method will reject the hypothesis that *LATE* for the African-American is negative at a significance level of 0.025.

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TABLE 1. Estimation result under extensions Assumption 2.1

Group	NS,NM,NB	NS,M,NB	S,NM,NB	S,NM,B	S,M,NB	S,M,B	B-Group only
$Pr(Z_i = 1)$	0.464	0.879	0.349	0.322	0.608	0.802	0.6188
Observations	429	1191	307	314	380	246	703
$LATE^{ID}_{\tilde{A}}(F)$	0.5599	0.1546	0.2524	0.4773	0.5276	0.4358	1.0993
CI for $LATE^{ID}_{\tilde{A}}(F)$	[0.01, 1.11]	[−0.51, 0.82]	[−1.22, 1.73]	[−0.09,1.04]	[−2.54, 3.59]	[−5.15, 6.02]	[0.58, 1.62]
$P(\mathcal{Y}_1, 1) - Q(\mathcal{Y}_1, 1)$	0.1120	0.1084	0.0265	0.0739	0.0164	0.0338	0.0375
$Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)$	0.1148	0.0960	0.0684	0.1495	0.0922	0.0308	0.1583
$LATE^{Wald}(F)$	0.5976	0.0761	−6.4251	1.1873	−1.5412	17.9620	5.0499
CI for $LATE^{Wald}(F)$	[−0.20 1.39]	[−1.24, 1.39]	[−105,92]	[−0.53,2.90]	[−5.09,2.01]	[−1.7e4,1.7e4]	[−6.16,16.26]
$E[D_i \| Z_i = 1]$ $-E[D_i \| Z_i = 0]$	0.1080	0.1084	−0.0070	0.0692	−0.0697	0.0002	0.0317

3. A GENERAL THEORY

In the previous section, the construction of relaxed assumptions via minimal defiers or minimal marginal dependence demonstrates several advantages: the relaxed assumptions are non-refutable, and the identified *LATE* coincides with the Wald ratio whenever the original IA-M assumption is not rejected. We aim to develop a general theory for extending refutable assumptions using the insights from the *LATE* application. In the discussion, we shall explore the general conditions to ensure we have a well-behaved relaxed assumption, and we also discuss some subtle issues when constructing a relaxed assumption.

Besides the *LATE* example in Section 2, we also consider an additional moment inequality model that contributes to the illustration of the general theory.

Example 1. Suppose the econometrician has access to the outcome Y_i, a binary treatment D_i, and control variables Z_i as data. Each individual i also bears an unobserved heterogeneity η_i. The outcome Y_i is related to (Z_i, D_i, η_i) via $Y_i = f^*(Z_i, D_i, \eta_i)$ for some unknown function f^*. The econometrician imposes two assumptions that facilitate the discussion.

1. A_1: The model is additively separable, $Y_i = f(Z_i, D_i; \kappa) + \eta_i$ for some known integrable function f and unknown finite dimensional parameter κ.
2. A_2: The unobserved heterogeneity has a negative conditional mean $-\infty < E[\eta_i|D_i] \leq 0$.

Two moment inequalities $E[Y_i - f(Z_i, 1; \kappa)|D_i = 1] \leq 0$ and $E[Y_i - f(Z_i, 0; \kappa)|D_i = 1] \leq 0$ can be derived from $A \equiv A_1 \cap A_2$.

3.1. Definitions

In economic models, there are typically two types of variables: the variables that are observed, denoted by X_i, which will be used for estimation, and the primitive variables, denoted by ϵ_i, that are used to generate the observed variables. In the moment inequality examples, $X_i = (Y_i, D_i, Z_i)$, and $\epsilon_i = (D_i, Z_i, \eta_i)$. The primitive variables are either determined outside the model, or their determinants are not the focus of the model.[12]

Before collecting data, the econometrician first has an idea about the spaces of X and ϵ that she will work with, and denote the spaces as $\mathcal{X}$ and Υ, respectively. We correspondingly define the observation space and the primitive distribution space.

DEFINITION 3.1. *The data distribution space $\mathcal{F}$ is the collection of all possible distributions $F(X)$ whose support is a subset of $\mathcal{X}$. The primitive distribution space $\mathcal{G}$ is the collection of all possible distributions $G(\epsilon)$ whose support is a subset of Υ.*

[12] For example, in the moment inequality model, D_i may be determined by Z_i, but we don't focus on the study of how D_i is related to Z_i.

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Note that we do not specify $\mathcal{X}$ as the support of X_i because the econometrician does not know F before data are collected. For example, if Y_i is an employee's income from an unknown firm, then a reasonable space for Y_i is the $[0,\infty)$, but the collected data distribution F may be supported on some bounded interval. However, since D_i is a known binary treatment, we only need to consider its space as $\{0,1\}$. For both the definitions of $\mathcal{F}$ and $\mathcal{G}$, econometricians can impose additional regularity conditions on $\mathcal{F}$ and $\mathcal{G}$: the regularity conditions may be relevant for the existence of moments of ϵ_i and X_i, or the existence of densities of the distributions. In the two-moment-inequality example, we may only consider a G^s such that $E_{G^s}[|\eta_i||D_i] < \infty$. Thinking of $\mathcal{F}$ and $\mathcal{G}$ in a general way rather than specifying ex-ante support would help the econometrician to lay out the data-dependent construction of relaxed assumptions later.

However, the observed data distribution F is not the ultimate goal for an econometric interpretation. Instead, we are interested in the features of the distribution of the primitive variables, and/or the mechanism of how ϵ_i determines X_i. Therefore, we follow the languages in Koopmans and Reiersol (1950) and Jovanovic (1989) to call a pair of a distribution of primitive variables and a mechanism as an econometric structure.

DEFINITION 3.2. *An econometric structure $s = (G^s, M^s)$ consists of a distribution G^s of ϵ, and an outcome mapping $M^s : \mathcal{G} \to \mathcal{F}$.*

Since M^s is a function, we also slightly abuse the notation to use $M^s(G^s)$ to denote the singleton set $\{F\}$.[13] In the *LATE* example, the mapping M^s is characterized by (2.1) which is the same across all structures, and we are mainly interested in G^s, the distribution of the potential outcomes. In contrast, the mapping M^s in the two-moment-inequality example can be summarized by the κ parameter, and we are more interested in κ.[14] The econometrician also specifies a paradigm for studying the model, which we call the structure universe.

DEFINITION 3.3. *A structure universe $\mathcal{S}$ is a collection of structures such that $\cup_{s\in\mathcal{S}} M^s(G^s) = \mathcal{F}$, and an assumption A is a subset of $\mathcal{S}$.*

Here, we explicitly distinguish the structure universe $\mathcal{S}$ and an assumption A, though either is a collection of structures. The structure universe $\mathcal{S}$ is the paradigm that can span different empirical contexts. On the other hand, an assumption A places constraints that are suitable for a particular empirical context, or convenient for empirical analysis. In the moment inequalities example, the structure universe considers a general class of functions without imposing any constraints on the functional form or the distributional relationship between D_i and η_i. Then A_1 imposes the functional form restriction and A_2 imposes distributional restrictions.

[13] We use such a notation in the *LATE* application, equation (2.4).

[14] Suppose $s = (G^s, \kappa)$, then $M^s(G^s) = \{F \in \mathcal{F} \big| F$ is the push-forward measure of G^s under $Y_i = f(Z_i, D_i, \eta_i; \kappa)\}$.

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The condition $\cup_{s\in\mathcal{S}} M^s(G^s) = \mathcal{F}$ ensures that the structure universe $\mathcal{S}$ is neither too small nor too large for the analysis of possible data distributions. We conclude this section with the definition of refutable assumptions.

DEFINITION 3.4 (Breusch, 1986). *An assumption A is called refutable if there exists an $F \in \mathcal{F}$ such that $F \notin \cup_{s\in A} M^s(G^s)$.*

In other words, if we can find an F that cannot be generated by any econometric structures in A, then this F refutes A as a proper assumption. In the two-moment-inequality example, if we cannot find an κ such that both inequalities hold simultaneously under F, then F refutes $A_1 \cap A_2$.

3.2. The Identification Problem

In many empirical studies, we want to find the value of a parameter of interest rather than a class of structures that are consistent with data. This parameter can be a moment of unobserved primitive variables or a counterfactual prediction of the structure.[15] We therefore present a formal definition of the identification of the parameter of interest.

DEFINITION 3.5. *A parameter of interest θ is a function $\theta : \mathcal{S} \to \Theta$, where Θ is the parameter space. The identified set for θ is a correspondence $\Theta_A^{ID} : \mathcal{F} \rightrightarrows \Theta$ such that*

$$\Theta_A^{ID}(F) = \{\theta(s) : s \in A \quad \text{and} \quad F \in M^s(G^s)\}. \tag{3.1}$$

Here, we define the identified set as a correspondence of F rather than the usual definition of a subset of the parameter space. Once we have access to the data distribution, we can derive the usual sense of the identified set as $\Theta_A^{ID}(F)$. There are two advantages when we think of the identified set as a correspondence rather than as a realized set.

First, we can connect the assumption refutation issue to the image of the identified set correspondence. Whenever an assumption is refuted by an observed data distribution F, Definition 3.5 leads to an empty set under F. The converse also holds: in the two-moment-inequality example, if for some F we cannot find a κ that simultaneously satisfies the two-moment inequalities, then the assumption $A_1 \cap A_2$ is refuted by F. This also warns us that the identification result under a refutable assumption cannot be naively used. For example, the identified set for *LATE* should be $\emptyset$ rather than the Wald ratio whenever F violates the testable implication. In contrast, whenever we use a non-refutable assumption $\tilde{A}$, the identified set will be

[15] In the two-moment-inequality model, suppose $\kappa = (\kappa_1, ..., \kappa_{d_\kappa})$, we may be interested in the counterfactual mean of Y_i when κ_1 is set to zero. The parameter of interest can be written as $\theta(s) = E_{G^s}[f(X_i, D_i; \tilde{\kappa}) + \eta_i]$, where $\tilde{\kappa}_j = \kappa_j$ for all $j > 1$ and $\tilde{\kappa}_1 = 0$.

non-empty for any $\mathcal{F}$, because we can find an $s \in \tilde{A}$ that rationalizes F, and $\theta(s)$ will be in the identified set.

Second, viewing the identified set as a correspondence allows us to evaluate the appropriateness of the imposed assumption A. Let us equip $\mathcal{F}$ and Θ with metrics $d_{\mathcal{F}}$ and d_θ, respectively. We may question the appropriateness of an assumption A if the identified set $\Theta_A^{ID}(F)$ is not continuous. On one hand, it is hard to rationalize why the identified set will change abruptly when we perturb the data distribution slightly. On the other hand, because of the sampling error, estimation and inference can incur extra complications when the true data distribution F_0 is close to the discontinuity point. For example, when the parameter is point identified, then the discontinuity of the identified quantity and the consequential non-differentiability implies the impossibility of finding locally asymptotic unbiased estimators or regular estimators (Hirano and Porter, 2012).

3.3. Minimal Deviation Relaxed Assumption

While an assumption A may be refutable, we impose A in the first place because it reflects the economic theory suitable for the empirical context. Therefore, we would consider a departure from A as abnormal and against the economic intuition behind A. We relax our assumption to allow for a minimal departure from the baseline assumption A, which is called the minimal deviation method. Formally, suppose the refutable assumption A can be written as an intersection of several assumptions: $A = \cap_{j=1}^J A_j$. This representation allows us to consider a departure from a particular A_j.

DEFINITION 3.6. *Fix an index $j \in \{1, 2, ...J\}$. Suppose $\cap_{l \neq j} A_l$ is not refutable. A relaxation measure of departure from A_j with respect to $\{A_l\}_{l \neq j}$ is a function $m_j : \mathcal{S} \to \mathbb{R}_+ \cup \{+\infty\}$ such that $m_j(s) = 0$ for all $s \in A_j$. We say m_j is well-behaved if for any $F \in \mathcal{F}$, there exists a structure $s^* \in \cap_{l \neq j} A_l$ such that*

$$m_j(s^*) = \inf\{m_j(s) : F \in M^s(G^s) \quad \textit{and} \quad s \in \cap_{l \neq j} A_l\} \equiv m_j^{min}(F),$$

where $m_j^{min}(F)$ is the minimal deviation for the observed data distribution F.

In Definition 3.6, we require $\cap_{l \neq j} A_l$ to be non-refutable so that it is possible to construct a non-refutable relaxed assumption out of $\cap_{l \neq j} A_l$. When the relaxation is well-behaved, then for any $F \in \mathcal{F}$, we should be able to find an econometric structure s^* with finite deviation amount such that $F \in M^{s^*}(G^{s^*})$. This is essential for the construction of an extension using m_j.[16]

In the two-moment-inequality example, A_1 alone is not refutable because we do not put any restriction on the error η_i. For the second assumption A_2, we can

[16] An example of an ill-behaved relaxation measure is available upon request. Roughly speaking, it can happen when $\mathcal{S}$ is an infinite dimensional space. We can find a sequence of underlying distributions G_n where each of $G_n \in \mathcal{G}$, but the pointwise limit of G_n is not in $\mathcal{G}$.

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consider a deviation measure from the A_2 condition as $m_2(s) = E_{G^s}[\eta_i|D_i = 1]_+ + E_{G^s}[\eta_i|D_i = 0]_+$, where $x_+ = \max\{0, x\}$. This deviation measure can be shown to be well-behaved if κ lies in a compact set and $f(X_i, D_i; \kappa)$ is continuous and dominated by some integrable functions.[17]

Now we construct the minimal deviation relaxed assumption $\tilde{A}$ which generalizes the minimal defiers and minimal marginal dependent instrument relaxed assumptions in the *LATE* application.

DEFINITION 3.7. *Fix an index $j \in \{1, 2, ..., J\}$ and a well-behaved relaxation measure m_j. We call $\tilde{A} = \cup_{F \in \mathcal{F}} \left\{ s \in \cap_{l \neq j} A_l : m_j(s) = m_j^{min}(F),\ F \in M^s(G^s) \right\}$ the minimal deviation relaxed assumption of A under m_j.*

Similar to the relaxed assumptions in the *LATE* application, $\tilde{A}$ is constructed via a data-dependent method. In our parallel paper (Liao, 2024), we show that the minimal deviation method is equivalent to a robust Bayesian method which specifies a certain class of priors. We now link the properties of the relaxation measure m_j to the properties of the identified set $\Theta^{ID}_{\tilde{A}}$ under $\tilde{A}$.

PROPOSITION 3.1. *Suppose m_j is a well-behaved relaxation measure with respect to $\{A_l\}_{l \neq j}$.*

1. *If m_j is a sharp characterization of A_j under $\cap_{l \neq j} A_l$, i.e.,*

   $\cap_{j=1}^J A_j = (\cap_{l \neq j} A_l) \cap \{s : m_j(s) = 0\},$

   then for any parameter of interest, $\tilde{A}$ preserves the identified set whenever F does not reject A, i.e.,

   $\Theta^{ID}_{\tilde{A}}(F) = \Theta^{ID}_{A}(F),\ \text{if } F \in \cup_{s \in A} M^s(G^s).$

2. *If $m_j(s) = 0$ does not induce a partition on the predictions of A_j under $\cap_{l \neq j} A_l$, i.e.,*

   $\left(\cup_{s: s \in A} M^s(G^s) \right) \cap \left(\cup_{s: s \in \tilde{A} \setminus A,\, m_j(s) = 0} M^s(G^s) \right) \neq \emptyset,$

   then there exists an F and a parameter of interest $\tilde{\theta}$ such that $\tilde{\Theta}^{ID}_{\tilde{A}}(F) \neq \tilde{\Theta}^{ID}_{A}(F)$.

Moreover, there always exists a well-behaved m_j that sharply characterizes A_j under $\cap_{l \neq j} A_l$.

In the *LATE* example, $m^d(s)$ is a sharp characterization of the "No Defiers" assumption while $m^{MI}(s)$ is not a sharp characterization of the independent IV

[17] For any θ^s in the parameter space, construct G^s as the distribution of $(D_i, Z_i, Y_i - f(Z_i, D_i; \kappa))$. This construction ensures that $s = (\kappa^s, G^s)$ generates F. Let s_n be a sequence of structures such that $m_2(s_n) \to m_2^{min}(F)$. Because κ lies in a compact space, let $k(n)$ be a subsequence such that $\kappa^{s_{k(n)}}$ converges to some $\tilde{\kappa}$. Then the dominated convergence theorem ensures the deviation measure at $\tilde{\kappa}$ is exactly $m_2^{min}(F)$.

assumption. Non-sharp characterization often arises when we fail to consider all features imposed by the assumption.[18]

The benefits of using an m_j that sharply characterizes A_j are remarkable: we get the preservation property for any parameter of interest. In different empirical settings, researchers' parameters of interest can differ. If an m_j that sharply characterizes A_j is found, researchers can use this m_j and the corresponding minimal deviation relaxed assumption $\tilde{A}$ across different empirical contexts to consider different parameters of interest. In the *LATE* application, it means that the minimal defiers relaxed assumption not only preserves the identified set for *LATE* but also for other parameters of interest such as *ATE*.

When we can find some econometric structure s' outside A but $m_j(s') = 0$ and s' can also make predictions as structures in A, then we fail to preserve the identified set when F does not refute A. An example is the m^{MI} in the *LATE* example: the $m^{MI}(s)$ does not induce a partition of predictions of A^{IV} under $A^{ND} \cap A^{ER}$, so we can find non-preserved parameters of interest,[19] because the $m^{MI}(s)$ is more narrowly focused on the properties of the *LATE* quantity.

3.4. Discussions

3.4.1. *Choosing Among Relaxed Assumptions.* The way to construct a relaxed assumption is not unique, and the econometrician has to decide which relaxed assumption to accommodate the empirical contexts. In this section, we discuss several subtle issues that result in multiple relaxed assumptions. Along the way, we also give some criteria for econometricians to choose the relaxed assumption.

The multiplicity issue arises when the econometrician starts to think about which of assumptions $A_1, ..., A_J$ to relax. We recommend that the econometrician considers the empirical appropriateness of each assumption and decides the choice of j. For the *LATE* example, if we are confident that the instrument is randomized and satisfies certain independence properties, we should seek to relax the "No Defiers" assumption in the first place.

After deciding the j, the econometrician needs to choose a relaxation measure m_j. We recommend that the econometrician considers an m_j that has an economic interpretation. We can then interpret the minimal deviation relaxed assumption as a model that has the least departure from the economic intuition behind A_j. For the *LATE* example, the measure $m^d(s)$ can be interpreted as the probability of

[18] In the moment inequalities example, $m_2(s)$ is a sharp characterization of A_2. However, we can also construct a non-sharp characterization $\tilde{m}_2(s) = E_{G^s}[\eta_i | D_i = 1]_+$. In this case, $\tilde{m}_2(s)$ only considers relaxing the first moment inequality.

[19] Consider the indirect effect of instrument on ATE: $\tilde{\theta}(s) = E[Y_i(1) - Y_i(0) | Z_i = 1] - E[Y_i(1) - Y_i(0) | Z_i = 0]$. Whenever the IA-M assumption A is not rejected by F, the identified set for $\tilde{\theta}$ is $\tilde{\Theta}^{ID}_A(F) = \{0\}$ because we maintain the independent IV assumption. However, if we use the extension $\tilde{A}'$ in Assumption 2.2, the identified set is not a singleton under F because $m^{MI}(s) = 0$ does not imply an independent instrument.

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defiers in the structure s. After constructing $\tilde{A}$ using m_j, we also recommend that the econometrician checks whether the identified set is continuous in the observed data distribution F. In Appendix B, we further discuss the high-level conditions to ensure the continuity of the identified set constructed in Definition 3.7.

The multiplicity of the relaxed assumption can also arise from the multiplicity of the representation of the original assumption. Suppose we have two ways to represent the original assumption: $A = A_j \cap (\cap_{l \neq j} A_l) = A_j \cap (\cap_{l \neq j} A_l')$. Fix a j, even if we use the same measure m_j, since the minimal deviation is defined with respect to $\{A_l\}_{l \neq j}$, the relaxed assumption can differ when using a different representation. In our *LATE* example, we can either use the A^{IV} or the $A^{TI} \cap A^{EM-NTAT}$. When we deviate from the "No Defiers" assumption, the meaning of $m^d(s) > 0$ under $A^{IV} \cap A^{ER}$ is different from its meaning under $A^{TI} \cap A^{EM-NTAT} \cap A^{ER}$. In the *LATE* application, we choose the alternative representation because $A^{IV} \cap A^{ER}$ can be refutable. When it is necessary to change the representation of the A assumption as in the *LATE* example, we recommend that the econometrician maintains the economic interpretation of the alternative representation.

3.4.2. *Using Non-Refutable Assumptions.* Popper's falsification criterion (Popper, 2005) requires that a scientific theory should be empirically testable and falsifiable. The minimal deviation relaxed assumption $\tilde{A}$ is non-refutable and one may worry that using an $\tilde{A}$ may fail Popper's falsification criterion. We now clarify the difference between using a non-refutable assumption and the requirement of falsifiability of a scientific theory.

A non-refutable assumption $\tilde{A}$ can be viewed as a collection of scientific theories with different model parameters, and each economic structure in $\tilde{A}$ is still empirically falsifiable because each $s \in \tilde{A}$ only predicts a unique data distribution. When the econometrician obtains a data distribution F, she picks out a subset of falsifiable economic structures that predicts F, and the rest of the economic structures in $\tilde{A}$ are refuted.

Moreover, non-refutable assumptions are widely imposed in econometrics to best utilize the available data. For example, in a nonparametric regression $Y_i = h(X_i) + \epsilon_i$, the conditional mean zero assumption $E[\epsilon_i | X_i] = 0$ is not refutable; In a simple potential outcome framework (Rubin, 1974) with $Y_i(1), Y_i(0)$ and $Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$, the independent IV assumption $D_i \perp Y_i(1), Y_i(0)$ cannot be refuted by any data distribution, if we only observe (Y_i, D_i).

3.4.3. *Incomplete Models.* In both the *LATE* and the moment inequalities model, the image of mapping M^s for any s is a singleton, and there is no ambiguity in the predicted observed data distribution. However, many economic models feature multiplicity and indeterminacy in the outcome variables: for a discrete game with multiple equilibria, there is ambiguity in the choice of the observed equilibrium (Bresnahan and Reiss, 1991; Tamer, 2003; Aradillas-Lopez and Tamer, 2008); For a binary discrete choice model, utilities of two choices may have a tie. If we do not specify how such a tie can be broken, and such a tie happens with a positive probability, the model is incomplete.[20]

In the incomplete model context, the mapping M^s is usually a multi-valued correspondence. Except for the necessity of a general definition of M^s and refutability, the analyses of the structure universe and identification problem remain unchanged. We can still follow the minimal deviation relaxed assumption approach to solve the refutable assumption issue.

However, one issue in incomplete models is quite different from that in complete models: in view of Proposition 3.1, in incomplete models, there may not exist any well-behaved m_j that sharply characterize A_j. As a result, we may not be able to find a non-refutable relaxed assumption $\tilde{A}$ that preserves the identified set whenever A is not refuted by F. To illustrate this issue, let us consider a binary choice model.

Example 2. There are two choices $j \in \{0,1\}$, and the utility of $j=0$ is normalized to zero, $U_0 = 0$. For choice $j=1$, we assume a decision maker i has a utility $U_1 = 1+\epsilon_i$, where ϵ_i is supported on $\{-1,0\}$. The econometrician has access to a sample of individual choices $X_i \in \{0,1\}$. The ϵ_i is the primitive variable.

We assume that decision makers choose $j=1$ if $U_1 > U_0$, choose $j=0$ if $U_0 > U_1$. We do not specify the decision makers' behavior when there is a tie $U_1 = U_0$. The decision makers' behavioral assumption implies a correspondence M from the distribution of ϵ_i to a set of distributions of X_i: for each G^s of ϵ_i, we have

$$M(G^s) = \{F : Pr_F(X_i = 1) \geq Pr_{G^s}(\epsilon_i = 0)\}.$$

Similar to the *LATE* example, we focus on the distribution of G^s since the mapping M is the same for all structures. Suppose the econometrician imposes an assumption: $A = \{G^s : Pr_{G^s}(\epsilon_i = 0) \geq 1/2\}$. This is an economic assumption imposing that the shock is more likely to realize a zero value. It is easy to see that any F such that $Pr_F(X_i = 1) < 1/2$ can refute A.

We now find a non-refutable relaxed assumption $\tilde{A}$. This $\tilde{A}$ must contain the distribution $G^{\tilde{s}}$ where $Pr_{G^{\tilde{s}}}(\epsilon_i = 0) = 0$. This is because, the observed data distribution $\tilde{F}$ such that $Pr_{\tilde{F}}(X_i = 0) = 1$ can only be generated by the $G^{\tilde{s}}$.

Consider a parameter of interest $\theta(G^s) = Pr_{G^s}(\epsilon_i = 0)$, and consider an F such that $Pr_F(X_i = 1) = 2/3$. This F does not refute the original assumption A, and the identified set under the original assumption is $\Theta^{ID}_A(F) = [1/2, 2/3]$. However, for any non-refutable assumption $\tilde{A}$, we must have $[1/2, 2/3] \cup \{0\} \subseteq \Theta^{ID}_{\tilde{A}}(F)$ because $G^{\tilde{s}} \in \tilde{A}$, and $G^{\tilde{s}}$ also predicts F. We lost the preservation property of the identified set when F does not refute A.

Example 2 shows that an incomplete model can have undesirable properties when we try to find a non-refutable relaxed assumption. A solution to incomplete models is to add a model completion which selects a unique outcome when there

[20] Specifying a tie-breaking rule, such as a random decision, is a way to complete the model.

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is ambiguous multiplicity. In the context of Example 2, we need to specify the way to select an outcome when a tie of utilities of two choices happens.

4. CONCLUSION

This article proposes a minimal deviation relaxation method for the identification of complete models with refutable assumptions. The relaxed assumption is constructed via a data-dependent method, which selects the econometric structures that minimize the deviation from the original assumption for each possible data distribution. We also discuss the properties of the identified set under the relaxed assumption and the criteria for choosing among different minimal deviation relaxed assumptions.

As a leading application of the minimal deviation relaxation method, we study the *LATE* model, where the "No Defiers" assumption, independent IV assumption, and exclusion restriction are jointly testable. We provide two minimal deviation relaxed assumptions: one focuses on relaxing the "No Defiers" assumption, while the other focuses on relaxing the independent IV assumption. We also emphasize the importance of using the conditional type independence assumption in these two relaxed assumptions.

We briefly discuss the issues that arise when extending the theory to incomplete models. For incomplete models, multiple distributions can be predicted by one econometric structure. Such a feature calls for more general definitions of identification, assumption testability, and more general methods to find a relaxed assumption. Constructing a relaxed assumption using the minimal deviation method is more challenging, and the properties of such constructions are left for future research.

APPENDICES

A. PROOFS

A.1. Proofs of Lemma 2.2

Proof. To avoid notation confusion, let $A' = A^{ER} \cap A^{TI} \cap A^{EM-NTAT} \cap A^{ND}$ be the alternative representation in Lemma 2.2, and let $A = A^{ER} \cap A^{IV} \cap A^{ND}$ be the original representation of the IA-M assumption in (2.6).

We first note that $A^{IV} \subseteq A^{TI} \cap A^{EM-NTAT}$. As a result, we have $A \subseteq A'$.

It remains to show that $A' \subseteq A$. Let $s \in A'$ be any econometric structure. It suffices to show the condition $\{Y_i(d,z), D_i(z)\}_{d,z\in\{0,1\}} \perp Z_i$ holds for s. For any B_1, B_0 set, by the exclusion restriction of s, we have $Y_i(d,1) = Y_i(d,0)$, so

$$
\begin{aligned}
&Pr_{G^s}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in B_0, D_i(1) = 1, D_i(0) = 0 | Z_i = 1) \\
=_{(1)}\ &Pr_{G^s}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in B_0 | D_i(1) = 1, D_i(0) = 0, Z_i = 1) \\
&\times Pr_{G^s}(D_i(1) = 1, D_i(0) = 0 | Z_i = 1)
\end{aligned}
$$

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$$=_{(2)} Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0|D_i(1)=1, D_i(0)=0, Z_i=1)$$
$$\times(1-Pr_{G^s}(D_i(1)=1, D_i(0)=1|Z_i=1)-Pr_{G^s}(D_i(1)=0, D_i(0)=0|Z_i=1))$$
$$=_{(3)} Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0|D_i(1)=1, D_i(0)=0, Z_i=0)$$
$$\times(1-Pr_{G^s}(D_i(1)=1, D_i(0)=1|Z_i=1)-Pr_{G^s}(D_i(1)=0, D_i(0)=0|Z_i=1))$$
$$=_{(4)} Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0|D_i(1)=1, D_i(0)=0, Z_i=0)$$
$$\times(1-Pr_{G^s}(D_i(1)=1, D_i(0)=1|Z_i=0)-Pr_{G^s}(D_i(1)=0, D_i(0)=0|Z_i=0))$$
$$=_{(5)} Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0, D_i(1)=1, D_i(0)=0|Z_i=0), \tag{A.1}$$

where (1) and (5) follow by the formula of conditional probability, (2) follows because $D_i(1)\geq D_i(0)$ almost surely under $s\in A'$, (3) follows due to the conditional type independence instrument assumption, and (4) follows due to the probabilities of always-takers and never-takers are independent of Z_i, i.e., the assumption $A^{EM-NTAT}$. The equality (A.1) shows the independence condition for the compliers type, since we start with $D_i(1)=1$ and $D_i(0)=0$. For the always-takers and never-takes, we use the same trick to show that for $d=0,1$, $A^{TI}\cap A^{EM-NTAT}$ implies:

$$Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0, D_i(1)=D_i(0)=d|Z_i=1)$$
$$=Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0|D_i(1)=D_i(0)=d, Z_i=1)$$
$$Pr(D_i(1)=D_i(0)=d|Z_i=1)$$
$$=Pr_{G^s}(Y_i(1,0)=Y_i(1,1)\in B_1, Y_i(0,0)=Y_i(0,1)\in B_0, D_i(1)=D_i(0)=d|Z_i=0). \tag{A.2}$$

We then use (A.1) and (A.2) to conclude the independent IV condition $\{Y_i(d,z), D_i(z)\}_{d,z\in\{0,1\}}\perp Z_i$ holds when there are no defiers. So $s\in A$ holds and $A'\subseteq A$. □

A.2. Proof of Proposition 2.1

To prove Proposition 2.1, we explicitly construct the econometric structures that achieve the minimal defiers and minimal marginal dependence respectively. Lemma A.1 below constructs the structure that achieves the minimal defiers, and Lemma A.2 below constructs the structure that achieves the minimal marginal dependence distance.

LEMMA A.1. *Let F be any distribution of outcome, and let $p(y,d), q(y,d)$ be the Radon–Nikodym derivatives in (2.13). Consider the following G^s for all $d,z\in\{0,1\}$: let $Pr_{G^s}(Z_i=z)=Pr_F(Z_i=z)$, and for all $d,z\in\{0,1\}$:*

$$Pr_{G^s}(Y_i(d,z)\in B_{dz}, D_i(1)=1, D_i(0)=1|Z_i=1)=Pr_{G^s}(Y_i(d,z)\in B_{dz}, D_i(1)=1, D_i(0)=1|Z_i=0)$$
$$=G^a(Y_i(0,0)\in B_{00}\cap B_{01})\times\int_{B_{11}\cap B_{10}}(\min\{p(y,1),q(y,1)\})d\mu_F(y), \tag{A.3}$$

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where G^a is any probability measure, and $Y_i(d,z)$ in (A.3) holds for all $d,z \in \{0,1\}$. Similarly, let

$$\begin{aligned} &Pr_{G^s}(Y_i(d,z) \in B_{dz}, D_i(1)=0, D_i(0)=0|Z_i=1) = Pr_{G^s}(Y_i(d,z) \in B_{dz}, D_i(1)=0, D_i(0)=0|Z_i=0) \\ &= G^n(Y_i(1,1) \in B_{11} \cap B_{10}) \times \int_{B_{00}\cap B_{01}} (\min\{p(y,0), q(y,0)\}) d\mu_F(y), \end{aligned} \quad \textbf{(A.4)}$$

where G^n is any probability measure. Let

$$\begin{aligned} &Pr_{G^s}(D_i(1)=1, D_i(0)=0|Z_i=1) = P(\mathcal{Y}_1,1) - Q(\mathcal{Y}_1,1), \\ &Pr_{G^s}(D_i(1)=1, D_i(0)=0|Z_i=0) = Q(\mathcal{Y}_0,0) - P(\mathcal{Y}_0,0), \\ &Pr_{G^s}(D_i(1)=0, D_i(0)=1|Z_i=0) = Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1), \\ &Pr_{G^s}(D_i(1)=0, D_i(0)=1|Z_i=1) = P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0), \end{aligned} \quad \textbf{(A.5)}$$

be the Z_i-conditional probabilities of the compliers and defiers, and construct

$$\begin{aligned} &Pr_{G^s}(Y_i(d,z) \in B_{dz}|D_i(1)=1, D_i(0)=0, Z_i=1) = Pr_{G^s}(Y_i(d,z) \in B_{dz}|D_i(1)=1, D_i(0)=0, Z_i=0) \\ &= \frac{\int_{B_{00}\cap B01} \max\{q(y,0)-p(y,0),0\} d\mu_F(y) \times \int_{B_{11}\cap B_{10}} \max\{p(y,1)-q(y,1),0\} d\mu_F(y)}{(P(\mathcal{Y}_1,1)-Q(\mathcal{Y}_1,1)) \times (Q(\mathcal{Y}_0,0)-P(\mathcal{Y}_0,0))}, \end{aligned} \quad \textbf{(A.6)}$$

and

$$\begin{aligned} &Pr_{G^s}(Y_i(d,z) \in B_{dz}|D_i(1)=0, D_i(0)=1, Z_i=1) = Pr_{G^s}(Y_i(d,z) \in B_{dz}|D_i(1)=0, D_i(0)=1, Z_i=0) \\ &= \frac{\int_{B_{00}\cap B01} \max\{p(y,0)-q(y,0),0\} d\mu_F(y) \times \int_{B_{11}\cap B_{10}} \max\{q(y,1)-p(y,1),0\} d\mu_F(y)}{(Q(\mathcal{Y}_1^c,1)-P(\mathcal{Y}_1^c,1)) \times (P(\mathcal{Y}_0^c,0)-Q(\mathcal{Y}_0^c,0))}. \end{aligned} \quad \textbf{(A.7)}$$

Then the constructed G^s satisfies: (1) G^s is a probability measure; (2) $G^s \in A^{TI} \cap A^{ER} \cap A^{EM-ATNT}$; (3) $F \in M(G^s)$; and (4) $m^d(s) = m^{min,d}(F)$, with $m^{min,d}(F)$ defined in Assumption 2.1.

Proof. We first check that G^s is a probability measure. Since the marginal distribution of Z_i under G^s coincides with the distribution of outcome F, it suffices to check the measure of $Y_i(d,z), D_i(1), D_i(0)$ is a probability measure conditional on $Z_i = 1$ and $Z_i = 0$. To do this, we have

$$\begin{aligned} &\sum_{d_1,d_0\in\{0,1\}} Pr_{G^s}(Y_{dz} \in \mathcal{Y}, D_i(1)=d_1, D_i(0)=d_0|Z_i=1) \\ &=_{(1)} \int_{\mathcal{Y}} \min\{p(y,1), q(y,1)\} d\mu_F(y) + \int_{\mathcal{Y}} \min\{p(y,0), q(y,0)\} d\mu_F(y) \\ &+ (P(\mathcal{Y}_1,1)-Q(\mathcal{Y}_1,1)) + (P(\mathcal{Y}_0^c,0)-Q(\mathcal{Y}_0^c,0)) \\ &=_{(2)} (P(\mathcal{Y}_1^c,1)+Q(\mathcal{Y}_1,1)) + (P(\mathcal{Y}_0,0)+Q(\mathcal{Y}_0^c,0)) + (P(\mathcal{Y}_1,1)-Q(\mathcal{Y}_1,1)) + (P(\mathcal{Y}_0^c,0)-Q(\mathcal{Y}_0^c,0)) \\ &= P(\mathcal{Y},1)+P(\mathcal{Y},0) = 1, \end{aligned}$$

where equality (1) holds by construction of G^s, and equality (2) holds by the definition of $\mathcal{Y}_1$ and $\mathcal{Y}_0$. Similarly, for the measure conditional on $Z_i = 0$, we have

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$$
\begin{aligned}
&\sum_{d_1, d_0 \in \{0,1\}} Pr_{G^s}(Y_{dz} \in \mathcal{Y}, D_i(1) = d_1, D_i(0) = d_0 | Z_i = 0) \\
&= \int_{\mathcal{Y}} \min\{p(y,1), q(y,1)\} d\mu_F(y) + \int_{\mathcal{Y}} \min\{p(y,0), q(y,0)\} d\mu_F(y) \\
&+ (Q(\mathcal{Y}_0, 0) - P(\mathcal{Y}_0, 0)) + (Q(\mathcal{Y}_1^c, 1) - Q(\mathcal{Y}_1^c, 1)) \\
&= Q(\mathcal{Y}, 0) + Q(\mathcal{Y}, 1) = 1.
\end{aligned}
$$

This checks that G^s is a probability measure.

The conditional type independence condition $s \in A^{TI}$ follows directly by the construction of G^s in (A.3)–(A.7). The property that the probabilities of always-takers and never-takers are independent of Z also follows directly from the construction of G^s in (A.3) and (A.4).

To show that exclusion restriction ($Y_i(1,1) = Y_i(0,1)$ and $Y_i(0,1) = Y_i(0,0)$ a.s.), we check the conditions in Lemma E.1 in the Supplementary Material. By construction,

$$
\begin{aligned}
&Pr_{G^s}(Y_i(1,1) \in B_{11}, Y_i(1,0) \in B_{10} | Z_i = 1) = Pr_{G^s}(Y_i(1,1) \in B_{11}, Y_i(1,0) \in B_{10} | Z_i = 0) \\
&= \int_{B_{11} \cap B_{10}} (\min\{p(y,1), q(y,1)\}) d\mu_F(y) + Pr_{G^n}(Y_i(1,1) \in B_{11} \cap B_{10}) \\
&+ \int_{B_{11} \cap B_{10}} \max\{p(y,1) - q(y,1), 0\} d\mu_F(y) + \int_{B_{11} \cap B_{10}} \max\{q(y,1) - p(y,1), 0\} d\mu_F(y),
\end{aligned} \tag{A.8}
$$

where the right-hand side of (A.8) depends only on the set $B_{11} \cap B_{10}$. Therefore, by Lemma E.1 in the Supplementary Material, $Y_i(1,1) = Y_i(1,0)$ almost surely. Similarly, we can use Lemma E.1 in the Supplementary Material to check $Y_i(0,1) = Y_i(0,0)$ almost surely. As a result, the exclusion restriction holds.

Then we check G^s can generate the data distribution F, i.e., $F \in M(G^s)$. To do this, we check that the model-predicted observable distribution coincides with the observed data distribution.

$$
\begin{aligned}
\underbrace{Pr_{M(G^s)}(Y_i \in B, D_i = 1 | Z_i = 1)}_{\textit{Model Predicted Outcome Distribution}} &=_{(3)} \sum_{j=0}^{1} Pr_{G^s}(Y_i(1,1) \in B, Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = j | Z_i = 1) \\
&=_{(4)} [Q(B \cap \mathcal{Y}_1, 1) + P(B \cap \mathcal{Y}_1^c, 1)] + [P(B \cap \mathcal{Y}_1, 1) - Q(B \cap \mathcal{Y}_1, 1)] \\
&= P(B, 1) \\
&=_{(5)} \underbrace{Pr_F(Y_i \in B, D_i = 1 | Z_i = 1)}_{\textit{Observed Outcome Distribution}},
\end{aligned} \tag{A.9}
$$

where equality (3) holds by the potential outcome framework (2.1), (4) holds by the construction of G^s, and (5) holds by the definition of $P(B,1)$. Similarly,

$$
\begin{aligned}
\underbrace{Pr_{M(G^s)}(Y_i \in B, D_i = 0 | Z_i = 1)}_{\textit{Model Predicted Outcome Distribution}} &= \sum_{j=0}^{1} Pr_{G^s}(Y_i(0,1) \in B, Y_i(1,1) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = j | Z_i = 1) \\
&= [P(B \cap \mathcal{Y}_0, 0) + Q(B \cap \mathcal{Y}_0^c, 0)] + \left[P(B \cap \mathcal{Y}_0^c, 0) - Q(B \cap \mathcal{Y}_0^c, 0)\right]
\end{aligned}
$$

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$$= P(B,0)$$
$$= \underbrace{Pr_F(Y_i \in B, D_i = 0|Z_i = 1)}_{\text{Observed Outcome Distribution}}.$$

Similar relations between G^s and Pr_F also hold when $Z_i = 0$. This checks $F \in M(G^s)$.

In the last step, we check that G^s achieves the minimal probability of defiers. We first find a lower bound for $m^{min,d}(F)$, and show that $m^d(s)$ achieves this lower bound.

Consider any $s^* \in A^{ER} \cap A^{TI} \cap A^{EM-ATNT}$, and $F \in M(G^{s^*})$. We have

$$\begin{aligned}
&Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 1|Z_i = 1)\\
&=_{(6)} Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}|D_i(1) = 1, D_i(0) = 1, Z_i = 1)Pr(D_i(1) = 1, D_i(0) = 1|Z_i = 1)\\
&=_{(7)} Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}|D_i(1) = 1, D_i(0) = 1, Z_i = 0)Pr(D_i(1) = 1, D_i(0) = 1|Z_i = 0)\\
&= Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 1|Z_i = 0),
\end{aligned} \tag{A.10}$$

where equality (6) holds by Bayes' rule, (7) holds by $s^* \in A^{ER} \cap A^{TI} \cap A^{EM-ATNT}$. Now we consider the following decompositions:

$$\begin{aligned}
P(B_1,1) &= Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 1|Z_i = 1)\\
&\quad + Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 0|Z_i = 1),\\
Q(B_1,1) &= Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 1|Z_i = 0)\\
&\quad + Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0),
\end{aligned} \tag{A.11}$$

and use (A.10) to get:

$$\begin{aligned}
P(B_1,1) - Q(B_1,1) &= Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 0|Z_i = 1)\\
&\quad - Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in B_1, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0).
\end{aligned} \tag{A.12}$$

Take $B_1 = \mathcal{Y}_1^c$, we have

$$\begin{aligned}
&Pr(D_i(1) = 0, D_i(0) = 1|Z_i = 0)\\
&\geq Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in \mathcal{Y}_1^c, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0)\\
&= Pr_{G^{s^*}}(Y_i(1,0) = Y_i(1,1) \in \mathcal{Y}_1^c, Y_i(0,0) = Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 0|Z_i = 1) + Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1)\\
&\geq Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1),
\end{aligned} \tag{A.13}$$

and similarly, we can show $Pr(D_i(1) = 0, D_i(0) = 1|Z_i = 1) \geq P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)$. So the total measure of defiers satisfies

$$m^d(s^*) \geq [Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1)]Pr(Z_i = 0) + [P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)]Pr(Z_i = 1).$$

Therefore, we get a lower bound $m^{min,d}(F) \geq [Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1)]Pr(Z_i = 0) + [P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)]Pr(Z_i = 1)$.

On the other hand, by the construction of G^s, the measure of defiers under G^s is

$$m^d(s) = [Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1)]Pr(Z_i = 0) + [P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)]Pr(Z_i = 1).$$

So the constructed s achieves the minimal measure of defiers. □

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LEMMA A.2. *Let F be any distribution of outcome, and let $p(y,d), q(y,d)$ be the Radon–Nikodym derivatives with respect to μ_F. Consider the following G^s:*

$$Pr_{G^s}(Y_i(d,z) \in B_{dz} \quad \forall d,z \in \{0,1\}, D_i(1)=1, D_i(0)=1|Z_i=z)$$
$$= \begin{cases} G^a(Y_i(0,0) \in B_{00} \cap B_{01}) \times \int_{B_{10} \cap B_{11}} \min\{p(y,1), q(y,1)\} d\mu_F(y) & \text{if} \quad z=1, \\ G^a(Y_i(0,0) \in B_{00} \cap B_{01}) \times \int_{B_{10} \cap B_{11}} q(y,1) d\mu_F(y) & \text{if} \quad z=0, \end{cases} \tag{A.14}$$

where G^a is any probability measure, and

$$Pr_{G^s}(Y_i(d,z) \in B_{dz} \quad \forall d,z \in \{0,1\}, D_i(1)=0, D_i(0)=0|Z_i=z)$$
$$= \begin{cases} G^n(Y_i(0,0) \in B_{00} \cap B_{01}) \times \int_{B_{10} \cap B_{11}} \min\{p(y,0), q(y,0)\} d\mu_F(y) & \text{if} \quad z=0, \\ G^n(Y_i(0,0) \in B_{00} \cap B_{01}) \times \int_{B_{10} \cap B_{11}} p(y,0) d\mu_F(y) & \text{if} \quad z=1, \end{cases} \tag{A.15}$$

where G^n is any probability measure. Let

$$\begin{aligned} &Pr(D_i(1)=1, D_i(0)=0|Z_i=1) = P(\mathcal{Y}_1,1) - Q(\mathcal{Y}_1,1) \\ &Pr(D_i(1)=1, D_i(0)=0|Z_i=0) = Q(\mathcal{Y}_0,0) - P(\mathcal{Y}_0,0), \end{aligned} \tag{A.16}$$

and let:

$$Pr_{G^s}(Y_i(d,z) \in B_{dz} \quad \forall d,z \in \{0,1\}|D_i(1)=1, D_i(0)=0, Z_i=z),$$
$$= \frac{\int_{B_{00} \cap B_{01}} \min\{q(y,0)-p(y,0),0\} d\mu_F(y) \times \int_{B_{10} \cap B_{11}} \min\{p(y,1)-q(y,1),0\} d\mu_F(y)}{(P(\mathcal{Y}_1,1)-Q(\mathcal{Y}_1,1)) \times (Q(\mathcal{Y}_0,0)-P(\mathcal{Y}_0,0))}, \tag{A.17}$$

$$Pr_{G^s}(Y_i(d,z) \in B_{dz} \quad \forall d,z \in \{0,1\}, D_i(1)=0, D_i(0)=1|Z_i=z) \equiv 0. \tag{A.18}$$

Then the following results hold: (1) G^s is a probability measure; (2) $G^s \in A^{ER} \cap A^{ND} \cap A^{TI-CP}$; (3) $F \in M(G^s)$; and (4) $m^{MD}(s) = m^{min,MI}(F)$, where $m^{min,MI}(F)$ is defined in Assumption 2.2.

Proof. We first check that G^s is a probability measure.

$$\begin{aligned} &\sum_{d_1,d_0 \in \{0,1\}} Pr_{G^s}(Y_i(d,1) \in \mathcal{Y}, D_i(1)=d_1, D_i(0)=d_0|Z_i=1) \\ &= \int_{\mathcal{Y}} \min\{p(y,1), q(y,1)\} d\mu_F(y) + \int_{\mathcal{Y}} p(y,0) d\mu_F(y) + (P(\mathcal{Y}_1,1) - Q(\mathcal{Y}_1,1)) \\ &= (P(\mathcal{Y}_1^c,1) + Q(\mathcal{Y}_1,1)) + P(\mathcal{Y},0) + (P(\mathcal{Y}_1,1) - Q(\mathcal{Y}_1,1)) \\ &= P(\mathcal{Y},1) + P(\mathcal{Y},0) = 1. \end{aligned}$$

We can check that the quantities also sum up to one for $Z_i = 0$, which checks that G is a probability measure.

Checking $F \in M^s(G^s)$ is similar to the proofs in Lemma A.1. The conditional type independence for compliers, "No Defiers" assumptions hold for G^s by construction. Exclusion restriction holds by Lemma E.1 in the Supplementary Material.[21]

[21] See Lemma A.1 for the procedures for proof of this statement.

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We now show a lower bound for the $m^{min,d}(F)$. Let s^* be any structure in $A^{ER} \cap A^{ND} \cap A^{TI-CP}$ and $F \in M(G^{s^*})$. We use the following decomposition:

$$\begin{aligned}
P(B_1,1) &= Pr_{G^{s*}}(Y_i(1,1) \in B_1, D_i(1)=1, D_i(0)=1|Z_i=1) \\
&\quad + Pr_{G^{s*}}(Y_i(1,1) \in B_1, D_i(1)=1, D_i(0)=0|Z_i=1), \\
Q(B_1,1) &= Pr_{G^{s*}}(Y_i(1,0) \in B_1, D_i(1)=1, D_i(0)=1|Z_i=0) \\
&\quad + Pr_{G^{s*}}(Y_i(1,0) \in B_1, D_i(1)=0, D_i(0)=1|Z_i=0) \\
&=_{(1)} Pr_{G^{s*}}(Y_i(1,0) \in B_1, D_i(1)=1, D_i(0)=1|Z_i=0),
\end{aligned} \tag{A.19}$$

where equality (1) follows due to the “No Defiers” condition. Take the Radon–Nikodym derivatives with respect to μ_F on both sides to get

$$\begin{aligned}
p(y,1) &= g^{s^*}_{y1}(y,1,1|Z_i=1) + g^{s^*}_{y1}(y,1,0|Z_i=1) \\
q(y,1) &= g^{s^*}_{y0}(y,1,1|Z_i=0).
\end{aligned}$$

Take the difference between $p(y,1)$ and $q(y,1)$ to get

$$g^{s^*}_{y1}(y,1,1|Z_i=1) - g^{s^*}_{y0}(y,1,1|Z_i=0) = p(y,1) - q(y,1) - g^{s^*}_{y1}(y,1,0|Z_i=1). \tag{A.20}$$

Since $g^{s^*}_{y1}(y,1,0|Z_i=1) \geq 0$, we have[22]

$$\left[g^{s^*}_{y1}(y,1,1|Z_i=1) - g^{s^*}_{y0}(y,1,1|Z_i=0)\right]^2 \geq \max\{-(p(y,1)-q(y,1)),0\}^2.$$

Similarly, using the decomposition of $Q(B,0)$ and $P(B,0)$, we have

$$\left[g^{s^*}_{y0}(y,0,0|Z_i=0) - g^{s^*}_{y1}(y,0,0|Z_i=1)\right]^2 \geq \max\{-(q(y,0)-p(y,0)),0\}^2.$$

So the measure of deviation from marginal independence equals:

$$\begin{aligned}
m^{MI}(s^*) &= \int \left[g^{s^*}_{y1}(y,1,1|Z_i=1) - g^{s^*}_{y0}(y,1,1|Z_i=0)\right]^2 \\
&\quad + \left[g^{s^*}_{y0}(y,0,0|Z_i=0) - g^{s^*}_{y1}(y,0,0|Z_i=1)\right]^2 d\mu_F(y) \\
&\geq \int \max\{-(p(y,1)-q(y,1)),0\}^2 + \max\{-(q(y,0)-p(y,0)),0\}^2 d\mu_F(y) \\
&= m^{MI}(s),
\end{aligned}$$

where the last equality holds by the construction of G^s, which shows that s achieves $m^{min,MI}(F)$. □

A.3. Proof of Theorem 1

Proof. The expressions for the identified minimal defiers amount and the minimal marginal dependence distance are proved in Lemmas A.1 and A.2, respectively.

Since Theorem 1 involves both the minimal defiers relaxed assumption and the minimal marginal dependent instrument relaxed assumption, we separate the proof into two major

[22] Note that $(x-t)^2 \geq (\max\{-x,0\})^2$ when $t \geq 0$ holds.

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parts. Because $P(B,d)$ and $Q(B,d)$ are absolutely continuous with respect to some dominating measure μ_F, equation (2.4) implies that $Pr_{G^s}(Y_i(d,z) \in B, D_i(z) = d, D_i(1-z) = d'|Z_i = z)$ are also absolutely continuous with respect to μ_F for all $d, z, d' \in \{0,1\}$. So we will use the Radon–Nikodym density of G^s with respect to μ_F throughout this proof: we use $g^s_{yz}(y,d,d'|Z_i)$ to denote the density of $Pr_{G^s}(Y_i(d,z) \in B_{dz}, D_i(1) = d, D_i(0) = d'|Z_i)$.

For both $\tilde{A}$ and $\tilde{A}'$, our goal is to show that the point identification of the conditional density of $Y_i(1,1)$ given $D_i(1) = 1$, $D_i(0) = 0$ and $Z_i = 1$:

$$\frac{g^s_{y1}(y,1,0|Z_i = 1)d\mu_F(y)}{\int_{\mathcal{Y}} g^s_{y1}(y,1,0|Z_i = 1)d\mu_F(y)},$$

and the identification of the conditional density of $Y_i(0,0)$ given $D_i(1) = 1, D_i(0) = 0,$ and $Z_i = 0$:

$$\frac{g^s_{y0}(y,1,0|Z_i = 0)d\mu_F(y)}{\int_{\mathcal{Y}} g^s_{y0}(y,1,0|Z_i = 0)d\mu_F(y)}.$$

Using the above expressions, for any structure s such that $F \in M(G^s)$, from the definition of $LATE$, we can derive:

$$LATE(s) = \frac{\int_{\mathcal{Y}} y g^s_{y1}(y,1,0|Z_i = 1)d\mu_F(y)}{\int_{\mathcal{Y}} g^s_{y1}(y,1,0|Z_i = 1)d\mu_F(y)} - \frac{\int_{\mathcal{Y}} y g^s_{y0}(y,1,0|Z_i = 0)d\mu_F(y)}{\int_{\mathcal{Y}} g^s_{y0}(y,1,0|Z_i = 0)d\mu_F(y)}, \quad \textbf{(A.21)}$$

where the denominator $\int_{\mathcal{Y}} g^s_{yz}(y,1,0|Z_i = z)d\mu_F(y)$ represents the probability of compliers conditioned on $Z_i = z$. Therefore, the point identification of g^s_{y1} and g^s_{y0} implies the point identification of $LATE(s)$. Both $\tilde{A}$ and $\tilde{A}'$, we will show the identification results:

$$\begin{aligned} &g^s_{y1}(y,1,0|Z_i = 1) = \max\{p(y,1) - q(y,1), 0\}, \\ \text{and} \quad &g^s_{y0}(y,1,0|Z_i = 0) = \max\{q(y,0) - p(y,0), 0\}. \end{aligned} \quad \textbf{(A.22)}$$

Plug (A.22) into (A.21) we get the expression (2.14).

For the minimal defiers relaxed assumption $\tilde{A}$. First, $\tilde{A}$ satisfies Assumption 2.1, and the construction of G^s in Lemma A.1 implies the minimal measure of defiers:

$$m^{min,d}(F) = [Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1)]Pr(Z_i = 0) + [P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)]Pr(Z_i = 1).$$

We now show that the quantity $m^{min,d}(F)$ can only be achieved when we specify the g^s_{y1} and g^s_{y0} as in (A.22). To start, we look at the probability of defiers given $Z_i = 1$:

$$\begin{aligned} &Pr(D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &= Pr_{G^s}(Y_i(1,0) \in \mathcal{Y}_1^c, Y_i(0,0) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &+ Pr_{G^s}(Y_i(1,0) \in \mathcal{Y}_1, Y_i(0,0) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &\geq_{(1)} Pr_{G^s}(Y_i(1,0) \in \mathcal{Y}_1^c, Y_i(0,0) \in \mathcal{Y}, D_i(1) = 0, D_i(0) = 1|Z_i = 0) \\ &=_{(2)} Pr_{G^s}(Y_i(1,1) \in \mathcal{Y}_1^c, Y_i(0,1) \in \mathcal{Y}, D_i(1) = 1, D_i(0) = 0|Z_i = 1) + Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1) \\ &\geq_{(3)} Q(\mathcal{Y}_1^c,1) - P(\mathcal{Y}_1^c,1), \end{aligned}$$

where the first inequality (1) holds with equality if and only if $g^s_{y0}(y,0,1|Z = 0) = 0$ for all $y \in \mathcal{Y}_1$, equality (2) holds by (A.11)–(A.13) in the proof of Lemma A.1, where we use the

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A^{TI} and $A^{EM-NTAT}$. Inequality (3) holds with equality if and only if $g^s_{y1}(y,1,0|Z_i=1)=0$ for all $y \in \mathcal{Y}_1^c$.

Similarly, we can write the condition for $Z_i = 1$:

$$
\begin{aligned}
&Pr(D_i(1)=0, D_i(0)=1|Z_i=1)\\
&= Pr_{G^s}(Y_i(0,1) \in \mathcal{Y}_0^c, Y_i(1,1) \in \mathcal{Y}, D_i(1)=0, D_i(0)=1|Z_i=1)\\
&+ Pr_{G^s}(Y_i(0,1) \in \mathcal{Y}_0, Y_i(1,1) \in \mathcal{Y}, D_i(1)=0, D_i(0)=1|Z_i=1)\\
&\geq Pr_{G^s}(Y_i(0,1) \in \mathcal{Y}_0^c, Y_i(1,1) \in \mathcal{Y}, D_i(1)=0, D_i(0)=1|Z_i=1)\\
&= Pr_{G^s}(Y_i(0,1) \in \mathcal{Y}_0^c, Y_i(1,1) \in \mathcal{Y}, D_i(1)=1, D_i(0)=0|Z_i=0) + P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0)\\
&\geq P(\mathcal{Y}_0^c,0) - Q(\mathcal{Y}_0^c,0),
\end{aligned}
$$

where the first inequality with equality holds if and only if $g^s_{y1}(y,0,1|Z=1)=0$ for all $y \in \mathcal{Y}_0$ and the last inequality holds with equality if and only if $g^s_{y0}(y,1,0|Z_i=0)=0$ for almost all $y \in \mathcal{Y}_0^c$. Therefore, $m^d(s) = m^{min,d}(F)$ if and only if the density conditions hold:

$$
\begin{aligned}
&g^s_{y0}(y,0,1|Z_i=0)=0 \quad \forall y \in \mathcal{Y}_1, \quad g^s_{y1}(y,1,0|Z_i=1)=0 \quad \forall y \in \mathcal{Y}_1^c,\\
&g^s_{y1}(y,0,1|Z_i=1)=0 \quad \forall y \in \mathcal{Y}_0, \quad g^s_{y0}(y,1,0|Z_i=0)=0 \quad \forall y \in \mathcal{Y}_0^c.
\end{aligned} \tag{A.23}
$$

Now, take Radon–Nikodym derivatives of (A.12) with respect to μ_F, we have

$$
p(y,1) - q(y,1) = g^s_{y1}(y,1,0|Z_i=1) - g^s_{y0}(y,0,1|Z=0). \tag{A.24}
$$

Combine the expression (A.24), the definition of $\mathcal{Y}_1, \mathcal{Y}_0$ and equation (A.23), we have $g^s_{y1}(y,1,0|Z=1) = \max\{p(y,1)-q(y,1),0\}$ must hold for all $s \in \tilde{A}$. We can symmetrically get $g_{y0}(y,0,1|Z=0) = \max\{q(y,0)-p(y,0),0\}$ must hold for all $s \in \tilde{A}$.

For the minimal defiers relaxed assumption $\tilde{A}'$. Suppose $\tilde{A}'$ satisfies Assumption 2.2. By Lemma A.2, the $m^{min,MI}(F)$ is:

$$
m^{min,MI}(F) = \int \Big(\max\{(p(y,1)-q(y,1))^2,0\} + \max\{(q(y,0)-p(y,0))^2,0\}\Big)\, d\mu_F(y).
$$

For any $s^* \in \tilde{A}'$ and $F \in M^{s^*}(G^{s^*})$, by (A.20), we have

$$
\begin{aligned}
(g^{s^*}_{y1}(y,1,1|Z_i=1) - g^{s^*}_{y0}(y,1,1|Z_i=0))^2 &= [p(y,1)-q(y,1)-g^{s^*}_{y1}(y,1,0|Z_i=1)]^2\\
&\geq_{(*)} \max\{-p(y,1)+q(y,1),0\}^2,
\end{aligned}
$$

where the inequality $(*)$ holds with equality if and only if

$$
g^{s^*}_{y1}(y,1,0|Z_i=1) = \max\{p(y,1)-q(y,1),0\}.
$$

Similarly,

$$
\begin{aligned}
(g^{s^*}_{y1}(y,0,0|Z_i=1) - g^{s^*}_{y0}(y,0,0|Z_i=0))^2 &= (q(y,0)-p(y,0)-g^{s^*}_{y0}(y,1,0|Z_i=0))^2\\
&\geq_{(**)} \max\{-q(y,0)+p(y,0),0\}^2,
\end{aligned}
$$

where the inequality $(**)$ holds with equality if and only if $g^{s^*}_{y0}(y,1,0|Z_i=0) = \max\{q(y,0)-p(y,0),0\}$. Since $s^* \in \tilde{A}'$ achieves the $m^{min,MI}(F)$, $(*)$ and $(**)$ hold with equality, and (A.22) must hold. □

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A.4. Proof of Lemma 2.3

Proof. By the bounded density condition, the Lebesgue measure of set $\{y : \mathbb{1}(-b_n \leq f(y,d) \leq b_n)\}$ is less than $CM_0 b_n$. Therefore

$$\int_{M_l}^{M_u} |f(y,d)| \mathbb{1}(-b_n \leq f(y,d) \leq b_n) dy \leq CM_0 b_n^2.$$

So Assumption 2.9 implies Assumption 2.8. □

A.5. Proofs in Section 2.5

A.5.1. *Lemmas for Theorem 2.* We first define the following objects:

$$\begin{aligned} f_n^{l,m}(y) &\equiv \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{Y_i - y}{h_n}\right) \mathbb{1}(D_i = l, Z_i = m), \\ \bar{f}_n^{l,m}(y) &\equiv \frac{1}{h_n} E\left[K\left(\frac{Y_i - y}{h_n}\right) \mathbb{1}(D_i = l, Z_i = m)\right]. \end{aligned} \tag{A.25}$$

LEMMA A.3. *Let $h_n = n^{-\gamma}$ for some $\gamma \in (0,1)$, such that $\frac{nh_n}{|\log h_n|} \to \infty$. Define $a_n = \min\left\{\sqrt{\frac{nh_n}{\log h_n^{-1}}}, h_n^{-2}\right\}$. Suppose Assumptions 2.6 and 2.7 hold, then there exists a constant C such that for all $d, z \in \{0,1\}$, we have*

$$\lim_{n\to\infty} \sup a_n \sup_y \left| \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{Y_i - y}{h_n}\right) \mathbb{1}(D_i = d, Z_i = z) - p(y,1) Pr(Z_i = z) \right| \leq C \quad a.s.. \tag{A.26}$$

LEMMA A.4. *Let $h_n = n^{-\gamma}$ for some $\gamma \in (0,1)$, such that $\frac{nh_n}{|\log h_n|} \to \infty$ and let $a_n = \min\left\{\sqrt{\frac{nh_n}{\log h_n^{-1}}}, h_n^{-2}\right\}$. If there exists a constant $c > 0$ such that $Pr(Z_i = 1) \in [c, 1-c]$, and $\sup_y [\max(p(y,d), q(y,d)] < \infty$, then for any $\epsilon > 0$ such that*

$$\begin{aligned} n^{-\epsilon} a_n \sup_y |f_h(y,1) - (p(y,1) - q(y,1))| &= o_p(1) \\ n^{-\epsilon} a_n \sup_y |f_h(y,0) - (q(y,0) - p(y,0))| &= o_p(1). \end{aligned} \tag{A.27}$$

LEMMA A.5. *(Limit Distribution of Infeasible Components) Recall $f(y,1) = p(y,1) - q(y,1)$ and $f(y,0) = q(y,0) - p(y,0)$. Suppose $E[||Y_i||^{2+\delta}] < \infty$ for some $\delta > 0$. Define the infeasible trimming set*

$$\mathcal{Y}_d^{infsb}(b_n) = \{y \in \mathcal{Y} : f(y,d) \geq b_n \quad y \in [M_l, M_u]\} \cup \mathcal{Y}_d^{ut} \cup \mathcal{Y}_d^{lt}.$$

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Let $\boldsymbol{X}_i(b_n)$ and Σ be

$$\boldsymbol{X}_i(b_n)=\begin{pmatrix} \mathbb{1}(Z_i=0)\\ \mathbb{1}(Z_i=1)\\ Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\\ Y_i\mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\\ Y_i\mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0^{infsb}(b_n))\\ Y_i\mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0^{infsb}(b_n))\\ \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\\ \mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\\ \mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0^{infsb}(b_n))\\ \mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0^{infsb}(b_n)) \end{pmatrix} \quad and \quad \Sigma=Var\begin{pmatrix} \mathbb{1}(Z_i=0)\\ \mathbb{1}(Z_i=1)\\ Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1)\\ Y_i\mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1)\\ Y_i\mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0)\\ Y_i\mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0)\\ \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1)\\ \mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_1)\\ \mathbb{1}(D_i=0,Z_i=0)\mathbb{1}(Y_i\in\mathcal{Y}_0)\\ \mathbb{1}(D_i=0,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_0) \end{pmatrix}. \tag{A.28}$$

Then for any $b_n\to 0$, $\frac{1}{\sqrt{n}}\sum_{i=1}^n(\boldsymbol{X}_i(b_n)-E[\boldsymbol{X}_i(b_n)])\to_d N(0,\Sigma)$.

LEMMA A.6. *Let $h_n=n^{-1/5}$ and $b_n=n^{-1/4}/\log n$, and Assumptions 2.3–2.8 hold, then for $d,k\in\{0,1\}$:*

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^n Y_i\left[\frac{\mathbb{1}(D_i=d,Z_i=d)}{Pr(Z_i=d)}-\frac{\mathbb{1}(D_i=d,Z_i=1-d)}{Pr(Z_i=1-d)}\right](\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\right|\\
&\quad\equiv Term\ 1=o_p(1/\sqrt{n}),\\
&\left|\frac{1}{n}\sum_{i=1}^n \left[\frac{\mathbb{1}(D_i=d,Z_i=d)}{Pr(Z_i=d)}-\frac{\mathbb{1}(D_i=d,Z_i=1-d)}{Pr(Z_i=1-d)}\right](\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\right|\\
&\quad\equiv Term\ 2=o_p(1/\sqrt{n}),\\
&\left|\frac{1}{n}\sum_{i=1}^n Y_i\mathbb{1}(D_i=d,Z_i=k)(\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\right|\equiv Term\ 3=o_p(1),\\
&\left|\frac{1}{n}\sum_{i=1}^n \mathbb{1}(D_i=d,Z_i=k)(\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\right|\equiv Term\ 4=o_p(1).
\end{aligned} \tag{A.29}$$

Proof. We prove the case for $d=k=1$, the rest holds similarly. We look at $\sqrt{n}\times$*(Term 1)* first. For any $\epsilon>0$,

$$\begin{aligned}
&Pr\Bigg(\Bigg|\frac{\sqrt{n}}{n}\sum_{i=1}^n Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right]\\
&\quad\times(\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\Bigg|>\epsilon\Bigg)\\
&\le Pr(\sup_y|f_h(y,1)-f(y,1)|\ge cn^{-2/5+\epsilon})
\end{aligned}$$

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$$+Pr\Bigg(\frac{\sqrt{n}}{n}\sum_{i=1}^{n}\Bigg|Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right]$$

$$\times\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u])\Bigg|>\epsilon\Bigg), \tag{A.30}$$

where the inequality holds because on the event $\sup_y|f_h(y,1)-f(y,1)|<cn^{-2/5+\epsilon}$:

$$\left|\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n))\right|\le\mathbb{1}\left(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u]\right).$$

Note that

$$Var\left(\left|\frac{\sqrt{n}}{n}\sum_{i=1}^{n}Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right](\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u]))\right|\right)$$

$$\le E\left(\left|\frac{\sqrt{n}}{n}\sum_{i=1}^{n}Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right](\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},Y_i\in[M_l,M_u]))\right|^2\right)$$

$$\le\underbrace{E\left[\left|Y_i^2\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right]^2(\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u]))\right|\right]}_{\text{Term }A}$$

$$\underbrace{+(n-1)E\left[|Y_i|\left|\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right|\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u]\right]^2}_{\text{Term }B}.$$

Term $A=o(1)$ by the dominated convergence theorem since

$$\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u])\to 0,$$

and the second moment of Y_i is bounded by assumption. For term B, by Assumption 2.8,

$$E\left[Y_i\left|\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right|\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u])\right]$$

$$\le\max\{|M_l|,|M_u|\}E\left[\left|\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)}-\frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right|\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u])\right]$$

$$=O((b_n+cn^{-2/5+\epsilon})^2)=O(1/(\sqrt{n}\log^2 n)),$$

therefore $B=(n-1)O(\frac{1}{n\log^4 n})=o(1)$. Therefore, the last term in (A.30) is $o(1)$ by mean squared error convergence. Since $Pr(\sup_y|f_h(y,1)-f(y,1)|\ge cn^{-2/5+\epsilon})\to 0$ by Lemma A.4, $\sqrt{n}\times$*Term 1* is $o_p(1)$. The proof of *Term 2* is similar to that of *Term 1*.

Then we look at Term 3:

$$Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\mathbb{1}(D_i=1,Z_i=1)(\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n)))-\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n)))\right|>\epsilon\right)$$

$$\le Pr(\sup_y|f_h(y,1)-f(y,1)|\ge cn^{-2/5+\epsilon})$$

$$+Pr\left(\frac{1}{n}\sum_{i=1}^{n}\left|Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ \ Y_i\in[M_l,M_u])\right|>\epsilon\right).$$

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Note that

$$
\begin{aligned}
&Var\left(\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ Y_i\in[M_l,M_u]))\right|\right)\\
&\le E\left(\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ Y_i\in[M_l,M_u]))\right|^2\right)\\
&\le \underbrace{\frac{1}{n}E\left(\left|Y_i^2\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ Y_i\in[M_l,M_u])\right|\right)}_{C}\\
&+\underbrace{\frac{n-1}{n}\left(E\left|Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(|f(Y_i,1)|<b_n+cn^{-2/5+\epsilon},\ Y_i\in[M_l,M_u])\right|\right)^2}_{D}.
\end{aligned}
$$

Term $C\to 0$ and Term $D\to 0$ by dominated convergence theorem, since $b_n+cn^{-2/5+\epsilon}\to 0$. The result for Term 3 in the lemma holds by mean squared error convergence. The result for Term 4 holds by similar argument. □

LEMMA A.7. *(Asymptotic Linear Expansion of Numerators and Denominators of (2.16)) Let $h_n\asymp n^{-1/5}$, $b_n\asymp n^{-1/4}/\log n$, $c_n\asymp n^{-2/5+\epsilon}$ and $0<\epsilon<2/5-1/4$ as in Lemma A.6, and Assumptions 2.3–2.8 hold, then for $d,z\in\{0,1\}$, and $Y_i^{\#}\equiv Y_i$ for all i, or $Y_i^{\#}\equiv 1$ for all i:*

$$
\begin{aligned}
&\Bigg|\Bigg\{\frac{1}{n}\sum_{i=1}^{n}Y_i^{\#}\left[\frac{\mathbb{1}(D_i=d,Z_i=z)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=z)}-\frac{\mathbb{1}(D_i=d,Z_i=1-z)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1-z)}\right]\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_d(b_n))\\
&\qquad -\int_{\mathcal{Y}_d}y(p(y,d)-q(y,d))dy\Bigg\}\\
&-\frac{E[Y_i^{\#}\mathbb{1}(D_i=d,Z_i=z)\mathbb{1}(Y_i\in\mathcal{Y}_d)]}{Pr(Z_i=1)Pr(Z_i=0)}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1-z)-Pr(Z_j=1-z)\right]\\
&+\frac{E[Y_i^{\#}\mathbb{1}(D_i=d,Z_i=1-z)\mathbb{1}(Y_i\in\mathcal{Y}_d)]}{Pr(Z_i=1)Pr(Z_i=0)}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=z)-Pr(Z_j=z)\right]\\
&-\frac{1}{nPr(Z_i=z)}\sum_{i=1}^{n}\Big(Y_i^{\#}\mathbb{1}(D_i=d,Z_i=z)\mathbb{1}(Y_i\in\mathcal{Y}_d^{infsb}(b_n))\\
&\qquad -E[Y_i^{\#}\mathbb{1}(D_i=d,Z_i=z)\mathbb{1}(Y_i\in\mathcal{Y}_d^{infsb}(b_n))]\Big)\\
&+\frac{1}{nPr(Z_i=1-z)}\sum_{i=1}^{n}\Big(Y_i^{\#}\mathbb{1}(D_i=d,Z_i=1-z)\mathbb{1}(Y_i\in\mathcal{Y}_d^{infsb}(b_n))\\
&\qquad -E[Y_i^{\#}\mathbb{1}(D_i=d,Z_i=1-z)\mathbb{1}(Y_i\in\mathcal{Y}_d^{infsb}(b_n))]\Big)\Bigg|\\
&=o_p(1/\sqrt{n}).
\end{aligned}
\tag{A.31}
$$

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Remark A.1. For $Y_i^{\#} \equiv Y_i$ (resp., $Y_i^{\#} \equiv 1$), (A.31) is the term for the numerator (resp., denominator) of the estimator of (2.14).

Proof. We prove the first statement (A.31) for $d = z = 1$ and $Y_i^{\#} \equiv Y_i$. The rest of the statements hold similarly by changing the value of d, z, and $Y_i^{\#}$.

We look at the expansion

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)}\right]\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n)) - \int_{\mathcal{Y}_1} y(p(y,1)-q(y,1))dy \\
&= \underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_j=1)}\right]\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))}_{A_1} \\
&\underbrace{-\frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_j=0)}\right]\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))}_{A_2} \\
&\underbrace{+\frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right](\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n)) - \mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n+c_n)))}_{B} \\
&\underbrace{+\frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{Pr(Z_i=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{Pr(Z_i=0)}\right]\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n+c_n))}_{C_1} \\
&\underbrace{-\int y(p(y,1)-q(y,1))\mathbb{1}(y\in\mathcal{Y}_1^{infsb}(b_n+c_n))dy}_{C_2} \\
&\underbrace{+\int_{\mathcal{Y}} y(p(y,1)-q(y,1))(\mathbb{1}(y\in\mathcal{Y}_1^{infsb}(b_n+c_n)) - \mathbb{1}(y\in\mathcal{Y}_1))dy}_{D}.
\end{aligned} \tag{A.32}$$

The expansion above holds by adding and subtracting the same terms repeatedly. For term A_1, we can write it as

$$\begin{aligned}
A_1 &= \frac{1}{n}\sum_{i=1}^{n} Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\hat{\mathcal{Y}}_1(b_n))\left[\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{1}{Pr(Z_j=1)}\right] \\
&=_{(1)} \left[\frac{1}{n}\sum_{i=1}^{n} Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n+c_n) + o_p(1)\right] \\
&\qquad\times\left[\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{1}{Pr(Z_j=1)}\right] \\
&=_{(2)} \left[E[Y_i\mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i\in\mathcal{Y}_1^{infsb}(b_n+c_n))] + o_p(1)\right] \\
&\qquad\times\left[\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)} - \frac{1}{Pr(Z_j=1)}\right]
\end{aligned}$$

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$$
\begin{aligned}
&=_{(3)} \frac{E[Y_i \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i \in \mathcal{Y}_1^{infsb}(b_n+c_n))]+o_p(1)}{Pr(Z_i=1)Pr(Z_i=0)} \\
&\qquad \times \left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)-Pr(Z_j=0)\right] \\
&\times \frac{Pr(Z_i=1)Pr(Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_i=1)\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_i=0)} \\
&=_{(4)} \frac{E[Y_i \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i \in \mathcal{Y}_1^{infsb}(b_n+c_n))]+o_p(1)}{Pr(Z_i=1)Pr(Z_i=0)} \\
&\qquad \times \left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)-Pr(Z_j=0)\right]\times(1+o_p(1)) \\
&=_{(5)} \frac{E[Y_i \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i \in \mathcal{Y}_1)]+o_p(1)}{Pr(Z_i=1)Pr(Z_i=0)} \\
&\qquad \times \left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)-Pr(Z_j=0)\right]\times(1+o_p(1)) \\
&=_{(6)} \frac{E[Y_i \mathbb{1}(D_i=1,Z_i=1)\mathbb{1}(Y_i \in \mathcal{Y}_1)]}{Pr(Z_i=1)Pr(Z_i=0)}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)-Pr(Z_j=0)\right]+o_p(1/\sqrt{n})
\end{aligned} \tag{A.33}
$$

where equality (1) holds by *Term 1* of Lemma A.6 with $d=1$ and $c_n=o(b_n)$; equality (2) holds by the Glivenko–Cantalli theorem for changing class of sets; equality (3) holds because we multiply and divide the same term; equality (4) holds by the continuous mapping theorem; equality (5) holds by dominated convergence theorem since $\mathbb{1}(y \in \mathcal{Y}_1^{infsb}(b_n+c_n)) \to \mathbb{1}(y \in \mathcal{Y}_1)$; equality (6) holds by observing that $\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)-Pr(Z_j=0)=o_p(1/\sqrt{n})$ and then we apply Slutsky's theorem to get the equality.

Similarly, apply Lemma A.6 *Term 3* with $d=1,k=0,$ we have

$$
A_2=\frac{E[Y_i \mathbb{1}(D_i=1,Z_i=0)\mathbb{1}(Y_i \in \mathcal{Y}_1)]}{Pr(Z_i=1)Pr(Z_i=0)}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)-Pr(Z_j=1)\right]+o_p(1/\sqrt{n}). \tag{A.34}
$$

By Lemma A.6, $B=o_p(1/\sqrt{n})\times$ *Term 1*. By Assumption 2.8 and the choices of b_n,c_n,

$$
D \le \max\{|M_u|,|M_l|\}O((b_n+c_n)^2)=O_p(n^{-0.5}/\log^2 n)=o_p(1/\sqrt{n}).
$$

The result follows since A_1 in (A.33), A_2 in (A.34), and C_1-C_2 terms correspond to the terms in (A.31).

The rest of the equalities in Lemma A.7 hold by applying different values of d,k in Lemma A.6. □

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A.5.2. *Proof of Theorem 2.*

Proof. Let $\boldsymbol{X}_i(b_n)$ be the vector in Lemma A.5. Now let

$$\hat{\pi} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{N} Y_i\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(Z_j=0)}\right]\mathbb{1}(Y_i \in \hat{\mathcal{Y}}_1(b_n)) \\ \frac{1}{n}\sum_{i=1}^{n} Y_i\left[\frac{\mathbb{1}(D_i=0,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} - \frac{\mathbb{1}(D_i=0,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)}\right]\mathbb{1}(Y_i \in \hat{\mathcal{Y}}_0(b_n)) \\ \frac{1}{n}\sum_{i=1}^{N}\left[\frac{\mathbb{1}(D_i=1,Z_i=1)}{\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(Z_j=1)} - \frac{\mathbb{1}(D_i=1,Z_i=0)}{\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(Z_j=0)}\right]\mathbb{1}(Y_i \in \hat{\mathcal{Y}}_1(b_n)) \\ \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\mathbb{1}(D_i=0,Z_i=0)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=0)} - \frac{\mathbb{1}(D_i=0,Z_i=1)}{\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}(Z_j=1)}\right]\mathbb{1}(Y_i \in \hat{\mathcal{Y}}_0(b_n)) \end{pmatrix},$$

$$\pi = \begin{pmatrix} \int_{\mathcal{Y}_1} y(p(y,1)-q(y,1))dy \\ \int_{\mathcal{Y}_0} y(q(y,0)-p(y,0))dy \\ \int_{\mathcal{Y}_1} (p(y,1)-q(y,1))dy \\ \int_{\mathcal{Y}_0} (q(y,0)-p(y,0))dy \end{pmatrix}.$$

By Lemma A.7,

$$\sqrt{n}(\hat{\pi}-\pi) = o_p(1) + \Gamma\sqrt{n}(\boldsymbol{X}_i(b_n) - E[\boldsymbol{X}_i(b_n)]),$$

where Γ matrix is specified in Theorem 2. Moreover, we notice that $\widehat{LATE} = \frac{\hat{\pi}_1}{\hat{\pi}_3} - \frac{\hat{\pi}_2}{\hat{\pi}_4}$, and $\widetilde{LATE}^{ID} = \frac{\pi_1}{\pi_3} - \frac{\pi_2}{\pi_4}$, and the Π in Theorem 2 is the Jacobian matrix of function $f(\pi) = \frac{\pi_1}{\pi_3} - \frac{\pi_2}{\pi_4}$. The result follows due to the delta method. □

A.6. Proofs of Proposition 3.1

Proof. Let $F \in \cup_{s\in A} M^s(G^s)$ hold. By definition of the identified set

$$\Theta_{\tilde{A}}^{ID}(F) = \left\{\theta(s) : F \in M^s(G^s),\, s \in \tilde{A}\right\}. \tag{A.35}$$

Since F does not refute the original assumption A, there is an $s_0 \in A$ such that $F \in M^{s_0}(G^{s_0})$ and $m_j(s_0) = 0$ hold by Definition 3.6. As a result, $m_j^{min}(F) = 0$ must hold. We can then rewrite (A.35):

$$\Theta_{\tilde{A}}^{ID}(F) = \left\{\theta(s) : F \in M^s(G^s),\, s \in \tilde{A},\, m_j(s) = 0\right\}. \tag{A.36}$$

Since m_j is a sharp characterization of the predictions of A_j, $\tilde{A} \cap \{s : m_j(s) = 0\} = A$. We can then write (A.36) as

$$\Theta_{\tilde{A}}^{ID}(F) = \left\{\theta(s) : F \in M^s(G^s),\, s \in A\right\} = \Theta_A^{ID}(F),$$

where the last equality holds by the definition of the identified set under assumption A. This proves the first statement of the proposition.

Suppose m_j does not induce a partition on the predictions of A_j under $\cap_{l\neq j} A_l$, then by definition, we can find an F' such that

$$F' \in \left(\cup_{s:s\in A} M^s(G^s)\right) \cap \left(\cup_{s:s\in \tilde{A}\setminus A,\, m_j(s)=0} M^s(G^s)\right).$$

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Since $\cap_{l\neq j} A_l$ is non-refutable, we must have an $s' \notin A$ such that $F' \in M^{s'}(G^{s'})$. Let $\tilde{\theta}$ be the econometric structure itself, i.e., $\tilde{\theta}(s) = s$. Then we have $s' \in \tilde{\Theta}^{ID}_{\tilde{A}}(F')$ but $s' \notin \tilde{\Theta}^{ID}_{A}(F')$. This proves the second statement.

It remains to show that a well-behaved sharp characterization always exists. Consider the following set:

$$\mathcal{H}^{nf}_{\mathcal{S}}(A) = \left\{ s : M^s(G^s) \cap \left(\cup_{s' \in A} M^{s'}(G^{s'}) \right) \neq \emptyset \right\},$$

so $\mathcal{H}^{nf}_{\mathcal{S}}(A)$ collects all econometric structures that are observationally equivalent to some structures in A. We consider the relaxed assumption

$$\tilde{A}^{max} \equiv \left(A \cup [\mathcal{H}^{nf}_{\mathcal{S}}(A)]^c \right) \cap \left(\cap_{l \neq j} A_l \right).$$

Correspondingly, we define a relaxation measure m_j such that: (1) $m_j(s) = 0$ for all $s \in A$; (2) $m_j(s) = 1$ for all $s \in \tilde{A}^{max} \backslash A$; and (3) $m_j(s) = +\infty$ for all $s \in \mathcal{S} \backslash \tilde{A}^{max}$. This m_j sharply characterizes A_j.

We want to show that m_j is well-behaved. Consider any F: if F does not refute A, then there is some $s \in A$ such that $F \in M^s(G^s)$ and $m_j(s) = 0$ achieves the minimal deviation; if F refutes A, then no structures in $\mathcal{H}^{nf}_{\mathcal{S}}(A)$ can predict F because all structures in $\mathcal{H}^{nf}_{\mathcal{S}}(A)$ are observationally equivalent to A. To predict F, we must find $s \in [\mathcal{H}^{nf}_{\mathcal{S}}(A)]^c \cap \left(\cap_{l \neq j} A_l \right)$.[23] Such an s must have $m_j(s) = 1$ and achieves the minimal deviation $m^{min}_j(F) = 1$. □

B. Continuity of an Identified Set

As we discussed in the main text, we recommend that the econometrician checks the continuity of the identified set. In this section, we fix a parameter of interest θ and further discuss some high-level conditions that ensure the continuity property of the identified set. We endow the observed data distribution space $\mathcal{F}$ with a metric $d_{\mathcal{F}}$. We briefly recall the definition of a continuous correspondence.

Property 1 (Identified Set Continuity). The identified set $\Theta^{ID}_A(F) : \mathcal{F} \rightrightarrows \Theta$ is called upper hemincontinuous at a point F_0 if for any open neighborhood V of $\Theta^{ID}_A(F_0)$, there exists a neighborhood U of F_0 such that for all $F' \in U$, $\Theta^{ID}_A(F')$ is a subset of V. The identified set is called lower hemicontinuous at a point F_0 if for any open set V intersecting $\Theta^{ID}_A(F_0)$, there exists a neighborhood U of F_0 such that $\Theta^{ID}_A(F')$ intersects V for all $F' \in U$. A continuous identified set is both upper and lower hemicontinuous.

If the identified set is an interval, then it is continuous if and only if both the upper and lower bounds are continuous functions of F. Without the continuity property, a consistent estimator of the identified set may not exist, and the identified set can be spuriously informative due to sampling error. Examples of discontinuous and continuous identified set correspondences can be found in Proposition B.3. The following is a sufficient condition to check Property 1 for the minimal deviation relaxed assumption in Definition 3.7.

[23] Finding such an s is possible, otherwise, $\left(\cap_{l \neq j} A_l \right)$ is refutable.

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PROPOSITION B.1. *Let $(\mathcal{F}, d_F)$ and (Θ, d_θ) be metric spaces, and let $\mathcal{S}$ be a structure universe. Let $\tau_{\mathcal{F}}$ be the topology on $\mathcal{F}$ induced by d_F. Let $A = A_j \cap (\cap_{l\neq j} A_l)$, let A_j be the assumption that we want to relax, and let m_j be the well-behaved relaxation measure in Definition 3.7. We equip the $\cap_{l\neq j} A_l$ space with the weak topology $\tau_{A_{-j}}$ induced by the mapping $h(s) = M^s(G^s)$[24]:*

$$\tau_{A_{-j}} \equiv \{O \subseteq \cap_{l\neq j} A_l : O = h^{-1}(P) \cap (\cap_{l\neq j} A_l) \text{ for some } P \in \tau_{\mathcal{F}}\}.$$

Suppose: (1) $\theta(s) : \cap_{l\neq j} A_l \to \Theta$ is a continuous function and (2) the function $m_j^{min}(F)$ in Definition 3.6 is continuous. If $m_j^{-1} : \mathbb{R} \rightrightarrows \cap_{l\neq j} A_l$ is an upper (resp., lower) hemicontinuous correspondence, then $\Theta^{ID}_{\tilde{A}}(F)$ is an upper (resp., lower) hemicontinuous correspondence from $(\mathcal{F}, d_F)$ to (Θ, d_θ).

Here is the rationale behind Property 1: we may want a continuous relation between the structure universe $\mathcal{S}$ and the observation space $\mathcal{F}$. When the true structure s changes a little, the predicted observation distribution should not change drastically. Similarly, the parameter of interest θ should also be continuous with respect to change in the true structure. The relation can be represented as $\mathcal{F} \xleftarrow{M^s(G^s)} \mathcal{S} \xrightarrow{\theta(s)} \Theta$. Unfortunately, there may not exist a natural topology embedded in $\mathcal{S}$. The weak topology defined in Proposition B.1 is the smallest topology such that the mapping $M^s(G^s)$ is continuous. The construction of $\tau_{A_{-j}}$ in Proposition B.1 uses the inverse of $M^s(G^s)$ to induce a topology on the structure universe $\mathcal{S}$. With this construction, the relations become: $\mathcal{F} \xrightarrow{(M^s(G^s))^{-1}} \mathcal{S} \xrightarrow{\theta(s)} \Theta$. The identified set can then be viewed as the composite mapping of $(M^s(G^s))^{-1}$ and θ, defined on the relaxed assumption $\tilde{A} \subseteq \mathcal{S}$. If $\theta(s)$ is continuous, and the relaxation that we make is continuous, i.e., m_j^{-1} is continuous, then the composite map should also be continuous. Therefore, Property 1 can be viewed as a consequence of the continuity of $\mathcal{F} \to \mathcal{S}$, the continuity of $\mathcal{S} \to \Theta$, and a continuous relaxation of the original assumption A via m_j.

An Example of a Failure of Property 1

We now consider an alternative relaxed assumption in the *LATE* application where the corresponding identified set is not continuous. We focus on the case where $\mathcal{Y}$ is a bounded subset of $\mathbb{R}$. Let us consider the case that A^{ER} and A^{ND} hold but we relax the independent IV assumption. Instead of using the m^{MI} as the relaxation measure, we construct the relaxed assumption in another way.

First, we consider the set of econometric structures whose prediction intersects with the predictions of the IA-M assumption:

$$\mathcal{H}^{nf}_{\mathcal{S}}(A) = \left\{s : M^s(G^s) \cap \left(\cup_{s' \in A} M^{s'}(G^{s'})\right) \neq \emptyset\right\}.$$

In other words, $\mathcal{H}^{nf}_{\mathcal{S}}(A)$ collects all econometric structures that are observationally equivalent to some structures in the original assumption A. We consider a relaxed assumption

[24]This is a slight abuse of the notation since $h(s)$ is a single-valued correspondence and its image space is $2^{\mathcal{F}}$. We abuse the notation and use $h(s)$ to denote the $M^s(G^s)$ as a function since $M^s(G^s)$ is a singleton in the complete model setting.

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$\tilde{A}^{max} \equiv \left(A \cup [\mathcal{H}_{\mathcal{S}}^{nf}(A)]^c\right) \cap A^{ER} \cap A^{ND}$. This $\tilde{A}^{max}$ is the construction used in the proof of Proposition 3.1.

PROPOSITION B.2. *The closure of the identified set for LATE under $\tilde{A}^{max}$ is*

$$\overline{LATE^{ID}_{\tilde{A}^{max}}(F)} = \begin{cases} \frac{E[Y_i|Z_i=1]-E[Y_i|Z_i=0]}{E[D_i|Z_i=1]-E[D_i|Z_i=0]} & \text{if } F \text{ does not refute } A, \\ \left[\underline{\mathcal{Y}}_{P(B,1)} - \bar{\mathcal{Y}}_{Q(B,0)}, \bar{\mathcal{Y}}_{P(B,1)} - \underline{\mathcal{Y}}_{Q(B,0)}\right] & \text{otherwise,} \end{cases}$$

where for $V \in \{P, Q\}$, $\underline{\mathcal{Y}}_{V(B,0)}$ is the lower bound of the support of Y_i under measure $V(B,0)$, and $\bar{\mathcal{Y}}_{V(B,1)}$ is the upper bound of the support of Y_i under measure $V(B,1)$.

The intuition of the identified set is clear: when F does not refute A, we maintain the original IA-M assumption, and when F refutes A, we simply give up the independent IV assumption fully. As a result, the identified set for *LATE* is very unstable when F satisfies $p(y,1) - q(y,1) = 0$ for some positively measured set of y. Whenever we perturb F slightly such that $p(y,1) - q(y,1) < 0$ and F refutes A, the identified set for *LATE* explodes. Second, the identification set $LATE^{ID}_{\tilde{A}^{max}}(F)$ is not any better than the $LATE^{Wald}_{A}(F)$ in equation (2.8). In terms of interpretation, an uninformative identified set[25] for *LATE* is not different from an empty identified set.

We now compare the identified set for *LATE* under the maximal extension in Proposition B.2 and the identified set for *LATE* in Theorem 1 in terms of Property 1. We equip $\mathcal{F}$ with the Sobolev norm: $||F||_{1,\infty} \equiv \max_{i=0,1} ||F^{(i)}||_\infty$, where $F^{(i)}$ is the i-th Radon–Nikodym density of F with respect to μ_F.

PROPOSITION B.3. *Let $\tilde{A}_1$ be the relaxed assumption in Proposition B.2 and let $LATE^{ID}_{\tilde{A}_1}(F)$ be the corresponding identified set. Let $\tilde{A}_2$ be the minimal defiers relaxed assumption defined in Assumption 2.1 and $LATE^{ID}_{\tilde{A}_2}(F)$ be the corresponding identified set defined in (2.14). Suppose $\forall F \in \mathcal{F}$, the support of Y_i is bounded above by M_s^u and bounded below by M_s^l, then $LATE^{ID}_{\tilde{A}_1}(F)$ is not upper hemicontinuous with respect to the Sobolev norm $||\cdot||_{1,\infty}$, and $LATE^{ID}_{\tilde{A}_2}(F)$ is continuous with respect to $||\cdot||_{1,\infty}$.*

Supplementary Material

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[25] Note that the identification result in Proposition B.2 contains only support information when the IA-M assumption is refuted.

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